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Hard-core scattering for N-body systems

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We prove propagation properties (maximal and minimal velocity bounds) for pseudo-resolvents associated with N-body Hamiltonians with short-range potentials that are infinite on a star-shaped domain centered at the origin. Motivated by the fact that the invariance principle holds for usual N-body systems, we define the cluster wave operators in terms of pseudo-resolvents and prove that they exist and are asymptotically complete. For any cluster decomposition a these operators intertwine the hard-core pseudo-selfadjoint Hamiltonians, corresponding to the pair of pseudo-resolvents R, R_a, and equal the Abel operators constructed in terms of Hamiltonians.

Contents:

1. Introduction

One of the most important goals in scattering theory is the study of the asymptotic behavior (when $t \to \pm\infty$) of $e^{-itH}\psi$, where ψ is an arbitrary state on the orthogonal complement of the space of eigenvectors of the Hamiltonian H. More precisely, we are interested in finding a family $\{H_a\}$ of selfadjoint operators with simpler (and known) spectral and evolution properties, such that for any state ψ a family of vectors $\{\psi_a^\pm\}$ should exist, for which the convergences

$$\left\| e^{-itH}\psi - \sum_a e^{-itH_a}\psi_a^\pm \right\| \overset{t\to\pm\infty}{\to} 0 \tag{1.1}$$

are satisfied. If this takes place, then we say that the system is *asymptotically complete.*

The peculiarity of the N-body Hamiltonians is that they are a sum of a differential operator (with excellent dispersion properties) and a perturbation that does not vanish (when $|x| \to \infty$) along certain directions of the configuration space X. This makes us

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think that, if asymptotic completeness holds for such systems, then $e^{-itH_a}\psi_a^{\pm}$ should be asymptotically localized within some cones centered at the classical trajectories.

These geometrical ideas allowed V. Enss to show that (1.1) is true for three-body quantum systems with potentials that decay slightly faster than the Coulomb interaction. After that, asymptotic completeness for N-body short-range quantum systems was proved in 1987 by I. Sigal and A. Soffer [33], and in the succeeding years many people tried to simplify or extend their proof for more complicated many-body problems. Indeed, there was first an effort of making the theory more "readable", done by J. Derezinski in [13]. Then, G. M. Graf, using jointly ideas not only from the earlier works of V. Enss [15-18] and from [33] but also from more recent papers of Sigal and Soffer ([35] and especially [34]), succeeded in giving in [22] a remarkable time-dependent-like proof of the quoted result, which differed from the previous proofs by several (important) aspects. We will emphasize only the fact that in [22] some of the main propagation properties were obtained without the use of the Mourre estimate, i.e. independently of intimate knowledge of the spectral properties of the Hamiltonian. These properties were sufficient for showing the existence of the cluster wave operators but not for their completeness. Indeed, for the latter result, a propagation property involving jointly a time-dependent localization in position and a localization in the total energy was needed, and for proving it a good knowledge of the spectrum of the N-body Hamiltonian was crucial. Actually, this is the only place where Graf invokes the Mourre estimate (in order to obtain (local) positivity for the commutator of the Hamiltonian with the generator of the dilations group) and by this means he eliminates the decay hypothesis imposed in [33] on the second derivative of the potential. Further, using refined results on the Mourre theory due to W. Amrein, Anne Boutet de Monvel, and V. Georgescu (see [1] and also [10] for optimality), we have shown in [26], on the lines of [22], that no condition on the derivatives of the potential was needed to prove completeness of the Agmon-type systems. Moreover, since locally the potentials were allowed to be as singular as the kinetic energy permits it, the question of validity of the statement on asymptotic completeness for much singularly perturbed systems (as the hard-core N-body quantum systems) arises naturally. Indeed, interest in such problems is rather old, going back, e.g., to the works of W. Hunziker [25] and especially of D. W. Robinson, P. Ferrero, and O. de Pazzis (see [31, 20]), where, under rather restrictive assumptions on the geometry of the potentials (spheric symmetry, the supports of the singularities where cylinders centered on the subspaces of the relative movement of the clusters) and on the forces (repulsivity), the absence of the singular continuous spectrum and the existence and completeness of the wave operators, corresponding to the *elastic* channel, were established. But this is, of course, a very simplified case, because even if the problem was posed in an N-body context, the above hypothesis transformed it into a one-channel scattering problem. Very recently, Anne Boutet de Monvel, V. Georgescu, and A. Soffer, using both the locally conjugate operator method and an algebraic approach (which appears naturally in the N-body context), have succeeded in giving in [11] a complete spectral analysis for this type of highly singular Hamiltonians. More precisely, it was proven that under quite reasonable smoothness conditions, imposed on the border of the supports of the singularities, the generator of the dilations group is (in some weak sense) conjugated to the hard-core N-body Hamiltonian, which proves to be sufficient to obtain a limiting absorbtion principle (even in an optimal form). Then, absence of the singular-continuous part of the spectrum and local decay follow in a standard way. Our task is to continue this work by studying the scattering properties of these systems.

We will begin by describing the geometrical particularities of the configuration space X (an Euclidean space) related to the N-body problem. Let us denote by L a finite partially ordered index set and demand it to be a lattice. Take then a family $\{ X^a \}_{a \in L}$ of subspaces of X such that $X^{\sup(a,b)} = X^a + X^b$, and $\{ 0 \}$ and X correspond to $\min L \equiv a_{\min}$ and $\max L \equiv a_{\max}$, respectively. In the usual N-body situation, where X is the space of the configurations of the set of N particles relative to the center of mass coordinate system, L is the lattice of partitions of the set $\{ 1,...,N \}$, X^a is the subspace of X consisting of the configurations which describe the internal motion of the clusters (fragments) of the partition a and, finally, X_a (the orthogonal of X^a in X with respect to a well chosen scalar product) can be identified with the space of configurations of the relative motions of the clusters. Let us denote for any $a \in L$ by π^a the orthogonal projection on X^a. Further, according to S. Agmon (see [3]), an N-body type Hamiltonian is defined as the sum of the (positive) Laplace–Beltrami operator Δ and a family $\{ \widetilde{V}(a) \}_{a \in L}$ of operators which are factorized as $\widetilde{V}(a) = \widetilde{V}^a \circ \pi^a$. Here we considered the simplest case where $\widetilde{V}^a$ is the operator (in H (X^a)) of multiplication by a function having a good decay at infinity in all directions of X^a. Although this assumption is currently used in many of the papers dedicated to this subject, in the more recent ones it is shown that the same results can be obtained if $\widetilde{V}^a$ is a reasonable differential operator. As for the hard-core systems, the physical picture of clusters formed by particles that cannot get arbitrarily close to each other is modeled by (positive) singularities of the potentials, having as supports compacts of X^a; of course, a short and long range parts can be added to these singular potentials. A precise definition is obtained by using a limiting process, in which the hard-core Hamiltonian H is seen as a limit, in the strong resolvent sense, of the family of self-adjoint operators in H (X)

$$H_\alpha = \Delta + \sum_{a \in L} \left(V(a) + \alpha\chi(a) \right) = \widetilde{H} + \alpha \sum_{a \in L} \chi^a \circ \pi^a . \tag{1.2}$$

We have denoted by $\widetilde{H}$ a standard N-body Hamiltonian which, under natural assumptions on the symmetric operators V^a, becomes a selfadjoint, bounded from below operator whose form-domain is the order one Sobolev space $\mathrm{H}^1(X)$. Notice that, when tending to infinity, the parameter $\alpha \geq 0$ will increase the value of the cylindrically supported perturbation $\chi(a)$, where χ^a denotes the operator of multiplication with the characteristic function of a compact K^a from X^a.

It has been proven (in [11], Lemma 3.7 and Proposition 3.8) that for *each* z in $\mathbb{C} \setminus [\inf \sigma(\widetilde{H}), \infty)$ the limit $R(z) \equiv \lim_{\alpha \to \infty} (z - H_\alpha)^{-1}$ exists in the strong sense in $B(\mathrm{H}^{-1}, \mathrm{H}^1)$ and in the norm of each of the Banach spaces $B(\mathrm{H}^s, \mathrm{H}^t)$ for $-1 \leq s \leq t \leq 1$ and $t - s < 2$. Actually, $R(z)$ is a *self-adjoint pseudo-resolvent family*, i.e. it satisfies the first resolvent identity and $R(z^*) = R(z)^*$. It is known (see [23]) that the closure in H of Ran $R(z)$ is a *proper* subspace of H (let us denote it by H_∞) which does not depend on z and coincides with the closure of the domain of a self-adjoint operator H

for which $R(z)\,|_{\mathrm{H}_\infty} = (z - H)^{-1}$ and $R(z)\,|_{\mathrm{H}\ominus\mathrm{H}_\infty} = 0$. We will call H a *pseudo-selfadjoint* operator on $\dot{\mathrm{H}}$ and this will be the Hamiltonian modeling a hard-core Agmon-type problem. Nevertheless, we shall refer to its spectral properties as those of the selfadjoint operator H which acts in the proper subspace H_∞.

We emphasize that one of the difficulties in studying the spectral and scattering properties of such operators is the fact that they are not densely defined. Moreover, we cannot say a priori that Hamiltonians constructed as above are factorized in the same tensor product form as those from the family $\{H_a\}$. Actually this holds. Indeed, as in the usual N-body problem, using the limiting process described before, it is possible to construct for each $a \in \mathrm{L}$ a pseudo-selfadjoint Hamiltonian H^a, which corresponds to the hard-core problem relative to the sublattice $\mathrm{L}_a = \{b \in \mathrm{L} \mid b \le a\}$. On the other hand, it is shown that for any $a \in \mathrm{L}$ the family of resolvents $\{R_{a,\alpha}\}_{\alpha \ge 0}$ of the (genuine) selfadjoint operators

$$H_{a,\alpha} = H^a_\alpha \otimes_a 1 + 1 \otimes_a (\pi_a \nabla)^2$$

tends in norm in $B(\mathrm{H}^{-1}(X), \mathrm{H}^\theta(X))$, for any $\theta < 1$, to a pseudo-resolvent R_a (to which the pseudo-selfadjoint sub-Hamiltonian H_a corresponds), whenever the convergence $H^a_\alpha \to H^a$ takes place in the norm-resolvent sense in $B(\mathrm{H}^{-1}(X^a), \mathrm{H}^\theta(X^a))$. This implies

$$H_a = H^a \otimes_a 1 + 1 \otimes_a \Delta_a, \tag{1.3}$$

where Δ_a denotes the Laplace-Beltrami operator in $\mathrm{H}(X^a)$ and the above tensor product sum is an operator defined in the proper subspace $\overline{D(H^a)} \otimes \mathrm{H}(X_a)$ of $\mathrm{H}(X)$, with the operator domain $D(H_a) = D(H^a) \otimes \mathrm{H}^2(X_a)$.

Then, the set of *thresholds* of H is defined as

$$\tau(H) = \bigcup_{a \in \mathrm{L}\setminus\{a_{\max}\}} \sigma_p(H^a), \tag{1.4}$$

whereas the set of *critical values* of H, denoted by $\xi(H)$, will be the union of $\tau(H)$ with $\sigma_p(H)$ being the point spectrum of H.

The main ingredient for the study of the spectral and scattering properties of the Hamiltonian is the Mourre estimate, which states (see (1.5) below) local positivity for the commutator of H with the generator of the dilations group A, defined as

$$A = \frac{1}{2}(P\cdot Q + Q\cdot P) = \frac{1}{2}\sum_{j=1}^{\dim X}(P_j Q_j + Q_j P_j),$$

where Q_j is the operator of multiplication by the coordinate x_j (w.r.t. some orthonormal basis of X) and $P_j \equiv -i\frac{\partial}{\partial x_j} = \mathrm{F}^* Q_j \mathrm{F}$, with F being the Fourier transform. Actually, this estimate stresses *strict* positivity of the lower semicontinuous function ρ^A_H: $\mathrm{R} \to (-\infty, +\infty]$ on an open subset of R, where for all $\lambda \in \mathrm{R}$

$$\rho_H^A(\lambda) \overset{def}{=} \sup\{\mu \in \mathrm{R} \mid \exists f \in C_0^\infty(\mathrm{R};\mathrm{R}), f(\lambda) \neq 0 \text{ s.t. } f(H)[iH,A]f(H) \geq \mu f(H)^2\}. \quad (1.5)$$

In [2,8, 27] an extensive study of this function is made. It is also important to point out that under the assumption of strong-C^1 regularity of the Hamiltonian w.r.t. A, even if H is a pseudo-selfadjoint operator, the identity

$$[A,R(z)] = R(z)[A,H]R(z) \quad (1.6)$$

is valid for any $z \in \mathrm{C}\backslash\sigma(H)$. We refer to §5 of [11] for the precise definitions of the above commutators and for the proof of the Mourre estimate in the context of the hard-core N-body systems. In fact, the result we needed and that we will intensely use is the strict positivity of $\rho_R^A(\lambda)$ on the set $\mathrm{R}\backslash\xi(R)$.

Another difficulty arising from the fact that the limit of $\{H_\alpha\}$ is only pseudo-selfadjoint in H comes from how we have to interpret the limit of $\{e^{itH_\alpha}\}$ and, correspondingly, how we have to define the cluster wave operators. It is not a trivial fact (see [12]) that, for any $a \in \mathrm{L}$, the family of evolution groups generated by $\{H_{a,\alpha}\}$ has a limit, but this limit exists only on $\mathrm{H}_{a,\infty}$, the closure of Ran R_a in H. Moreover, taking into account the inclusion $\mathrm{H}_{b,\infty} \subseteq \mathrm{H}_{a,\infty}$, true for any $a,b \in \mathrm{L}$ with $a \leq b$, and Theorem 3.23 (ii) [12], we see that the domain of the limit $\lim_{\alpha\to+\infty} e^{itH_\alpha}$ is a priori included in the range of $\lim_{\alpha\to+\infty} e^{itH_{a,\alpha}}$ even when this one is applied to the vectors of H_∞.

As for the cluster wave operators, in the usual N-body context (i.e. for any α finite) they are defined as

$$\Omega_{a,\alpha}^\pm \equiv \Omega^\pm\left(H_\alpha, H_{a,\alpha}; \mathrm{E}_{a,\alpha}\right) = s - \lim_{t\to\pm\infty} e^{itH_\alpha} e^{-itH_{a,\alpha}} \mathrm{E}_{a,\alpha}. \quad (1.7)$$

Notice that since $\mathrm{E}_{a,\alpha} = E_{pp}(H_\alpha^a) \otimes {}_a 1$ commutes with $H_{a,\alpha}$ (which has a purely absolutely continuous spectrum) and also with bounded functions of $H_{a,\alpha}$ as a consequence of the Hilbert space isomorphisms

$$\mathrm{H}(X) \cong L^2\left((X_a, d\xi_a); \mathrm{H}(X^a)\right) \cong \int_{X_a}^{\oplus} \mathrm{H}(X^a) d\xi_a$$

(see [11 and 21], then the limits (1.7) can be considered as wave operators with an identifier $\mathrm{E}_{a,\alpha}$ (see [5]). Moreover, for any $z \in (-\infty, \inf\sigma(H))$, the equality $E_{pp}(H_\alpha^a) = E_{pp}\left(R_\alpha^a(z)\right)$ is true, so the identification operator is the same for the wave operators constructed in terms of Hamiltonians and for those constructed in terms of resolvents. For the hard-core case, these identifiers are no more equal, and we will use the symbol E_a for $E_{pp}(H^a) \otimes {}_a 1$, which projects H into H_∞. Then $\mathrm{E}_a = E_{H_a}(\mathrm{R})\mathrm{E}_a = E_{R_a}(\mathrm{R}\backslash\{0\})\mathrm{E}_a$, where $E_A(\Delta)$ denotes the spectral measure of the operator A on $\Delta \subset \mathrm{R}$.

It appears that we should try to redefine the hard-core wave operators in terms of the known objects. This is also suggested by the invariance principle, which is true in the usual N-body case at least for the admissible function (see [5] for definitions) $(z - .)^{-1}$, with z chosen as above. Indeed, suppose first that *both* strong limits $\Omega^{\pm}\left(H_{\alpha}, H_{a,\alpha}; E_{a,\alpha}\right)$ and $\Omega^{\pm}\left(R_{\alpha}, R_{a,\alpha}; E_{a,\alpha}\right)$ *exist*. Then, the corresponding absolute Abelian limit also exists, and they are equal to the strong ones (see Corollary 6.14 in [5]). Thus, we can use the *weak* form of the invariance principle (see Theorem 11.25 in [5]) for this pair of operators in order to get their *equality*, and thus equality of the strong limits.

Thus it seems quite natural to *define* the wave operators corresponding to the hard-core case in terms of resolvents, and to prove the existence and asymptotic completeness for $\Omega^{\pm}\left(R, R_a; E_a\right) \equiv \Omega_a^{\pm}$.

As we will see in the next sections, the way we do this uses an algebraic framework which is proper also for the usual N-body resolvents, so, in what follows, we will automatically prove not only the existence of $\Omega_a^{\pm}$, but also the existence of $\Omega^{\pm}\left(R_{\alpha}, R_{a,\alpha}; E_{a,\alpha}\right)$, which was previously taken as a hypothesis for the weak invariance principle in the case of usual N-body systems.

Obviously, the intertwining property $\Omega_a^{\pm} = E_R(\mathrm{R}\setminus\{0\})\Omega_a^{\pm}$ is valid, and since for all $a \in \mathrm{L}$, $(z - H_a)R_a = E_{H_a}(\mathrm{R})$, $\Omega_a^{\pm}$ are partial isometries with the final domain H_{∞}, which intertwine the pair H, H_a on the closure of the range of R_a. The connection between $\Omega_a^{\pm}$ and the wave operators defined in terms of Hamiltonians as limits of

$$\mathrm{W}_a(t) = e^{itH}E_H(\mathrm{R})e^{itH_a}\mathrm{E}_a,$$

is made with the strong form of the invariance principle. Indeed, let $\mathrm{W}_a^{\pm}$ be the absolute Abelian limit of $\mathrm{W}_a(t)$, i.e.

$$\lim_{\varepsilon \to +0} \varepsilon \int_0^{\infty} e^{-\varepsilon t}\left\|\mathrm{W}_a(\pm t)\psi - \mathrm{W}_a^{\pm}\psi\right\|^2 dt = 0. \tag{1.8}$$

Then, in Appendix 6.2 we will show that the existence of $\Omega_a^{\pm}$ implies the existence of $\mathrm{W}_a^{\pm}$ and their equality. This shows that the strong limits of $\mathrm{W}_a(t)$, if they exist, are partial isometries with the final domain H_{∞}. Actually, this can be shown also directly, using an argument similar to that described in the proof of Lemma 4.1.

Finally, we would like to remark that despite their boundedness, the resolvents are not always comfortable objects to work with, mainly because of their non-local character. This feature becomes critical when one wants to prove, on the lines of [22, 14, 39], the following propagation theorem, that we consider to be the main result of the paper.

Theorem 1.1. *Let $a \in \mathrm{L}$ be arbitrarily chosen. If $\theta \in C_0^{\infty}\left(\mathrm{R}\setminus\xi(R_a)\right)$ and if $J \in C_0^{\infty}(X)$ has its support localized sufficiently close to the origin and outside any of the subspaces of the family $\{X_b\}_{b \in \mathrm{L}\setminus\mathrm{L}_a}$, then the estimate*

$$\int_1^{\infty} \frac{dt}{t}\left\|J\left(\frac{Q}{t}\right)\theta(R)e^{-itR}\psi\right\|^2 \le C\|\psi\|^2 \tag{1.9}$$

is true for some positive constant C and for all $\psi \in \mathrm{H}(X)$.

In Section 5, a more precise form of this theorem is stated and proved, and it is also shown that as a consequence of a particular case of it, the *asymptotic completeness* statement

$$\sum_{a \in \mathrm{L} \setminus \{a_{\max}\}} \Omega_a^{\pm} (\Omega_a^{\pm})^* = E_c(H) \tag{1.10}$$

is valid.

Let us review the contents of the following sections. In the next section we expose (following [6, 8, 9, 11] the algebraic framework related to an N-body type problem, putting emphasis on the results we need in our paper. We also briefly remind the construction of the partition of unity in the configuration space due to G. M. Graf and list some of the properties of the vector field attached to this family of functions. Section 3 is devoted to the proofs of those propagation estimates that can be deduced without the use of the Mourre estimate. It is shown in Section 4 that the existence of the limits $\Omega_a^{\pm}$ is a consequence of these estimates. Finally, Section 5 is mainly devoted to the proof of Theorem 1.1.

2. Algebraic and geometric frameworks

As we have already mentioned in the first section, for any $a \in \mathrm{L}$ the direct sum $X^a \oplus X_a$ determines the canonical isomorphisms of the Hilbert spaces $\mathrm{H}(X) \cong \mathrm{H}(X^a) \otimes \mathrm{H}(X_a) \cong L^2(X_a; \mathrm{H}(X^a))$. Let us denote by $\mathrm{K}(X^a)$ the C^*-algebra of compact operators on $\mathrm{H}(X^a)$ and let $\mathrm{T}^*(X_a)$ be the C^*-algebra naturally associated with the translation group in $B\left(\mathrm{H}(X_a)\right)$, i.e., the norm-closure (in $B\left(\mathrm{H}(X_a)\right) \equiv B(X_a)$) of the $\star$-subalgebra of operators of the form $f(\pi_a P) = \mathrm{F}^*_{X_a} f(\pi_a Q)\, \mathrm{F}_{X_a}$, with $f \in C_\infty(X_a)$. Then, the norm-closure in $B(X)$ of the linear space generated by the operators $S \otimes_a T$ which correspond (through the first of the above isomorphisms) to $S \otimes T \in \mathrm{K}(X^a) \otimes \mathrm{T}^*(X_a)$ will be a $C^{\star}$-subalgebra of $B(X)$, named the algebra of a-semicompact operators and denoted by

$$\mathrm{T}(a) = \mathrm{K}(X^a) \otimes_a \mathrm{T}^*(X_a)\,. \tag{2.1}$$

Further, with the aid of the family $\{\mathrm{T}(b)\}_{b \in \mathrm{L}}$ the vector space sum

$$\mathrm{T}_a = \sum_{b \in \mathrm{L}_a} \mathrm{T}(b) \tag{2.2}$$

(with $\mathrm{L}_a = \{ b \in \mathrm{L} \mid b \le a \}$) is constructed for each $a \in \mathrm{L}$. It was shown (see [6, 8] that the above sum is direct in the topological sense and that the canonical projections $\mathrm{P}(b)$: $\mathrm{T} \to \mathrm{T}(b)$ (which assign to any $T \in \mathrm{T} \equiv \mathrm{T}_{a_{\max}}$ a unique $T(b) \in \mathrm{T}(b)$) are norm-continuous and satisfy $\mathrm{P}(b)[T^*] = T(b)^*$. We will refer to $\mathrm{T}(b)$ as the *b-connected component* of T. Moreover, for any $a \in \mathrm{L}$ the projection $\mathrm{P}_a \overset{def}{=} \sum_{b \in \mathrm{L}_a} \mathrm{P}(b)$ is a $\star$-homomorphism between

T and its C^*-subalgebra T_a. For any $a \in L \setminus \{a_{max}\}$ a concrete expression for P_a was given by W. N. Polyzou (see (6) in [29], and also [30, 6, 8]. Notice that using the Möbius inversion formula (see Theorem 4.18 in [4]) we can retrieve any of the operators $T(a) \in \mathrm{T}(a)$ as a weighted sum of elements of the family $\{T_b\}_{b \in L_a}$, the weight being the Möbius function $\mu(b,a)$.

The second important property of the family of algebras of semicompact operators is its *graduation* with respect to the semi-lattice L, which means that for any $a,b \in L$, the inclusion

$$\mathrm{T}(a)\,\mathrm{T}(b) \subseteq \mathrm{T}(\sup\{a,b\}) \tag{2.3}$$

is valid. We summarize all these properties by saying that T is an *L-graded C^*-algebra.*

Let us go back now to the pseudo-selfadjoint operators H defined in Section 1. According to [8], we will say that H *is affiliated to the C^*-algebra* T iff *all* the realizations of the $*$-homomorphism $\phi : C_\infty(\mathbb{R}) \to B(X)$ belong to T. But since the Stone–Weierstrass Theorem ensures the existence of a bijection between the $*$-homomorphisms $\phi : C_\infty(\mathbb{R}) \to B(X)$ and pseudo-resolvents (such that for any $f \in C_\infty(\mathbb{R})$ we get $\phi(f)\big|_{H_\infty} = f(H)$ and $\phi(f)\big|_{H \oplus H_\infty} = 0$), and since H does not depend on z from the pseudo-resolvent w.r.t. which it has been defined, then an equivalent definition for the affiliation of H to T is: $R(z) \in \mathrm{T}$ for *some* $z \in \mathbb{C} \setminus \sigma(H)$. Remark that if H is affiliated to T, then any of the members of the family $\{H_a\}_{a \in L}$ is affiliated to a corresponding element of $\{T_a\}_{a \in L}$. Indeed, given two C^*-subalgebras T and $\widetilde{\mathrm{T}}$ of $B(X)$ and a pseudo-selfadjoint operator H affiliated to T, there exists a unique pseudo-selfadjoint operator P $[H]$, which is affiliated to $\widetilde{\mathrm{T}}$ and for which the relation P $[\phi] = \widetilde{\phi}$ is true on all $C_\infty(\mathbb{R})$. The operator P $[H]$ is related to H via the $*$-homomorphism $\mathrm{P} : \mathrm{T} \to \widetilde{\mathrm{T}}$. In [11] several affiliation criteria are given and it is also proved that the hard-core N-body Hamiltonian is affiliated to the N-body algebra T. As a consequence of this, an HVZ theorem is obtained for this context.

We are particularly interested in establishing the extension of the first resolvent identity to the context of L-graded C^*-algebras. For the case of the usual N-body systems, this is called the Weinberg-Van Winter equation (see [29, 6]. Actually, as we will see below, borrowing the algebraical-combinatorial technique of deducing such kind of identity, it is possible to give (primarily for N-body Hamiltonians $\widetilde{H}$ with bounded perturbations), for any $a,b \in L$, a precise meaning to the difference $\widetilde{R}_a - \widetilde{R}_b$ and to the a-connected component of $\widetilde{R} \in \mathrm{T}$, in terms of a sum of regularizing operators. These results will be important in obtaining useful commutation relations between hard-core pseudo-resolvents and multiplication operators.

Let us begin by introducing some special subsets of the lattice L. We will call a totally ordered subset ξ *a maximal chain in* L, iff for any $a,b \in \xi$, $b < a$, for which there is no $c \in \xi \setminus \{a,b\}$ such that $b < c < a$, there is also no other $d \in L \setminus \{a,b\}$ such that $b < d < a$. We define then the rank of the finite lattice L as $\max\{\operatorname{card}\xi \mid \xi \subseteq L, \xi \text{ maximal}\} \equiv |L|$. In the N-body case, L is the lattice of partitions of a set of N elements, for which $|L| = N$

Since L is a union of maximal chains ξ, any $a \in L$ will belong to at least one of those which satisfy $\max\xi = \max L$. Then, by convention, the *rank* of a in L is taken to be the number $|a|_{\mathrm{L}} \equiv \max\{\operatorname{card}\xi \mid \max\xi = \max \mathrm{L}, \min\xi = a, \xi \text{ maximal}\}$. This notion allows us to define for any $1 \le n \le |\mathrm{L}| \equiv N$, the n-th *level* of L as $\mathrm{L}(n) = \{b \in \mathrm{L} \mid |b|_{\mathrm{L}} = n\}$. Thus, we have $\mathrm{L} = \coprod_{n=1}^{N} \mathrm{L}(n)$, where $\coprod$ stands for 'disjoint union of sets'. Finally, we introduce, for some arbitrarily fixed $a \in \mathrm{L}$ and any $b \in \mathrm{L}_a$, two special subsets of the lattice L_a:

$$\mathrm{L}_a^b \equiv (\mathrm{L}_a)^b = \{c \in \mathrm{L}_a \mid c \ge b\}, \tag{2.4}$$

$$\mathrm{L}_{ba} = \{c \in \mathrm{L} \setminus \mathrm{L}_b \mid \sup\{b,c\} = a\}. \tag{2.5}$$

It is clear that L_a^b is a lattice in L_a and that L_{ba} is a bilateral ideal in L_a. Notice also that if L^a denotes $\mathrm{L}_{a_{\max}}^a$, then for all $a \in \left(\mathrm{L} \setminus \mathrm{L}^b\right) \bigcup \{b\}$ we have $\mathrm{L}_{ba} = 0$ and for all $a \in \mathrm{L}^b \setminus \{b\}$ we have $\{a\} \subseteq \mathrm{L}_{ba}$. Let us put the sign $\not\leftarrow$ between two elements of the lattice L whenever they are incomparable. The proof of the following identity will give us more intuition on the sets introduced at (2.5):

$$\mathrm{L}_a \setminus \mathrm{L}_b = \coprod_{c \in \mathrm{L}_a^b \setminus \{b\}} \mathrm{L}_{bc}. \tag{2.6}$$

Indeed, for any $c \in \mathrm{L}_a^b \setminus \{b\}$, since for all $d \in \mathrm{L}_a^b \setminus \{c\}$ we have $\sup\{d,b\} = d \ne c$, we get $\mathrm{L}_a^b \bigcap \mathrm{L}_{bc} = \{c\}$. Besides, to each of these c the remainder of the set L_{bc} corresponds in $\{d \in \mathrm{L}_a \mid d \not\leftarrow b\}$, which is disjoint of any of the sets $\mathrm{L}_{b\tilde{c}}$ corresponding to another $\tilde{c}$ of $\mathrm{L}_a^b \setminus \{b\}$. For, supposing the existence of some $d \in \mathrm{L}_{bc} \bigcap \mathrm{L}_{b\tilde{c}}$, $\{b,d\}$ will form a pair having (in the case of $c \not\leftarrow \tilde{c}$) no least upper bound in L. But this would mean that L is not a lattice. Finally, in order to prove the inclusion $\subseteq$ in (2.6) notice that we have already shown that the r.h.s. of (2.6) includes $\mathrm{L}_a^b \setminus \{b\}$, and that for any $d \in \mathrm{L}_a$, $d \not\leftarrow b$, we have $\sup\{b,d\} \in \mathrm{L}_a^b$ and $d \in \mathrm{L}_{b,\sup\{b,d\}}$. The inverse inclusion is trivial.

Further, corresponding to the set L_{ba} we construct the following bilateral ideal of L_a:

$$\mathrm{T}_{ba} = \sum_{c \in \mathrm{L}_{ba}} \mathrm{T}(c), \tag{2.7}$$

and denote by P_{ba} the canonical projection of T onto it. Then, the resolvent identity

$$\widetilde{R}_a - \widetilde{R}_b = \widetilde{R}_b\left(\widetilde{H}_a - \widetilde{H}_b\right)\widetilde{R}_a$$

can be written using (2.6) as

$$\widetilde{R}_a - \widetilde{R}_b = \sum_{c \in \mathrm{L}_a} \widetilde{R}_b \mathrm{I}_{bc} \widetilde{R}_a,$$

where I_{bc} is the sum over L_{bc} of the symmetric operators $\widetilde{H}(d)$ which all, except $\Delta = \widetilde{H}_{a_{\min}} = \widetilde{H}(a_{\min})$ (which is affiliated to $\mathrm{T}_{a_{\min}}$), belong to $\mathrm{T}(d)$. Then, iterating the above formula and taking it into account that, as a consequence of the definition of L_{bc}, $\mathrm{I}_{bc} = 0$ if $c \le b$ or $c \not\leftarrow b$, we get $\widetilde{R}_a - \widetilde{R}_b = \Sigma\, \widetilde{R}(\xi)$, with

$$\widetilde{R}(\xi) = \widetilde{R}_b \mathrm{I}_{bb_1} \widetilde{R}_{b_1} \mathrm{I}_{b_1 b_2} \widetilde{R}_{b_2} \ldots \mathrm{I}_{b_{n-1} b_n} \widetilde{R}_{b_n}, \tag{2.8}$$

where the sum is taken over all the (not necessarily maximal) chains ξ of L_a having the g.l.b. b and as l.u.b. any of the elements of L_a. Moreover, using inductive argument, it was shown (see Lemma 3.11 in [11]) that for any chain ξ, $\widetilde{R}(\xi) \in \mathrm{T}_{\min \xi \max \xi}$. This together with

$$\mathrm{P}_{ba} \left(\sum_{d \in \mathrm{L} \setminus \mathrm{L}_a} \mathrm{P}(d) + \sum_{d \in \mathrm{L}_b} \mathrm{P}(d) \right) = 0$$

shows that $\mathrm{P}_{ba}[\widetilde{R}]$ can be computed as

$$\widetilde{R}_{ba} = \mathrm{P}_{ba}[\widetilde{R}_a - \widetilde{R}_b] = \sum \widetilde{R}(\xi), \tag{2.9}$$

where this time the sum is performed over all the chains $\xi \subset \mathrm{L}$ with $\min \xi = b$ and $\max \xi = a$. Notice that in the case of $b = a_{\min}$ we have $\mathrm{T}_{a_{\min} a} = \mathrm{T}(a)$ and thus $\widetilde{R}(a) = \widetilde{R}_{a_{\min} a}$, which tells us that the a-connected component of the hard-core resolvent is a regularizing operator on all $\mathrm{H}(X)$, given by the sum (without repetitions) $\Sigma \widetilde{R}(\xi)$ performed over all the chains of the sublattice L_a.

We are now prepared to state two important consequences of the properties described above, that we will use throughout the paper. They were proved in [11] for hard cores, by extension from the particular usual N-body context, by the limiting procedure. The first one, is the following commutation relation (see Proposition 3.13 in [11])

$$[g, R(a)] = \sum_{b \in \mathrm{L}_a} R(b)[g, \Delta] R_{ba} + R(a)[g, \Delta] R_a, \tag{2.10}$$

true for any multiplication operator with functions $g \in C^2(X)$ having first and second order bounded derivatives (we are particularly interested in the case where g is the identity function). Notice that (2.10), (2.9), and (2.8) show that the multiple commutators $\mathrm{ad}_Q^k(R) = [\ldots[R,Q],Q],\ldots Q]$ are in $B(\mathrm{H}(X))$ for any finite order k.

The second result (Theorem 6.5 in [11]) states, for any $a \in \mathrm{L}$, decay of the a-connected component of the hard-core resolvent along all the directions of X^a, with the same rate as that imposed by the hypothesis on the non-hard-core part of the potentials. Since in this paper we are interested only in short-range N-body potentials, the only decay condition we impose on the functions V^a (see (1.2) and the comments following it) is: for any $a \in \mathrm{L} \setminus \{a_{\min}\}$, there is some $\mu > 1$ such that $\langle Q^a \rangle^\mu V^a$ belongs to $B(\mathrm{H}^1(X^a), \mathrm{H}^{-1}(X^a))$. We used $\langle x \rangle$ as an abbreviation of $\sqrt{1 + x^2}$. Then, the precise result we need states that

$$\langle Q^a \rangle^\mu R(a) \in B(\mathrm{H}^{-1}(X), \mathrm{H}^1(X)). \tag{2.11}$$

In the rest of this section we shall briefly review (following [22, 14], those geometric particularities of the configuration space X which are specific to the N-body problem. Indeed, the need to put in evidence some privileged directions for the non-propagation, which are closely related to the particular structure of the potentials, suggests the division of X into disjoint sets, all except one being neighborhoods (cone or semi-cylinder-shaped) of these directions. Moreover, a smooth vector field is constructed on X (by convolution or averaging) from the first distributional derivative of the convex, locally Lipschitz application $\varrho : X \to \mathbb{R}$

$$\varrho = \frac{1}{2} \max_{a \in \mathbf{L}} \{ \mid \pi_a(\cdot) \mid^2 + \nu_a \}. \tag{2.12}$$

In the above definition each of the parameters ν_a belongs to some interval $(\nu_{|a|}, \tilde{\nu}_{|a|})$ with positive bounds conveniently chosen, such that for any $1 \le n < m \le N$ the inequalities $\nu_m < \tilde{\nu}_m < \nu_n < \tilde{\nu}_1$ are satisfied. In this sense, ϱ can be treated also as a function of the argument $\nu \in \prod_{a \in \mathbf{L}} (\nu_{|a|}, \tilde{\nu}_{|a|})$ having each of its coordinates double-indexed, firstly by the number of the level of $\mathbf{L}$ to which each element a belongs and secondly by the number assigned to this element within the level. Thus, the total number of coordinates will be a card $\mathbf{L}$.

The identity (2.12) tells us that ϱ is an almost everywhere differentiable function on X, having a classical derivative a.e. defined on X which coincides with ϱ', the distributional derivative of ϱ. Actually, $\varrho' \in L^\infty_{\text{loc}}(X)$. Moreover, it can be shown (see Theorem 4.1.7 in [24]) that the second distributional derivative of ϱ is a measure on X, taking as values *positive* matrices from $B(X)$.

In order to see in which way the vector field ϱ' is a distortion of the identical mapping on X, we make the connection between (2.12) and the 'privileged directions' X_a by means of a finite, disjoint, a.e.-covering $\{ \Xi_a \}_{a \in \mathbf{L}}$ of X, defined as

$$\Xi_a = \bigcap_{b \neq a} \{ x \in X \mid \mid \pi_a(x) \mid^2 + \nu_a > \mid \pi_b(x) \mid^2 + \nu_b \} =$$
$$= \{ x \in X \mid \mid \pi_a(x) \mid^2 + \nu_a > \max_{b \neq a} \{ \mid \pi_b(x) \mid^2 + \nu_b \} \}. \tag{2.13}$$

This allows us to calculate explicitly ϱ (and its derivatives) using the partition of the unity $\{ J_a \}_{a \in \mathbf{L}}$ subordinated to the above open a.e.-covering of X. Indeed,

$$\varrho'(x) = \sum_{a \in \mathbf{L}} J_a(x) \pi_a(x) \tag{2.14}$$

holds as X-valued distributions and a.e. as functions. Moreover, since ϱ'' is a positive measure, then

$$\varrho''(x) \ge \sum_{a \in \mathbf{L}} J_a(x) \pi_a \tag{2.15}$$

is satisfied for *all* $x \in X$.

To obtain smooth partition of unity and vector field, we construct them from J_a, resp. ϱ, either as in [22, 14], by convolution with a C_0^∞-function φ (supported in a neighborhood of the origin in X, and satisfying $\int \varphi(x)dx = 1$, $\int x\varphi(x)dx = 0$), or, as in [38], by an averaging process, performed on the Cartesian product of intervals $\prod_{n=1}^{N} \prod_{i=1}^{\text{card}\, \mathbf{L}(n)} (\nu_{ni}, \tilde{\nu}_{ni})$ previously defined, as in

$$g(x) = \int_{\mathbf{R}^{\mathrm{card}\,\mathrm{L}}} \tilde{\varphi}(\nu) G(x,\nu) d\nu .$$

Here $\tilde{\varphi}$ is a positive C_0^∞ function supported in the above product set, with $\|\tilde{\varphi}\|_{L^1} = 1$, and in our case $G(.,\nu)$ will be replaced by J_a and ϱ (whose dependence on ν is given in (2.13) and (2.12), respectively). We will denote the new, smooth objects by j_a and r, respectively, and it is not difficult to show that all the properties previously enumerated (especially (2.14) and (2.15)) hold for them also. It is also shown that if Id denotes the identity function, then the mappings $r' - Id$, $r'' \cdot Id - r'$ and r''' are bounded in the L^∞ norm.

In Appendix 6.1, properties of some subsets of X which play an important role in the spectral and scattering analysis, as the open cones (defined for any $0 \le d < 1$)

$$\Gamma_a(d) = \{x \in X \mid \min_{b \in \mathrm{L} \setminus \mathrm{L}_a} | \pi^b(x) | = \operatorname{dist}(x, \cup_{b \in \mathrm{L} \setminus \mathrm{L}_a} X_b) > d \mid x \mid \}, \qquad (2.16)$$

or the "cells" $\overset{\circ}{X}_a = X_a \setminus \cup_{b \in \mathrm{L} \setminus \mathrm{L}_a} X_b$ are given, and the relation between them is studied. To get an intuitive image of the link between $\overset{\circ}{X}_a$ and the characteristic function J_a of the set Ξ_a, let us mention only the fact that if the argument of J_a is multiplied by a positive parameter γ, then the support of $J_a(\gamma.)$ will tend to coincide with $\overset{\circ}{X}_a$ when $\gamma \to \infty$.

3. Propagation properties

We begin with the so-called maximal velocity bound theorem (see [33–35, 22, 36].

Proposition 3.1. *Let g be a $C_0^\infty(X \setminus \{0\})$ scalar function. Then there is a constant $\lambda > 0$ (which cannot be made arbitrarily small) and a positive constant C such that*

$$\int_1^\infty \frac{dt}{t} \left\| g\left(\frac{Q}{\lambda t}\right) e^{-itR} \psi \right\|^2 \le C \|\psi\|^2 . \qquad (3.1)$$

P r o o f. Let us choose the propagation observable of the form

$$\Phi = h\left(\frac{Q}{\lambda t}\right),$$

and let h be a $C_0^\infty(\mathbf{R})$ radial function, constant in a neighborhood of the origin and equal to zero at infinity. Then the computation of the Heisenberg derivative of Φ with respect to the approximating hard-core resolvent R_α gives

$$\mathrm{D}_{R_\alpha} \Phi = \frac{1}{2\lambda t} R_\alpha (P \cdot h' + h' \cdot P) R_\alpha - \frac{1}{t} \frac{Q}{\lambda t} \cdot h' , \qquad (3.2)$$

where the dot means the scalar product of two vector operators. We have also used the convention according to which, whenever no confusion is possible, we will omit the arguments of the multiplication operators with (eventually time-dependent) functions. In order to simplify the aspect of some rather complicated formulas, we will keep to this convention throughout this paper. Denote now by $\tilde{h}$ the scalar function $Id \cdot h'$ and $h^{\cdot}(r) \equiv \frac{d}{dr} h(r) \le 0$. Straightforward calculation shows that for all $x \in X$

$$\begin{cases} h'(x) = \omega h^{\cdot}(|x|) \\ \tilde{h}(x) = |x| h^{\cdot}(|x|), \end{cases} \tag{3.3}$$

where ω denotes the unit vector $x|x|^{-1}$. Finally, take $g(r) = \left(-h^{\cdot}(r)\right)^{1/2}$ and notice that g is a radial $C_0^\infty(X\backslash\{0\})$ function. Then, (3.2) becomes

$$\begin{aligned} \mathrm{D}_{R_\alpha} h = \frac{1}{t} \left| \frac{Q}{\lambda t} \right| g^2 - \frac{1}{t} g \frac{R_\alpha (P\cdot\omega + \omega\cdot P) R_\alpha}{2\lambda} g - \\ - \frac{1}{2\lambda t}\left([R_\alpha P, g] \cdot \omega\, g R_\alpha + \text{h.c.}\right) - \\ - \frac{1}{2\lambda t}\left(g R_\alpha P \cdot \omega [g, R_\alpha] + \text{h.c.}\right). \end{aligned} \tag{3.4}$$

The latter two terms above are of the order $O(t^{-2})$ uniformly with respest to α because of the obvious equality

$$[R_\alpha, g] = R_\alpha \left[\frac{P^2}{2}, g \right] R_\alpha ,$$

and the fact that for all $\alpha \ge 0$, $R_\alpha \in B(\mathrm{H}^{-1}, \mathrm{H}^{1})$.

Denoting $\langle e^{-itR_\alpha}\psi, . e^{-itR_\alpha}\psi \rangle$ by $\langle . \rangle_{t,\alpha}$, we estimate the expectation value of the first two terms from the r.h.s. of (3.4) as follows:

$$\frac{1}{t} \left\langle \left| \frac{Q}{\lambda t} \right| g^2 \right\rangle_{t,\alpha} \ge \frac{1}{t} \inf_{x \in supp\, g} |x| \, \left\| g e^{-itR_\alpha} \psi \right\|^2 ,$$

$$\frac{1}{\lambda t} \langle g R_\alpha (P\cdot\omega + \omega\cdot P) R_\alpha g \rangle_{t,\alpha} \le \frac{1}{t} \frac{\| R_\alpha (P\cdot\omega + \omega \cdot P) R_\alpha \|}{\lambda} \left\| g e^{-itR_\alpha} \psi \right\|^2 .$$

Replacing the above inequalities in (3.4), we get

$$\left(\inf_{x \in supp\, g} |x| - \frac{1}{\lambda} \sup_{\alpha \ge 1} \left\| R_\alpha \right\|^2_{-1,0} \right) \int_1^\infty \frac{dt}{t} \left\| g\left(\frac{Q}{\lambda t}\right) e^{-itR_\alpha} \psi \right\|^2 \le \tilde{C} \, \|h\|_\infty \|\psi\|^2 .$$

This shows that if λ is chosen (with respect to the support of g) such that

$$\lambda > \sup_{\alpha \ge 1} \left\| R_\alpha \right\|^2_{-1,0} \sup_{x \in supp\, g} |x|^{-1} , \tag{3.5}$$

then applying the usual Fatou lemma (for the integral over t) yields

$$\int_1^\infty \frac{dt}{t} \left\| g\left(\frac{Q}{\lambda t}\right) e^{-itR} \psi \right\|^2 = \int_1^\infty \frac{dt}{t} \liminf_{\alpha \to \infty} \left\| g\left(\frac{Q}{\lambda t}\right) e^{-itR_\alpha} \psi \right\|^2 \le$$

$$\leq \liminf_{\alpha\to\infty}\int_1^\infty \frac{dt}{t}\left\| g\left(\frac{Q}{\lambda t}\right) e^{-itR_\alpha}\psi \right\|^2 \leq C\|\psi\|^2.$$

Thus, the Proposition is proved.

Let us make some comments on this first a priori result. From the point of view of the physical interpretation, it is rather clear why the greatest lower bound of the support of g is important: the quantum system cannot delocalize arbitrarily fast in time since its energy R is a bounded, decreasing function. There is also the special case of the so-called "tails" (scattering states describing the system already localized at infinity at finite times) which have non null asymptotic probability and thus have to belong to the orthogonal complement of the range of the projector g. Then, a brutal cutoff introducing at least upper bound for *supp g* (as in the hypothesis of the above Proposition) would avoid the problem caused by these "tails" but will make us lose the information about the states describing a system with really large asymptotic velocities and low energy. This shows that the above result is far from optimal. Nevertheless, Sigal and Soffer established in [35] a finer one, which holds for states belonging to a dense set of vectors, in which the upper limiting cutoff is eliminated and where the norm in the r.h.s. of (3.1) is taken in a weighted Lebesgue space. But this result is no more an abstract nonsense, since a localization in the complement of the set of the critical values of the Hamiltonian is needed and the Mourre estimate has been used to prove it.

Let us notice also that the lower bound established in (3.5) for the value of λ is not optimal. One could have proved the Proposition directly, without using the approximating family $\{R_\alpha\}_{\alpha\geq 0}$ and obtaining the best λ possible, but this requires the result (2.10) concerning commutators with hard-core resolvents (see how it is used in the proof of the following Proposition). Moreover, we shall see that not all the results we need for proving asymptotic completeness can be deduced by working with a sequence of approximating Hamiltonians, the algebraic properties of the N-body (hard-core) resolvents being crucial in the proof of the below theorems.

Proposition 3.2. *i) Let $f\in C_0^\infty(X)$ be constant around the origin on a region with interior diameter not too small (i.e. proportional to the maximal velocity bound). Then there exists $\delta>0$ (depending on the short-range part of the hard-core potential) such that for all $a\in \mathrm{L}$ and all $\psi\in \mathrm{H}(X)$ one has*

$$\int_1^\infty \frac{dt}{t}\left\| j_{a,\delta}^{1/2}\left([iR,Q_a]-\frac{Q_a}{t}\right) f\left(\frac{Q}{t}\right) e^{-itR}\psi\right\|^2 \leq C\|\psi\|^2, \tag{3.6}$$

where $j_{a,\delta}=j_a\left(\frac{Q}{t}t^\delta\right)$ denotes the multiplication operator with the smoothed characteristic function of the set Ξ_a (see (2.13)).

ii) Moreover, if $\Gamma_a(0)=X\setminus\bigcup_{b\not\leq a}X_b$, then the following estimate is true for all $J\in C_0^\infty(\Gamma_a(0))$ having the support not greater than that of f:

$$\int_1^\infty \frac{dt}{t} \left\| \left| [iR,Q_a] - \frac{Q_a}{t} \right| J\left(\frac{Q}{t}\right) e^{-itR} \psi \right\|^2 \le C \| \psi \|^2 . \tag{3.7}$$

P r o o f: *i)* Let us first introduce a symbol concerning only the (vector or scalar) operators of multiplication with $C^r(X)$ functions $g: g_\beta$ will stand for $g(.t^{\beta-1})$ if $\beta > 0$. Notice that this is consistent with the meaning of $j_{a,\delta}$ and if $(g')_\beta$ is denoted by g'_β, then by iteration we get

$$g_\beta^{(\alpha)} \equiv (g^{(\alpha)})_\beta = t^{(1-\beta)|\alpha|} (g_\beta)^{(\alpha)}$$

for any multiindex α with $|\alpha| \le p \le \infty$.

Further, denote by A_δ the operator

$$\begin{aligned} A_\delta &= \frac{1}{2} \sum_{a \in \mathrm{L}} \left([iR,Q_a] \cdot Q_a j_{a,\delta} + \text{h.c.} \right) = \\ &= \frac{1}{2} \frac{t}{t^\delta} \left([iR,Q_a] \cdot r_\delta' + r_\delta' \cdot [iR,Q_a] \right), \end{aligned} \tag{3.8}$$

where $r_\delta' = \sum_{a \in \mathrm{L}} \pi_a j_{a,\delta}$ is the smooth vector field introduced in Section 2. Using this operator, we shall construct the propagation observable

$$\Phi_\delta = f\left(\frac{A_\delta}{t} - \frac{Q^2}{2t^2} \right) f \tag{3.9}$$

(the central part of Φ_δ will occasionally be called S_δ). Then, we calculate as usual the Heisenberg derivative of Φ_δ and get

$$\mathrm{D}_R \Phi_\delta = 2\mathrm{Re}\, (\mathrm{D}_R f) S_\delta f + f(\mathrm{D}_R S_\delta) f. \tag{3.10}$$

The term we need in the conclusion of the Proposition (part *i)*) will be furnished by the second term in the r.h.s. of the above equality, while the other terms will be proved as being integrable in t on $[1, \infty)$. Let us begin with the simplest one, viz.

$$\int_1^\infty dt \langle \mathrm{D}_R \Phi_\delta \rangle_t \le 2 \sup_{t \ge 1} \left| \langle f S_\delta f \rangle_t \right| , \tag{3.11}$$

where $\langle . \rangle_t$ stands for $\langle e^{-itR} \psi , . e^{-itR} \psi \rangle$. Since S_δ is obviously uniformly bounded in time on the support of f, it yields the integrability of $\mathrm{D}_R \Phi_\delta$.

Let us now pass to the second term in the r.h.s. of (3.10) and calculate

$$\mathrm{D}_R S_\delta = \left[iR, \frac{A_\delta}{t} \right] - \left[iR, \frac{Q^2}{2t^2} \right] + \frac{\partial}{\partial t} \left(\frac{A_\delta}{t} \right) + \frac{1}{t} \frac{Q^2}{2t^2} . \tag{3.12}$$

We will estimate each of these terms separately:

$$\frac{1}{t} \left[iR, A_\delta \right] = \frac{1}{2} \sum_{b \in \mathrm{L}} \left(\left[iR, [iR,Q_b] \right] \cdot \frac{Q_b}{t} j_{b,\delta} + \text{h.c.} \right) +$$

$$+\frac{1}{2t^{\delta}}\left([iR,Q]\cdot[iR,r'_{\delta}]+\text{h.c.}\right). \tag{3.13}$$

The last line in the above equality can be computed by the repeated use of the commutation relation (2.10) (notice also that according to it, the first double sum in (3.14) below is precisely $[iR,Q]$) as follows

$$[iR,r'_{\delta}]=\sum_{b\in\mathbf{L}}[iR(b),r'_{\delta}]=$$

$$=\sum_{b,c\in\mathbf{L}}\left(R(c)\left[i\frac{P^2}{2},r'_{\delta}\right]R_{cb}+R(b)\left[i\frac{P^2}{2},r'_{\delta}\right]R_b\right)=$$

$$=\frac{1}{t}t^{\delta}r''_{\delta}\cdot\sum_{b,c\in\mathbf{L}}\left(R(c)PR_{cb}+R(b)PR_b\right)-$$

$$-\frac{i}{2t^2}t^{2\delta}\sum_{b,c,d\in\mathbf{L}}R(d)G_{\delta}\left(R_{dc}PR_{cb}+R_{db}PR_b\right)-$$

$$-\frac{i}{2t^2}t^{2\delta}\sum_{b,c\in\mathbf{L}}R(c)\left(r'''_{\delta}+G_{\delta}R_cP\right)R_{cb}-$$

$$-\frac{i}{2t^2}t^{2\delta}\sum_{b\in\mathbf{L}}R(b)\left(r'''_{\delta}+G_{\delta}R_bP\right)R_b, \tag{3.14}$$

where G_{δ} stands for $P\cdot r'''_{\delta}+r'''_{\delta}\cdot P$. Then, because of the boundedness of r''' (in the L^{∞}-norm) and the fact that also P is bounded by the resolvents, the last three terms of the above relation will give rise to an $O(t^{-2(1-\delta)})$ contribution; as for the first one, it can be minorated with using

$$r''_{\delta}(x)\geq\sum_{a\in\mathbf{L}}j_{a,\delta}(x)\pi_a \tag{3.15}$$

(which is true for all $x\in X$ as $B(X)$-valued measures on X, and for all δ). Finally, we get

$$\frac{1}{t}\left[iR,A_{\delta}\right]\geq\frac{1}{2}\sum_{a\in\mathbf{L}}\left(\left[iR,[iR,Q_a]\right]\cdot\frac{Q_a}{t}j_{a,\delta}+\text{h.c.}\right)+$$

$$+\frac{1}{t}\sum_{a\in\mathbf{L}}[iR,Q_a]\cdot j_{a,\delta}[iR,Q_a]+O(t^{-2+\delta}). \tag{3.16}$$

We pass now to the second term of the r.h.s of (3.12): using the fact that the family $\{j_{a,\delta}\}_{a\in\mathbf{L}}$ forms a partition of unity for any $\delta\geq 0$, we get

$$\left[iR,-\frac{Q^2}{2t^2}\right]=-\frac{1}{2t}\sum_{a\in\mathbf{L}}\left(\frac{Q_a}{t}\cdot j_{a,\delta}[iR,Q_a]+\text{h.c.}\right)-$$

$$-\frac{1}{2t^{1+\delta}}\sum_{a\in\mathbf{L}}\left(\frac{Q^a}{t}t^{\delta}\cdot j_{a,\delta}[iR,Q^a]+\text{h.c.}\right). \tag{3.17}$$

The last term here is $O(t^{-1-\delta})$ because all the components of $\frac{x^a}{t} t^\delta$ are bounded on the support of $j_{a,\delta}$; in other words, if Id denotes the identity function, the last term above is

$$-\frac{1}{2t^{1+\delta}}([iR,Q]\cdot(r' - Id)_\delta + \text{h.c.})$$

and from the definition of the vector field we know that $(r' - Id)$ is in $L^\infty(X)$.

In a similar manner, computing the third term of the r.h.s. of (3.12) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{A_\delta}{t}\right) &= -\frac{1}{2t^{1+\delta}}\left([iR,Q]\cdot r'_\delta + \text{h.c.}\right) + \\ &+ \frac{\delta-1}{2t^{1+\delta}}\left([iR,Q]\cdot(r''\cdot Id - r')_\delta + \text{h.c.}\right).\end{aligned} \tag{3.18}$$

Since $(r''\cdot Id - r')$ is also in $L^\infty(X)$, the last term above gives also an $O(t^{-1-\delta})$ contribution. Finally, using the same trick, we compute

$$\frac{1}{t}\frac{\dot{Q}^2}{2t^2} = \frac{1}{t}\sum_{a\in \mathrm{L}} \frac{Q_a}{t}\cdot j_{a,\delta}\frac{\dot{Q}_a}{t} + O(t^{-(1+2\delta)}) \tag{3.19}$$

and summing up (3.16) and (3.19) we obtain the estimate

$$\begin{aligned}\mathrm{D}_R S_\delta \geq \frac{1}{t}\sum_{a\in\mathrm{L}} &\left([iR,Q_a] - \frac{Q_a}{t}\right)\cdot j_{a,\delta}\left([iR,Q_a] - \frac{Q_a}{t}\right) + \\ +\frac{1}{2}\sum_{a\in\mathrm{L}} &\left(\Big[iR,[iR,Q_a]\Big]\cdot\frac{Q_a}{t}j_{a,\delta} + \text{h.c.}\right) + \\ &+ O(t^{-2+\delta}) + O(t^{-1-\delta}) + O(t^{-1-2\delta}).\end{aligned} \tag{3.20}$$

It remains to show that the second term of the r.h.s. of this inequality in integrable w.r.t. t, when taken in the mean value $\langle f\,.\,f\rangle_t$. Let us remark first that the product of the double commutator $\Big[iR,[iR,Q_a]\Big]$ with some multiplication operators with functions of the argument $\frac{x}{t}$ will appear quite frequently in this paper. We will prove below that if these functions are supported outside the subspaces X_b for all $b\in \mathrm{L}\setminus\mathrm{L}_a$, then the desired decay in t of the terms containing such products will be ensured. Roughly speaking, this is due to good decay along certain directions (depending on the given $b\in\mathrm{L}$) of the b-connected component of our Hamiltonian (i.e. the resolvent operator). Let us notice also that since the b-connected component of the N-body algebra T is (a semicompact operator algebra) of the form $\mathrm{T}(b) = \mathrm{K}(X^b)\otimes_b \mathrm{T}^*(X_b)$, its elements will commute with $P_a \equiv 1\otimes {}_a P_{X_a}$. Hence, taking it into account that T is the direct sum over L of T(b), we get

$$\begin{aligned}\Big[iR,[iR,Q_a]\Big]j_{a,\delta} &= \sum_{b\in\mathrm{L}} R[iR(b),P_a]Rj_{a,\delta} = \\ &= \sum_{b\in\mathrm{L}\setminus\mathrm{L}_a} R\left(R(b)P_a - P_aR(b)\right)\left(ij_{a,\delta}R - [iR,j_{a,\delta}]\right).\end{aligned} \tag{3.21}$$

Using the commutation formula (2.10) we will be able to compute $[iR, J_{a,\delta}]$ as

$$[iR, j_{a,\delta}] = \sum_{c \in \mathrm{L}} [iR(c), j_{a,\delta}] =$$

$$= \frac{t^{\delta}}{t} j'_{a,\delta} \sum_{c,d \in \mathrm{L}} \left(R(d) P R_{dc} + R(c) P R_c \right) + O(t^{-2+2\delta}) =$$

$$= \frac{t^{\delta}}{t} j'_{a,\delta} [iR, Q] + O(t^{-2+2\delta}), \tag{3.22}$$

and therefore (3.14) becomes

$$\left[iR, [iR, Q_a]\right] j_{a,\delta} = \sum_{b \in \mathrm{L} \setminus \mathrm{L}_a} RR(b) \left\{ (ij_{a,\delta} P_a + t^{\delta-1} j'_{a,\delta}) R + t^{\delta-1} j'_{a,\delta} P_a [iR, Q] \right\} -$$

$$- \sum_{b \in \mathrm{L} \setminus \mathrm{L}_a} RP_a R(b) \left(ij_{a,\delta} R + t^{\delta-1} j'_{a,\delta} [iR, Q] \right) + O(t^{-2+2\delta}). \tag{3.23}$$

The r.h.s of the above equality has the advantage that it contains only the products of $R(b)$ with the functions $j_{a,\delta}$ and $j'_{a,\delta}$. Notice also that for all $b \in \mathrm{L} \setminus \mathrm{L}_a$ there are strictly positive constants $C_{a,b}$ such that

$$\langle x^b \rangle \geq | x^b | \equiv \left| x^b_{a_{\min}} \right| \geq \left| x^b_a \right| \geq C_{a,b}$$

on the support of j_a. This shows that $\left(\langle x^b \rangle t^{\delta-1} \right)^{-\mu} j_{a,\delta} \leq C_{a,b}^{-\mu} j_{a,\delta}$, which together with (2.11) gives the desired decay in t of the r.h.s. of (3.23), as e.g. in

$$RR(b) j_{a,\delta} P_a R = t^{-\mu(1-\delta)} R \underbrace{R(b) \langle Q^b \rangle}_{O(1)} \underbrace{\left(\frac{\langle Q^b \rangle}{t} t^{\delta} \right)^{-\mu} j_{a,\delta} P_a R}_{O(1)}, \tag{3.24}$$

provided that $0 < \delta < 1$ satisfies the supplementary condition $\delta < 1 - \frac{1}{\mu}$.

Thus, it remains only to show that $\left| \langle (\mathrm{D}_R f) S_\delta f \rangle_t \right|$ is integrable in t. To do so, we choose an orthonormal basis in X and index by $k = 1, \ldots, \dim X$ the components of the vector operators as Q, Δf in this basis; actually, we shall denote by f_k' the components of Δf and notice that because of the choice we made on f all of them are negative scalar functions. Denote also by $\tilde{f}_k$ the $C_0^\infty(X \setminus \{0\})$-function $\sqrt{-f'_k}$ and by χ_k a smoothed characteristic function of $supp\, f$ satisfying $\chi_k \tilde{f}_k = \tilde{f}_k$. Then, commuting $\tilde{f}_k$ through $[iR, Q]$ and S_δ towards right, we get

$$(D_R f) S_\delta f = \frac{1}{t} \sum_k \underbrace{\tilde{f}_k \chi_k \left(\frac{Q_k}{t} - [iR, Q_k] \right) \chi_k S_\delta f \tilde{f}_k}_{O(1)} + O(t^{-2}), \tag{3.25}$$

where the $O(t^{-2})$ contribution is brought by the remainders of order two by the (already) $O(t^{-1})$ terms containing double commutators of the form $\left[[iR, Q], \tilde{f} \right]$. This shows that there are positive constants C_k and $\tilde{C}$ depending on the support of f (and thus on the maximal velocity bound) such that

$$\int_1^\infty dt \left| \langle (\mathrm{D}_R f) S_\delta f \rangle_t \right| \leq \sum_k C_k \int_1^\infty \frac{dt}{t} \left\| f\left(\frac{Q}{t}\right) e^{-itR} \psi \right\|^2 + \tilde{C} \| \psi \|^2 ,$$

which proves the statement *i)* of the Proposition via the maximal velocity bound theorem (Proposition 3.1).

We pass now to the proof of *ii)*: let us fix arbitrarily an a in L and *assume* first (and prove later) that for some J with support as in the hypothesis (i.e. depending on the chosen a) there is a $T > 0$ such that for all $t > T$ the equality

$$J\left(\frac{\cdot}{t}\right) = \left(\frac{\cdot}{t}\right) f^2\left(\frac{\cdot}{t}\right) \sum_{b \in \mathrm{L}_a} j_b\left(\frac{\cdot}{t} t^\delta\right) \tag{3.26}$$

is valid. We stress that this is sufficient for showing that (3.7) is a consequence of (3.6). Indeed, notice first that (replacing the letter a by b) an equivalent inequality for (3.6) is

$$\sum_{k=1}^{\dim X_b} \int_1^\infty \frac{dt}{t} \left\langle \left([iR,Q_k] - \frac{Q_k}{t} \right) f^2 j_{b,\delta} \left([iR,Q_k] - \frac{Q_k}{t} \right) \right\rangle_t \leq C \| \psi \|^2 .$$

Then, since $b \in \mathrm{L}_a$ means $X_a \leq X_b$, summation of the positive terms in the r.h.s. of inequality (3.27) below will be performed over a bigger index set than that of the l.h.s. Hence, using our assumption, we have the following:

$$\int_T^\infty \frac{dt}{t} \left\langle \left([iR,Q_a] - \frac{Q_a}{t} \right) \cdot J \left([iR,Q_a] - \frac{Q_a}{t} \right) \right\rangle_t =$$

$$\sum_{k=1}^{\dim X_a} \int_T^\infty \frac{dt}{t} \left\langle \left([iR,Q_k] - \frac{Q_k}{t} \right) J f^2 \sum_{b \in \mathrm{L}_a} j_{b,\delta} \left([iR,Q_k] - \frac{Q_k}{t} \right) \right\rangle_t \leq$$

$$\leq \| J \|_\infty \sum_{b \in \mathrm{L}_a} \sum_{k=1}^{\dim X_b} \int_1^\infty \frac{dt}{t} \left\langle \left([iR,Q_k] - \frac{Q_k}{t} \right) f^2 j_{b,\delta} \left([iR,Q_k] - \frac{Q_k}{t} \right) \right\rangle_t . \tag{3.27}$$

It remains to prove assumption (3.26). Notice that since the family $\{ j_{b,\delta} \}_{b \in \mathrm{L}}$ forms a partition of unity, it is enough to show that if $supp\, J \subset supp\, f$, then for all $t > T$, $x \in supp\, J$ implies $x \notin \bigcup_{b \in \mathrm{L} \setminus \mathrm{L}_a} supp\, j_{b,\delta}$. But this becomes obvious if one thinks of $j_{b,\delta}$ as a smoothed characteristic functions of semi-cylinders centered on the sets $\overset{\circ}{X}_b$, which shrink around these sets when $t \to \infty$ (with the "velocity" t^δ), and if one takes it also into account that the complementary set for $\Gamma_a(0)$ is precisely $\bigcup_{b \in \mathrm{L} \setminus \mathrm{L}_a} \overset{\circ}{X}_b$.

R e m a r k: In order to avoid some of the cumbersome calculations we made when estimating the terms related to the commutator $[iR,A_\delta]$ in the previous proof, we could work with the approximating resolvent R_α (instead of the hard-core resolvent R) and obtain estimates which are uniform with respect to the parameter α. Nevertheless, there will be a difficulty related to the terms containing the commutator $[iR_\alpha, P_a]$ for which the deeper result furnished by Theorem 6.5 in [11] has to be used. But this result concerns the hard-core resolvents, so we will have to pass first to the limit $\alpha \to \infty$ under the integral $\int_1^T dt$ for some $T > 1$, getting (for any $a \in \mathrm{L}$)

$$\int_1^T \frac{dt}{t} \left\| j_{a,\delta}^{1/2} \left([iR,Q_a] - \frac{Q_a}{t} \right) f(\tfrac{Q}{t}) e^{-itR} \psi \right\|^2 \le$$

$$\le \sum_{b \in \mathrm{L}} \int_1^\infty dt \left| \left\langle \left(R[iR,P_b] \left(it^{\delta-1} j'_{b,\delta} RP - j_{b,\delta} \right) R \frac{Q_b}{t} f \right\rangle_t \right| + O(1) \|\psi\|^2,$$

and finally we will let T go to ∞ and apply the above quoted theorem as in the previous proof (relation (3.24)).

Proposition 3.3. *If a is an arbitrary element of* L *and if $J \in C_0^\infty(\Gamma_a(0))$, then there is a positive constant C such that*

$$\int_1^\infty \frac{dt}{t} \left\| \, \left| [iR,Q_a] - \frac{Q_a}{t} \right|^{1/2} J(\tfrac{Q}{t}) \, e^{-itR} \psi \right\|^2 \le C \|\psi\|^2 \tag{3.28}$$

and all $\psi \in \mathrm{H}(X)$.

Before starting the proof let us introduce some notations that we shall use along all the rest of the paper: T_a will stand for $[iR,Q_a] - \frac{Q_a}{t}$ and if A is an unbounded (eventually time-dependent) operator, then its time-dependent regularization $\left(|A|^2 + t^{-4\beta} \right)^{1/2}$ will be denoted by $\langle A \rangle_\beta$.

P r o o f: First of all we shall reduce a little the context in which the proposition has to be proved by assuming that J belongs also to a particular class of functions that Derezinski denoted by F and defined as being formed by $C_0^\infty(X)$-functions f having the property that for any $a \in \mathrm{L}$ there is a neighborhood ν_a of the subspace X_a such that $f = f \circ \pi_a$ in ν_a. Obviously, this tells us that on some neighborhood of each X_a, the border of the support of each function in F will be perpendicular to X_a. It is also easy to check that F is a $*$-algebra which separates the points of X and is dense in C_0^∞ (X) in the L^∞-norm. Thus, it will be enough to prove the Proposition for $J \in C_0^\infty(\Gamma_a(0)) \cap \mathrm{F}$.

Since $\Gamma_a(0) = \bigcup_{b \in \mathrm{L}_a} \overset{\circ}{X}_b$, any compact set of $\Gamma_a(0)$ (and in particular the support of J) can be partitioned in ε-neighborhoods of a finite number of points of $\overset{\circ}{X}_b$ for some $b \in \mathrm{L}_a$. The arbitrary (but fixed) choice we make for J will determine the choice of the diameter of these neighborhoods and of the number n_b of those which are centered at points of $\mathring{X}_b$. Correspondingly, we will have a partition of unity on the support of J constructed with using a family $\{ j_{k_b} \mid k_b = 1,\dots,n_b, b \in \mathrm{L}_a \}$ and satisfying on X

$$\sum_{b \in \mathrm{L}_a} J \circ \pi_b \sum_{k_b=1}^{n_b} j_{k_b}^2 = J . \tag{3.29}$$

Notice that as a consequence of this choice we will also have $\nabla J = (\pi_b \nabla) J$ on the support of each j_{k_b}.

Secondly, it is clear that the estimate

$$\int_1^\infty \frac{dt}{t} \left\| \langle T_a \rangle_\beta^{1/2} J(\tfrac{Q}{t}) e^{-itR} \psi \right\|^2 \le C \| \psi \|^2 \tag{3.30}$$

implies (3.28), so it will be enough to prove the above inequality. To do so, let us choose the propagation observable

$$\Phi = J \langle T_a \rangle_\beta J \tag{3.31}$$

and compute its Heisenberg derivative

$$\mathrm{D_R} \Phi = 2\mathrm{Re}\, (\mathrm{D_R} J) \langle T_a \rangle_\beta J + J (\mathrm{D_R} \langle T_a \rangle_\beta) J . \tag{3.32}$$

We will show first that the second term in the r.h.s. of the above equality will furnish the term from the conclusion of the Proposition plus some integrable terms. For this, we shall use a particular form of the definition of the fractional power of a positive operator A (see [37])

$$A^\gamma = C \int_0^\infty \omega^{\gamma-1} (\omega + A)^{-1} A d\omega \tag{3.33}$$

which holds strongly on its domain for some positive constant C and for $\gamma \in (0,\frac{1}{2}]$. Then, easy computation gives

$$\mathrm{D_R} A^\gamma = C \int_0^\infty \omega^{\gamma-1} \left\{ \left(\mathrm{D}_R (\omega + A)^{-1} \right) A + (\omega + A)^{-1} \mathrm{D}_R A \right\} d\omega =$$
$$= C \int_0^\infty \omega^{\gamma-1} (\omega + A)^{-1} (\mathrm{D}_R A) \left(1 - (\omega + A)^{-1} A \right) d\omega . \tag{3.34}$$

In our case, we shall replace A by $\langle T_a \rangle_\beta^2$ and compute its Heisenberg derivative as

$$\mathrm{D}_R \langle T_a \rangle_\beta^2 = -\frac{2}{t} \left(\langle T_a \rangle_\beta^2 + (2\beta - 1) t^{-4\beta} \right) +$$
$$+ ([iR, [iR, Q_a]] \cdot T_a + h.c.). \tag{3.35}$$

Then, taking $\gamma = \frac{1}{2}$ and replacing (3.35) by (3.34) we get

$$D_R \langle T_a \rangle_\beta = -\frac{2C}{t}\left(\langle T_a \rangle_\beta^2 + (2\beta - 1)t^{-4\beta}\right) \times$$
$$\times \int_0^\infty \omega^{-\frac{1}{2}} (\omega + \langle T_a \rangle_\beta^2)^{-1} \{1 - \langle T_a \rangle_\beta^2 (\omega + \langle T_a \rangle_\beta^2)^{-1}\} d\omega +$$
$$+ C \int_0^\infty \omega^{\frac{1}{2}} (\omega + \langle T_a \rangle_\beta^2)^{-1} \{T_a \cdot [iR,[iR,Q_a]] + h.c.\} (\omega + \langle T_a \rangle_\beta^2)^{-1} d\omega . \tag{3.36}$$

The integral in the first term of the above equality is $\frac{1}{2} \langle T_a \rangle_\beta^{-1}$ (see [34], p. 132), so (3.36) becomes

$$D_R \langle T_a \rangle_\beta = -\frac{1}{t}\langle T_a \rangle_\beta + (1 - 2\beta)t^{-4\beta-1} \langle T_a \rangle_\beta^{-1} +$$
$$+ 2C\mathrm{Re} \int_0^\infty d\omega\, \omega^{\frac{1}{2}} (\omega + \langle T_a \rangle_\beta^2)^{-1}\, T_a \cdot [iR,[iR,Q_a]] (\omega + \langle T_a \rangle_\beta^2)^{-1} . \tag{3.37}$$

Notice that the second term in the r.h.s above is $O(t^{-1-2\beta})$ because of the obvious estimate $\| \langle T_a \rangle_\beta^{-1} \| \leq t^{2\beta}$, while the first one is exactly the term we need for the estimate (3.30). Thus, it will be enough to show integrability (w.r.t. t) of the expectation value $\langle J \,.\, J \rangle_t$ of the third term of (3.37). This will be performed as in the proof of the previous Proposition, i.e. by taking into account the decay in t of the double commutator $[iR,[iR,Q_a]]$ on the support of J. Indeed, the hypothesis $J \in C_0^\infty(\Gamma_a(0))$ suggests the existence, for all $b \in \mathrm{L}\backslash\mathrm{L}_a$, of some strictly positive constants C_b for which

$$\langle Q^b \rangle \geq | Q^b | \geq C_b t$$

on $supp\, J\left(\frac{\cdot}{t}\right)$ for all $t \geq 1$. This shows that

$$J\left(\frac{\langle Q_b \rangle}{t}\right)^{-\mu} \leq C_b^{-\mu} J ,$$

thus once J being brought nearby the double commutator we can apply the same argument as in (3.21) to (3.24) (take $\delta = 0$ for the present case) in order to obtain the $O(t^{-\mu})$ contribution. Nevertheless, besides the supplementary difficulties caused by the commutator of J with the resolvent $(\omega + \langle T_a \rangle_\beta^2)^{-1}$, there is also the problem of the boundedness of the components of T_a (which cannot be pulled out from the integral over ω because they do not commute with $\langle T_a \rangle_\beta$ of the resolvents). As we shall see, commuting J with $(\omega + \langle T_a \rangle_\beta^2)^{-1}$ gives rise to $O(t^{-1})$ factors (which are obviously not enough for integrability in t) containing derivatives of J but also operators T_a. The strategy will be to continue to commute J, J' through these resolvents until either the $O(t^{-2})$ terms or the products of these functions with $[iR,[iR,Q_a]]$ appear. Then, only the problems of the boundedness of the resulting T_a's and those of the integrability w.r.t. ω on both $[1,\infty)$ and $[0,1)$ will have to be solved. Actually, we shall see that the first two of them are related while in order to solve the third one we will need to "pick" a little part from the good decay we have obtained in the variable t and convert it in decay into ω . Let us first compute:

$$\left[(\omega+\langle T_a\rangle_\beta^2)^{-1},\ J\right]=$$
$$=\frac{2}{t}\mathrm{Re}\,(\omega+\langle T_a\rangle_\beta^2)^{-1}\,T_a\cdot\Big(J'\cdot[iR,Q]P_aR+\text{h.c.}+iRJ'R+O(t^{-1})\Big)(\omega+\langle T_a\rangle_\beta^2)^{-1}. \tag{3.38}$$

This allows us to give a precise formula for the operator which stands in (3.37) in the r.h.s. of (and in the product with) the $R(b)$'s (where $b\in \mathrm{L}\backslash\mathrm{L}_a$) yielding from $[iR,[iR,Q_a]]$, namely:

$$\begin{aligned}R(\omega+\langle T_a\rangle_\beta^2)^{-1}J&=(JR+J'[R,Q])(\omega+\langle T_a\rangle_\beta^2)^{-1}+\\&+\frac{1}{t}J'R(\omega+\langle T_a\rangle_\beta^2)^{-1}(T_a\cdot O(1)+\text{h.c.})(\omega+\langle T_a\rangle_\beta^2)^{-1}+\\&+O(t^{-2})\Big\{1+(\omega+\langle T_a\rangle_\beta^2)^{-1}\Big(T_a\cdot O(1)+\text{h.c.}\Big)\Big\}(\omega+\langle T_a\rangle_\beta^2)^{-1}+\\&+R(\omega+\langle T_a\rangle_\beta^2)^{-1}(T_a\cdot O(t^{-2})+\text{h.c.})(\omega+\langle T_a\rangle_\beta^2)^{-1}-\\&-\frac{1}{t}R\Big\{T_a\cdot\Big(\omega+\langle T_a\rangle_\beta^2\Big)^{-1}O(t^{-1})\Big(\omega+\langle T_a\rangle_\beta^2\Big)^{-1}O(1)\Big(\omega+\langle T_a\rangle_\beta^2\Big)^{-1}+\text{h.c.}\Big\}.\end{aligned} \tag{3.39}$$

Note that the last three lines of the r.h.s. above are $O(t^{-2})$, while the first two terms apparently have not the desired decay in t; but since J and J' will be next to some $R(b)$, they also will finally bring an $O(t^{-\mu})+O(t^{-1-\mu})$ contribution. Concerning boundedness of the components of those T_a from above which are taken in scalar product with the $O(1)$ terms, it will be safe to ensure it (uniformly in t and ω) with using the square root of the resolvent $(\omega+\langle T_a\rangle_\beta^2)^{-1}$. Nevertheless, this will *not* be the case for T_a which is taken in (3.37) in scalar product with $[iR,[iR,Q_a]]$, because to ensure integrability in ω over $[1,\infty)$, we need a norm of the resolvent $(\omega+\langle T_a\rangle_\beta^2)^{-1}$ of a power strictly superior to $\frac{3}{2}$. Thus, we prefer to bound it by a $\langle T_a\rangle_\beta^{-1}$ and commute the remaining $\langle T_a\rangle_\beta$ towards left, next to the J. Finally, using the Schwartz inequality with (3.39) replaced by (3.37), we obtain the estimate

$$\int_1^\infty dt\int_0^\infty d\omega\,\omega^{1/2}\left|\Big\langle J(\omega+\langle T_a\cdot\rangle_\beta^2)^{-1}T_a\cdot\ [iR,[iR,Q_a]](\omega+\langle T_a\rangle_\beta^2)^{-1}J\Big\rangle_t\right|\le$$

$$\leq C\|\psi\|\int_1^\infty dt\, O(t^{-\tilde{\mu}})\|\langle T_a\rangle_\beta Je^{-itR}\psi\|\sum_{k=1}^{\dim X_a}\|\langle T_a\rangle_\beta^{-1}T_k\|\times$$

$$\times\int_1^\infty d\omega\,\omega^{1/2}\sum_{j=0,1,3}\|(\omega+\langle T_a\rangle_\beta^2)^{-1}\|^{2+\frac{j}{2}}+ \tag{3.40}$$

$$+\tilde{C}\|J\|_\infty\|\psi\|^2\int_1^\infty dt\, O(t^{-\tilde{\mu}})\int_0^1 d\omega\,\omega^{1/2}\sum_{j=0,2,3}\|(\omega+\langle T_a\rangle_\beta^2)^{-1}\|^{3-\frac{j}{2}},$$

where C, $\tilde{C}$ are positive constants depending only on J, J' and $\tilde{\mu}$ stands for min $\{2,\mu\}$. Then, the second term in the r.h.s of (3.40) will be estimated by minorizing $\|(\omega+\langle T_a\rangle_\beta^2)^{-1}\|$ by $t^{4\beta}$, while for the first one we will take advantage of $\|(\omega+\langle T_a\rangle_\beta^2)^{-1}\|\leq\omega^{-1}$ in order to dominate the r.h.s. of (3.40) by

$$\|\psi\|\int_1^\infty dtO(t^{-\tilde{\mu}})\,\|\langle T_a\rangle_\beta Je^{-itR}\psi\|\int_1^\infty\omega^{-3/2}d\omega+\|\psi\|^2\int_1^\infty dtO(t^{12\beta-\tilde{\mu}})\int_0^1\omega^{1/2}d\omega. \tag{3.41}$$

Since the choice of $\beta>0$ is at our disposal, we will take it strictly inferior to $\frac{\tilde{\mu}-1}{12}$, which ensures integrability w.r.t. ω in the second term of the above sum. As for the first one, according to the Schwartz inequality, it will be dominated by

$$\|\psi\|\left(\int_1^\infty dtO(t^{1-2\tilde{\mu}})\right)^{1/2}\left(\int_1^\infty\frac{dt}{t}\|\langle T_a\rangle_\beta Je^{-itR}\psi\|^2\right)^{1/2}, \tag{3.42}$$

whose finiteness is a consequence of Proposition 3.2 (*ii*) and of the hypothesis of $\mu>1$.

It remains to estimate integrability of the expectation value $\langle\cdot\rangle_t$ of the first term of the r.h.s. of (3.32). For this, we will act exactly as in the proof of Proposition 3.2 (relation (3.25)) but use the partition of unity introduced at the beginning of this proof (see (3.29)). We have

$$\mathrm{D}_RJ=\frac{1}{t}\sum_{b\in\mathrm{L}_a}\sum_{k_b=1}^{n_b}\left([iR,Q_b]-\frac{Q_b}{t}\right)\cdot(\pi_b\nabla)\,Jj_{k_b}^2+\text{Remainder}, \tag{3.43}$$

where the "Remainder" will be shown to be an $O(t^{-2})$ term. Let us for the moment look at the first term above, and estimate

$$\left\langle\frac{1}{t}\sum_{b\in\mathrm{L}_a}\sum_{k_b=1}^{n_b}T_b\cdot J'j_{k_b}^2\langle T_a\rangle_\beta J\right\rangle_t\leq\frac{1}{t}\sum_{b\in\mathrm{L}_a}\sum_{k_b=1}^{n_b}\langle T_b\cdot J'j_{k_b}[j_{k_b},\langle T_a\rangle_\beta]J\rangle_t+$$

$$+\frac{1}{t}\sum_{b\in \mathbf{L}_a}\sum_{k_b=1}^{n_b}\langle [T_b, j_{k_b}]\cdot J'\langle T_a\rangle_\beta j_{k_b} J\rangle_t +$$

$$+\frac{1}{t}\sum_{b\in \mathbf{L}_a}\sum_{k_b=1}^{n_b}\langle \|J'\cdot T_b j_{k_b} e^{-itR}\psi\|\,\|\langle T_a\rangle_\beta J j_{k_b} e^{-itR}\psi\|. \tag{3.44}$$

Notice that as a consequence of

$$\|J'\cdot T_b j_{k_b} e^{-itR}\psi\| \le \sum_{l=1}^{\dim X_b}\|\partial_l J\|_\infty \|T_l\langle T_b\rangle_\beta^{-1}\|\,\|\langle T_b\rangle_\beta j_{k_b} e^{-itR}\psi\|$$

and of the Schwartz inequality (applied to the integral $\int dt$) we can dominate the last line of (3.44) by

$$C\sum_{b\in \mathbf{L}_a}\sum_{k_b=1}^{n_b}\left(\int_1^\infty \frac{dt}{t}\|\langle T_b\rangle_\beta j_{k_b} e^{-itR}\psi\|^2\right)^{1/2}\cdot\left(\int_1^\infty \frac{dt}{t}\|\langle T_a\rangle_\beta J j_{k_b} e^{-itR}\psi\|^2\right)^{1/2},$$

each of the above integrals being $\le C\|\psi\|^2$ by Proposition 3.2. Then, the commutator $[j_{k_b}, \langle T_a\rangle_\beta]$ will be computed with using (3.33) as

$$[j_{k_b}, \langle T_a\rangle_\beta] = 2C\,\mathrm{Re}\int_0^\infty d\omega\,\omega^{1/2}(\omega+\langle T_a\rangle_\beta^2)^{-1}\,T_a\cdot[[iR,Q_a],j_{k_b}](\omega+\langle T_a\rangle_\beta^2)^{-1}. \tag{3.45}$$

The above double commutator gives an $O(t^{-1})$ contribution, so we will continue to estimate the first two terms of the r.h.s. of (3.44) as before (relations (3.40) to (3.42)), the only difference being that $\tilde{\mu}$ will be replaced by 2. This shows that the l.h.s. of (3.44) is integrable in time; the Remainder of (3.43) can be computed by the Fourier spectral formula as

$$\frac{1}{t^2}\int_X dk\hat{J}''(k)e^{-\frac{ik}{t}Q}\int_0^1 d\tau\int_0^\tau d\sigma\, e^{-\frac{ik\sigma}{t}Q}[[iR,Q],Q]e^{+\frac{ik\sigma}{t}Q}, \tag{3.46}$$

where $\hat{J}''$ is a rapidly decreasing function (as Fourier transform of the smooth, compactly supported J'').

Finally, the integral over t of the l.h.s. of (3.32) is obviously dominated by

$$2\sup_{t\ge 1}|\langle J\langle T_a\rangle_\beta J\rangle_t| \le 2\sup_{t\ge 1}\|J\|_\infty \|\langle T_a\rangle_\beta J e^{-itR}\psi\| =$$

$$= 2\sup_{t\ge 1}\|J\|_\infty\Big\{\underbrace{\langle JT_a^2 J\rangle_t}_{O(1)} + t^{-4\beta}\|J\|_\infty^2\|\psi\|^2\Big\}^{1/2},$$

which completes the proof of the Proposition.

R e m a r k s: 1) The estimate (3.7) remains valid if one replaces Q_a by any of its components relative to some basis from X. Moreover, this is also true for the sharper estimate (3.28), as a consequence of the implication (see e.g. Proposition 6.2 in [19])

$$A_k^2 \leq \sum_{j=1}^{n} A_j^2 \Rightarrow A_k \leq \left(\sum_{j=1}^{n} A_j^2 \right)^{1/2},$$

valid for the set $\{A_j\}_{j=1,\dots,n}$ of (unbounded) positive self-adjoint operators.

2) Since Propositions 3.2 and 3.3 have been proved for an arbitrary lattice L, they are also true for any sublattices of L. More precisely, let a be an arbitrarily fixed element of L, and denote by $T_{b,a}$ the operator $[iR_a, Q_b] - \frac{Q_b}{t}$. Then, due to (2.2), the estimates t (3.7) and (3.28) will also be true if R is replaced by R_a, i.e. for any $b \in \mathrm{L}_a$ and any $J \in C_0^\infty(\Gamma_b(0))$ there is a positive constant C such that

$$\int_1^\infty \frac{dt}{t} \left(\left\| \, | T_{b,a} | J(\tfrac{Q}{t}) e^{-itR_a} \psi \right\|^2 + \left\| \, | T_{b,a} |^{1/2} J(\tfrac{Q}{t}) e^{-itR_a} \psi \right\|^2 \right) \leq C \|\psi\|^2 . \tag{3.47}$$

Moreover, taking it into account that $b \in \mathrm{L}_a$ implies $\Gamma_b(0) \subseteq \Gamma_a(0)$ and using the argument described in the proof of Proposition 3.2 (relation (3.24)), we see that the difference between $T_{b,a}$ and T_b is of the order of $O(t^{-\mu+1})$ on the support of J. Then, the obvious inequality $(A+B)^2 \leq 2(A^2+B^2)$ which holds for any pair of self-adjoint (not necessarily positive) operators A, B, shows that $T_{b,a}$ can be replaced by T_b in the first norm of the above inequality. In what follows we will see that the same is true for the second norm above, i.e. the estimate

$$\int_1^\infty \frac{dt}{t} \left\| \, \left| [iR, Q_b] - \frac{Q_b}{t} \right|^{1/2} J(\tfrac{Q}{t}) e^{-itR_a} \psi \right\|^2 \leq C \|\psi\|^2 \tag{3.48}$$

holds for all $b \in \mathrm{L}_a$, all $J \in C_0^\infty(\Gamma_b(0))$ and all $\psi \in \mathrm{H}(X)$. Indeed, using formula (3.33), an easy calculation gives

$$\langle T_b \rangle_\beta - \langle T_{b,a} \rangle_\beta = C \int_0^\infty d\omega\, \omega^{\frac{1}{2}} (\omega + \langle T_{b,a} \rangle_\beta^2)^{-1} \left[i(R - R_a), Q_b \right]^2 (\omega + \langle T_b \rangle_\beta^2)^{-1} +$$

$$+ C\mathrm{Re} \int_0^\infty d\omega\, \omega^{\frac{1}{2}} (\omega + \langle T_{b,a} \rangle_\beta^2)^{-1} T_{b,a} \cdot \left[i(R - R_a), Q_b \right] (\omega + \langle T_b \rangle_\beta^2)^{-1} . \tag{3.49}$$

This shows that we only have to commute the J's from the left or from the right through the resolvent $(\omega + \langle T_b \rangle_\beta^2)^{-1}$ in order to get either the products of J, J' with $R - R_a$ or $O(t^{-2})$ terms. The boundedness of the components of $T_{b,a}$ will be ensured exactly lik in the proof of the previous proposition (see the comments we made after equation (3.39)). In this way we show that the above difference is of integrable order, and thus (3.47) implies (3.48).

4. Wave operators

An important feature of the strategy that Sigal and Soffer devised to prove the existence of the cluster wave operators and asymptotic completeness of usual N-body short-range systems is that both these problems are treated as if they were of the same difficulty. Indeed, the existence of the strong limits of $\exp(itH)J_b\exp(-itH_b)$, where $\{J_b\}$ is a family of pseudodifferential operators verifying a partition of unity in the phase space has been proved for the first time. In our case, we will take, as in [22,14], a time-dependent family of identifiers and state:

Proposition 4.1. *If a is an arbitrary element of* L *and J is a* $C_0^\infty(\Gamma_a(0))$ *function, then the operator domain of the following limits:*

$$W^{\pm}(R,R_a;J) = s - \lim_{t\to\pm\infty} e^{itR}J(\tfrac{Q}{t})e^{-itR_a}, \tag{4.1}$$

$$W^{\pm}(R_a,R;J) = s - \lim_{t\to\pm\infty} e^{itR_a}J(\tfrac{Q}{t})e^{-itR} \tag{4.2}$$

is the whole H (X). *Moreover, the statement is true even when J is a bounded continuous function with support in* $\Gamma_a(0)$.

P r o o f: As in the proof of Proposition 3.3, for showing the first part of the Proposition it will be enough to take $J \in C_0^\infty(X)\cap$ F. We will also use the same partition of unity on the support of J constructed with using the family $\{ j_{k_b} \mid k_b = 1,\dots,n_b,\ b\in \mathrm{L}_a\}$.

To prove the existence of the limits (4.1) and (4.2) the Cook criterion will be used. More precisely, the computation

$$\frac{d}{dt}\left\langle \psi, e^{itR_a}Je^{-itR}\psi\right\rangle = \left\langle \psi, e^{itR_a}\left\{\mathrm{D}_RJ - i(R-R_a)\right\}e^{-itR}\psi\right\rangle$$

shows that a sufficient condition for the convergence of $e^{itR_a}Je^{-itR}$ in the expectation value on $\psi \in$ H (X) is

$$\int_1^\infty \left|\left\langle \psi, e^{itR_a}\mathrm{D}_RJe^{-itR}\psi\right\rangle\right| dt + \|\psi\|^2 \sum_{b\in \mathrm{L}\setminus \mathrm{L}_a} \int_1^\infty \|R(b)J\|\, dt \le C\,\|\psi\|^2. \tag{4.3}$$

Then this weaker type of convergence yields (in our special case) the strong convergence in a standard manner. Reasoning in the same way as in the proof of Proposition 3.2 (see relation (4.3)), we prove that the integrand in the second term of the r.h.s. of (3.21) is of the order of $O(t^{-\mu})$ for all $b \in \mathrm{L}\setminus\mathrm{L}_a$. Further, using formula (3.43), we show that the first term in (4.3) is dominated (modulo some $O(t^{-2})$ contributions) by

$$\int_1^\infty \frac{dt}{t} \sum_{b\in L_a} \sum_{k_b=1}^{n_b} \sum_{l=1}^{\dim X_b} \| T_l \langle T_l \rangle_\beta^{-1} \| \times$$

$$\times \| \langle T_l \rangle_\beta^{1/2} j_{k_b} e^{-itR_a} \psi \| \, \| \langle T_l \rangle_\beta^{1/2} j_{k_b} (\partial_l J) e^{-itR} \psi \| \le$$

$$\le C \sum_{b\in L_a} \sum_{k_b=1}^{n_b} \sum_{l=1}^{\dim X_b} \left(\int_1^\infty \frac{dt}{t} \| \langle T_l \rangle_\beta^{1/2} j_{k_b} e^{-itR_a} \psi \|^2 \right)^{1/2} \times$$

$$\times \left(\int_1^\infty \frac{dt}{t} \| \langle T_l \rangle_\beta^{1/2} j_{k_b} (\partial_l J) e^{-itR} \psi \|^2 \right)^{1/2},$$

where in the above estimate the Schwartz inequality has been used. Finally let us notice that each of the integrals in the above brackets is $\le C \|\psi\|^2$ as a consequence of the Remark following Proposition 3.3.

Suppose now that $J \in BC(\Gamma_a(0))$. Let χ be the smoothed characteristic function of a neighborhood of the origin in X, and denote by χ_γ the operator of multiplication by $\chi\left(\frac{\cdot}{\gamma^t}\right)$, where $\gamma > 0$ is a parameter. Then the product $J\chi_\gamma$ plays the rôle of the J's of the first part of the proposition, so it will be enough to prove that for any positive ε we can choose γ in such a way that

$$\sup_{t\ge 1} \| e^{itR_a} J(1-\chi_\gamma) e^{-itR} \psi \| < \varepsilon \|\psi\|_{H_1}$$

for all ψ belonging to the weighted Lebesgue space of order one $H_1(X)$. Moreover, using the obvious inequality $1 - \chi_\gamma \le Id$, true for γ sufficiently large, we see that a stronger condition is given by

$$\sup_{t\ge 1} \frac{1}{t} \| |Q| e^{-itR} \psi \| < \frac{\varepsilon\gamma}{\|J\|_\infty} \|\psi\|_{H_1}. \tag{4.4}$$

But (4.4) is a particular case of the result due to Radin and Simon (see Theorem 2.1 in [32]). Indeed, all we have to do is to mimic the proof of the quoted theorem (the approximating resolvent R_a taken as the Hamiltonian) and finally obtain

$$\| |Q| e^{-itR_a} \psi \| \le \| |Q| \psi \| + \int_0^1 d\tau \left\langle \widetilde{P}_a^2 \right\rangle_{t,a}^{1/2},$$

where $\{ \widetilde{P}_a \}_{a\ge 0}$ denotes the uniformly bounded family of operators $R_a P R_a$. Applying the Fatou lemma to the above inequality, we prove (4.4) provided that γ is chosen superior to $\varepsilon^{-1} \|J\|_\infty$.

Lemma 4.1. *If a is an arbitrary element of* L *and* E_a *denotes the projection* $E_{pp}(H^a) \otimes {}_a 1$, *then the limits*

$$\Omega_a^{\pm} = s - \lim_{t \to \pm\infty} e^{itR} e^{-itR_a} \mathrm{E}_a \tag{4.5}$$

exist on all H (X). *Moreover, if* $a \neq b$ *are two elements of* L, *then the ranges of the corresponding cluster wave operators are orthogonal.*

P r o o f: Since the existence of the operators $W^{\pm}(R,R_a;J)$ has been proved on H (X) for all $J \in BC(X)$ with support in $\Gamma_a(0)$, the $\varepsilon/3$ argument shows that in order to prove the existence of (4.5) the following convergence

$$\left\| \left(1 - J(\tfrac{Q}{t})\right) e^{-itR_a} u \right\| \overset{t \to \pm\infty}{\to} 0 \tag{4.6}$$

has to be true for all u belonging to a dense subset of E_a H (X). The choice we will make for J in (4.6) depends on the vector u.

For this, let us construct the dense set of u's by taking simple tensors of the form $w^a \otimes_a v_a$, where w^a belongs to $E_{pp}(H^a)$ H (X^a) (and thus w^a is an eigenvector of H^a for the eigenvalue λ^a) and $v_a = g_a(\tilde{P})v_a$ with $g_a \in C_0^\infty(\mathring{X}_a)$ and

$$\tilde{P}_a \equiv \left(z - \lambda^a - \frac{P_a^2}{2}\right)^{-1} P_a \left(z - \lambda^a - \frac{P_a^2}{2}\right)^{-1} . \tag{4.7}$$

As we explained before, we shall *choose* $J \in BC(\Gamma_a(0)) \cap$ F depending on each u by requiring:

$$\begin{cases} supp\, J \Big|_{X_a} \supset supp\, g_a \\ J \Big|_{X_a} = 1 \quad \text{on the support of } g_a . \end{cases} \tag{4.8}$$

Then, the l.h.s. of (4.6) is

$$\left\| \left(1 - J(\tfrac{Q}{t})\right) e^{-itR_a} w^a \otimes_a g_a(\tilde{P}_a) v_a \right\| = \left\| \left(1 - J(\tfrac{Q}{t})\right) e^{-it(z - \lambda^a - P_a^2/2)^{-1}} w^a \otimes_a g_a(\tilde{P}_a) v_a \right\| \tag{4.9}$$

because of relation (1.3), which in turn is a consequence of Theorem 3.10 of [11].

Let us take now one more cutoff function $h \in C_0^\infty(X^a)$ with $h = 1$ in a neighborhood of the subspace X_a. The precise choice we will make for the support of h will depend on the choice we make on J and hence on g_a, so for the moment the shape of the support of h is entirely at our disposal. In what follows we will denote by $A \lesssim B$ the inequality (between numbers) $A \leq B + o(1)$, where $o(1)$ is defined w.r.t. $t \to \pm\infty$. Then the r.h.s. of (4.9) will be majorized by:

$$\left\| \left(1 - J(\tfrac{Q}{t})\right) e^{-it(z - \lambda^a - P_a^2/2)^{-1}} h(\tfrac{Q}{t}) w^a \otimes_a g_a(\tilde{P}_a) v_a \right\| + \left\| \left(1 - h(\tfrac{Q}{t})\right) w^a \otimes_a v_a \right\| \lesssim$$

$$\leq \left\| \left(1 - J(\tfrac{Q}{t})\right) h(\tfrac{Q}{t}) g_a(\widetilde{P}_a) e^{-it(z - \lambda^a - P_a^2/2)^{-1}} w^a \otimes {}_a v_a \right\| \leq$$

$$\leq C \left\| \left(g_a(\widetilde{P}_a) - g_a(\tfrac{Q_a}{t})\right) e^{-it(z - \lambda^a - P_a^2/2)^{-1}} w^a \otimes {}_a v_a \right\| +$$

$$+ \left\| \left(1 - J(\tfrac{Q}{t})\right) h(\tfrac{Q}{t}) g_a(\tfrac{Q_a}{t}) e^{-it(z - \lambda^a - P_a^2/2)^{-1}} w^a \otimes {}_a v_a \right\|$$

where C is a positive constant depending on J and h and where, in the asymptotic inequality, it is supposed that h has the supplementary property $h = h \circ \pi^a$, so it commutes with the unitary group $\exp\{-it(z - \lambda^a - P_a^2/2)^{-1}\}$.

The first term in the above inequality tends to zero when $t \to \pm\infty$ as a consequence of Theorem 7.1.29 of Hörmander (see [24]), and thus

$$\left\| \left(1 - J(\tfrac{Q}{t})\right) e^{-itR_a} u \right\| \lesssim \left\| \left(1 - J(\tfrac{Q}{t})\right) h(\tfrac{Q}{t}) g_a(\tfrac{Q_a}{t}) e^{-it(z - \lambda^a - P_a^2/2)^{-1}} w^a \otimes {}_a v_a \right\|. \tag{4.10}$$

Since $J \in \mathrm{F}$, one can make our final hypothesis on h, namely: the support of h is chosen (as a function of J) to be included in the neighborhood of X_a for which $J = J \circ \pi_a$. Then, in the r.h.s. of (4.10) we will have

$$\left(1 - J(\tfrac{Q}{t})\right) h(\tfrac{Q}{t}) g_a(\tfrac{Q_a}{t}) = \left(1 - J(\tfrac{Q_a}{t})\right) g_a(\tfrac{Q_a}{t}) h(\tfrac{Q}{t}) = 0$$

because of the second hypothesis in (4.8), and thus the first statement of the lemma is proved.

It remains to show that for arbitrary $a \neq b$ in L and for any vectors $\varphi, \psi \in \mathrm{H}(X)$ we have $\langle \Omega_a^{\pm}\varphi, \Omega_b^{\pm}\psi \rangle = 0$. Actually, it will be enough to prove the convergence

$$\langle e^{-itR_a} \mathrm{E}_a \varphi, e^{-itR_b} \mathrm{E}_b \psi \rangle \overset{|t| \to \infty}{\longrightarrow} 0$$

for φ and ψ belonging to the dense sets previously introduced. This time we do not need any function g_a, hence we will take it equal to one. But for these vectors, the l.h.s. of the above relation is

$$\langle \mathrm{E}_a \varphi, e^{-it(z - \lambda^a - P_a^2/2)^{-1}} e^{it(z - \lambda^b - P_b^2/2)^{-1}} \mathrm{E}_b \psi \rangle$$

which converges to zero when t goes to ∞ as a consequence of the Riemann-Lebesgue lemma.

5. Proof of the minimal velocity theorem

Our main result, Theorem 1.1, can be stated in a more precise form as follows:

Theorem 5.1. *Let $a \in \mathrm{L}$ and $\varepsilon > 0$ be arbitrarily chosen. Let (J, θ) and $(\hat{J}, \hat{\theta})$ be two couples of functions, with $J, \hat{J} \in C_0^\infty(\Gamma_a(0))$ and $\theta, \hat{\theta} \in C_0^\infty(\mathrm{R})$, such that $\hat{J}$ and $\hat{\theta}$ are equal* *to one on the supports of J and θ, respectively. Suppose moreover that for arbitrary λ ∈ R one of the following two conditions:*

$$\inf\left\{\left(\rho^A_{R_a}(\lambda)-\varepsilon\right)\mu^2-\frac{x_a^2}{2}\ \Big|\ x\in supp\,\hat{J} \text{ and } \mu\in supp\,\theta\right\}>$$
$$>\sup_{x\in supp\,\hat{J}}\sum_{l=1}^{\dim X^a}\left\|[iR,Q_l]\right\|\frac{|x_l|}{2}, \tag{5.1}$$

$$\left\{\frac{2\inf\mu^2}{\sup\mu^2}\ \Big|\ \mu\in supp\,\theta\right\}\left(\rho^A_{R_a}(\lambda)-\varepsilon\right)>\sum_{l=1}^{\dim X}\left\|P_l\hat{\theta}(R)\right\|\sup_{x\in supp\,\hat{J}}|x_l| \tag{5.2}$$

is satisfied. Then, the estimate

$$\int_1^\infty\frac{dt}{t}\left\|J\left(\frac{Q}{t}\right)\theta(R)e^{-itR}\psi\right\|^2\le C\|\psi\|^2 \tag{5.3}$$

is true for some positive constant C and for all ψ ∈ H(X).

Notice that depending on the choice of the pair (J,θ), the above result may be treated either as a maximal or as a minimal velocity bound theorem. This result gives more information than we need to prove asymptotic completeness. Indeed, we will see later that the following corollary, which corresponds to the particular case $a=a_{\max}$ of the above theorem, is sufficient (via a standard induction argument) for the proof of (1.10). Let us denote by $\xi(R)$ the set of critical values of the operator R. Then:

Corollary 5.2. *Let* $\theta\in C_0^\infty(\mathrm{R}\setminus\xi(R))$ *and take* $J\in C_0^\infty(X)$ *with the support sufficiently close to the origin in X. Then,*

$$s-\lim_{t\to\pm\infty}e^{itR}J\left(\frac{Q}{t}\right)e^{-itR}\theta(R)=0. \tag{5.4}$$

P r o o f: It is sufficient to choose the support of J (depending on how close $supp\,\theta$ is to the points from $\xi(R)$) such that one of the two conditions (5.1) and (5.2) holds. Then (5.3) will be valid, which will imply

$$\liminf_{t\to\infty}\left\|e^{itR}J\left(\frac{Q}{t}\right)e^{-itR}\theta(R)\psi\right\|=0$$

for all $\psi\in\mathrm{H}(X)$. But according to Proposition 4.1, the full strong limits $W^{\mp}(R,R;J)$ exist on the entire $\mathrm{H}(X)$, so that they equal zero on all sets $\theta(R)\,\mathrm{H}(X)$ (whose union over all θ, as in the hypothesis, is dense in H_∞).

P r o o f of Theorem 5.1: Since the theorem should be true for any $a\in\mathrm{L}$, during the whole proof we will fix an arbitrarily chosen a and consider (unless otherwise specified) that all the operators of multiplication by functions are relative to the a (as it were be indexed by a). Then the proof will be performed by induction over the levels $L_a(n)$ ($n = 1, \dots,$ rank $L_a \equiv N_a$) of the sublattice L_a. Nevertheless, there is a particular case, namely $a = a_{\min}$, for which the proof is simpler and it works exactly in the same manner as for the first step of the inductive process corresponding to the cases $a \in L \setminus \{ a_{\min} \}$.

The induction hypothesis will be called (A_{n+1}) and announced as follows:

(A_{n+1}): *If for any $b \in L_a(n+1)$ and any $\varepsilon > 0$ there are two couples of functions (J, θ) and $(\hat{J}, \hat{\theta})$ with $J, \hat{J} \in C_0^\infty(\Gamma_b(0))$ and $\theta, \hat{\theta} \in C_0^\infty(\mathbf{R})$, such that $\hat{J}, \hat{\theta}$ are equal to one on the support of J and θ, respectively, and if for all $\lambda \in \mathbf{R}$, one of the following two conditions:*

$$\inf \left\{ \left(\rho_{R_b}^A(\lambda) - \varepsilon \right) \mu^2 - \frac{x_a^2}{2} \mid x \in supp\, \hat{J} \text{ and } \mu \in supp\, \theta \right\} > \sup_{x \in supp\, \hat{J}} \sum_{l=1}^{\dim X^a} \| [iR, Q_l] \| \frac{|x_l|}{2}, \tag{5.5}$$

$$\left\{ \frac{2 \inf \mu^2}{\sup \mu^2} \mid \mu \in supp\, \theta \right\} \left(\rho_{R_b}^A(\lambda) - \varepsilon \right) > \sum_{l=1}^{\dim X} \| P_l \hat{\theta}(R) \| \sup_{x \in supp\, \hat{J}} |x_l| \tag{5.6}$$

is fulfilled, then estimate (5.3) is true.

The first step of the (weak) inductive process, namely the validity of (A_{N_a}), will be given by the following lemma. As we said before, this lemma tells us also that Theorem 5.1 is true in the particular case $a = a_{\min}$ (see condition (5.7) below).

Lemma 5.1. *Let $b = a_{\min}$ in the hypothesis of (A_{n+1}) or suppose that for the same couples of functions the following strict inequality*

$$\inf \left\{ \left(\rho_{R_{a_{\min}}}^A(\lambda) - \varepsilon \right) \mu^2 - \frac{x^2}{2} \mid x \in supp\, \hat{J} \text{ and } \mu \in supp\, \theta \right\} > 0 \tag{5.7}$$

holds. Then (5.3) holds also.

P r o o f: Let us remind first that for a given vector operator S in $H(X)$, we will denote by S_a, S^a the operators $1 \otimes_a (\pi_a S)$, resp. $(\pi_a S) \otimes 1$ in $H(X)$, but we will not change the notation when the operators $(\pi_a S)$ acting in $H(X_a)$ and therefore $(\pi^a S)$ in $H(X^a)$ will be implicit. As stated before, whenever no confusion is possible, we will not mention the usual time-dependent argument x/t of the operators of multiplication by functions. Keeping in mind these conventions, let us choose the propagation observable

$$\Phi = \theta(R) J \left[iR, \frac{(Q^a)^2}{2t} \right] J \theta(R), \tag{5.8}$$

and compute as usual its Heisenberg derivative as

$$\mathrm{D}_R\Phi = \frac{1}{t}\,\mathrm{Re}\,\theta(R)(\mathrm{D}_R J)\left[iR,\,(Q^a)^2\right]J\theta(R) + \theta(R)J\left(\mathrm{D}_R\left[iR,\,\frac{(Q^a)^2}{2t}\right]\right)J\theta(R)\,. \qquad (5.9)$$

Then the obvious estimate

$$\int_1^\infty dt\,\langle\,\mathrm{D}_R\,\Phi\,\rangle_t \le 2\sup_{t\ge 1}\left|\,\langle\,\Phi\,\rangle_t\,\right| \le$$
$$\le \sum_{l=1}^{\dim X^a}\left\|\,[iR,Q_l]\,\right\|\sup_{x\in supp\,J}\,|\,x_l\,|\;\;\|J\,\|_\infty^2\,\|\theta\,\|_\infty^2\,\|\psi\,\|^2 \qquad (5.10)$$

shows that we only have to look at the terms from the r.h.s. of (5.9): we will begin by proving that the first one is integrable w.r.t. t. For this, let us take an orthonormal basis in X and denote by T_k the components of the vector operator $T_{a_{\min}} = T$ in this basis. Since the support of J is compact, there exists a finite family $\{j_i\}$ of C_0^∞-functions with supports contained in $supp\,\hat{J}$ and such that $J = J\,\Sigma_i j_i^2$. Then, proceeding as in the proof of the Proposition 3.3 (see relations (3.43) and (3.44)), we get

$$(\mathrm{D}_R J)\left[iR,\frac{(Q^a)^2}{t}\right]J = \frac{1}{t}\sum_{k=1}^{\dim X}\sum_i\, j_i\langle\,T_k\,\rangle_\beta^{1/2}\underbrace{\langle\,T_k\,\rangle_\beta^{-1}\,T_k}_{O(1)}\,B_k\langle\,T_k\,\rangle_\beta^{1/2}\,j_i + O(t^{-2}). \qquad (5.11)$$

Since $j_i \in C_0^\infty(\mathring{X}_{a_{\min}})$ for any i, we can use (3.30) in order to integrate the first term in the r.h.s. above, provided that

$$B_k \overset{\text{not}}{=} \langle\,T_k\,\rangle_\beta^{1/2}\,(\partial_k J)\left[iR,\,\frac{(Q^a)^2}{t}\right]J\,\langle\,T_k\,\rangle_\beta^{-1/2} \qquad (5.12)$$

is shown to be the sum of a uniformly bounded (in t) operator and some integrable terms. We have thus to commute $\langle\,T_k\,\rangle_\beta^{1/2}$ towards left. Since J and $\partial_k J$ bound the above commutator, we will only have to show the integrability of sums of the form

$$\frac{1}{t}\left[\langle\,T_k\,\rangle_\beta^{1/2},g\right][iR,Q_l]\tilde{g} + \frac{1}{t}\,g\left[\langle\,T_k\,\rangle_\beta^{1/2},[iR,Q_l]\right]\tilde{g}, \qquad (5.13)$$

where g, $\tilde{g}$ denote the operators of multiplication by the functions $\partial_k J$ or $(\partial_k J)\pi_l$ and $J\pi_l$ or J, respectively (their argument being as usual x/t). Using formula (3.33) with $\gamma = 1/4$ and the argument exactly as in the proof of Proposition 3.3 (see (3.45)) we can show that if β is appropriately chosen w.r.t. μ, the first term of (5.13) brings an integrable contribution. Then, in the same manner, the second term of the above sum is computed as

$$\frac{2C}{t}\,\mathrm{Re}\,gT_k\int_0^\infty d\omega\,\omega^{\frac{1}{4}}\left(\omega + \langle\,T_k\,\rangle_\beta^2\right)^{-1}\times$$

$$\times \left\{ [\,[iR,Q_k],[iR,Q_l]\,] + \left[[iR,Q_l],\frac{Q_k}{t}\right] \right\} (\omega + \langle T_k\rangle_\beta^2)^{-1}\,\tilde{g}.$$

It is clear that the second term in the curly brackets gives an $O(t^{-2})$ contribution and, for $\beta > 0$ sufficiently small, it will still be integrable w.r.t. t after integration in ω is performed. As for the first one, the formula

$$\frac{1}{t}\,[\,[iR,Q_k],[iR,Q_l]\,] = \frac{1}{2}\left[[iR,[iR,Q_l]],\frac{Q_k}{t}\right] + \frac{1}{2}\left[[iR,Q_k],iR],\frac{Q_l}{t}\right] \tag{5.14}$$

shows that we only have to prove

$$\int_1^\infty dt \int_0^\infty d\omega\,\omega^{\frac{1}{4}} \left\| \left(gT_k\omega + \langle T_k\rangle_\beta^2\right)^{-1} \frac{Q_l}{t}\,[iR,[iR,Q_k]] \times \right.$$

$$\left. \times \left(\omega + \langle T_k\rangle_\beta^2\right)^{-1} \tilde{g}\langle T_k\rangle_\beta^{-1/2} \right\| < \infty .$$

The above double commutator can be computed as in the proof of Proposition 3.2, with taking it into account that for any $c \in \mathrm{L}$ we have $[P_k,R(c)] = 0$ for all k for which $X_k \subseteq X_c$. Notice that we refer to $\subseteq$ as a vector space inclusion because X_k cannot belong to the lattice $\{X_b\}_{b\in \mathrm{L}}$. Thus, we have

$$[\,[iR,Q_k],R\,] = \sum_{\substack{c\in \mathrm{L}\\ X_k \nsubseteq X_c}} R\left(P_kR(c) - R(c)P_k\right)R .$$

Notice that for all $k = 1,\ldots,\dim X$ the inclusion $X_k \subset X_{a_{\min}}$ shows that the above sum will be performed over a subset of $\mathrm{L}\setminus\{a_{\min}\}$, and since the supports of g, $\tilde{g}$ are compacts from $\Gamma_{a_{\min}}(0)$ (for all $l = 1,\ldots,\dim X^a$), they are disjoint with any X_c, $c \in \mathrm{L}\setminus\{a_{\min}\}$, so it will be enough to commute g, $\tilde{g}$ towards right or left, respectively, in order to obtain either $O(t^{-2})$ terms, or products $gR(c) \sim O(t^{-\mu})$. As in the proof of Proposition 3.3 (see the comments made before (3.40)), there will be a tribute to pay in order to ensure integrability w.r.t. ω, but for a suitable (small, positive) β there will still remain enough decay in t for convergence of the above double integral.

We pass now to the second term in the r.h.s. of (5.9). Since yet, any of the hypothesis (5.5), (5.6) or (5.7) has been used; in what follows, this term will be estimated in three (somewhat) different ways, with each of these variants involving the Mourre estimate and only one of the mentioned hypothesis. Let us start by recalling that for any $b \in \mathrm{L}$ the differences $R - R_b$, $\theta(R) - \theta(R_b)$ and the double commutator $[\,[iR,Q_b],R\,]$ are of order $O(t^{-\mu})$ on $supp\ J \subset \Gamma_{a_{\min}}(0)$. Then the obvious equality

$$\left[iR,\left[iR,\frac{Q_a^2}{2t}\right]\right] = \mathrm{Re}\ [iR,[iR,Q_a]]\cdot\frac{Q_a}{t} + \frac{1}{t}\,|iR,Q_a|^2 \tag{5.15}$$

shows that

$$\theta(R)J\left(\mathrm{D}_R\left[iR,\frac{(Q^a)^2}{2t}\right]\right)J\theta(R) = -\frac{1}{t}\,\theta(R)J\left[iR,\frac{(Q^a)^2}{2t}\right]J\theta(R) +$$

$$+\frac{1}{t}JR_{a_{\min}}\theta(R_{a_{\min}})[iR_{a_{\min}},A]\theta(R_{a_{\min}})R_{a_{\min}}J-$$
$$-\frac{1}{t}\theta(R)J[iR,Q_a]^2J\theta(R)+O(t^{-\mu})+O(t^{-2}). \qquad (5.16)$$

We start with proving that (5.5) (taken with $b=a_{\min}$) implies (5.3). Since

$$[iR,Q_a]^2=T_a^2+2\mathrm{Re}\,T_a\cdot\frac{Q_a}{t}+\frac{Q_a^2}{t^2}, \qquad (5.17)$$

then of the mean value $\langle\cdot\rangle_t$ of the last term of the r.h.s. of (5.16) dominates the sum

$$-\frac{1}{t}\left\{\left\|\,|T_a|\,J\theta(R)e^{-itR}\psi\right\|^2+\sum_{l=1}^{\dim X_a}\sup_{x\in supp\,J}|x_l|\,\left\|\langle T_l\rangle_\beta^{1/2}J\theta(R)e^{-itR}\psi\right\|^2\right\}+$$
$$+O(t^{-2})\|\psi\|^2-\frac{1}{t}\sup_{x\in supp\,J}x_a^2\,\left\|J\theta(R)e^{-itR}\psi\right\|^2, \qquad (5.18)$$

in which the first line is integrable in t as a consequence of the propagation theorems 3.2 and 3.3. Further, the first term in the r.h.s. of (5.16) is minorized by

$$-\frac{1}{t}\sup_{x\in supp\,J}\sum_{l=1}^{\dim X^a}\left\|[iR,Q_l]\right\|\,|x_l|\,\left\|J\theta(R)e^{-itR}\psi\right\|^2, \qquad (5.19)$$

whereas for the second term of the r.h.s. of (5.16) we apply the Mourre estimate (see def. (1.5) and also (1.6)) so that it be dominated by

$$\frac{2}{t}\left(\rho^A_{R_{a_{\min}}}(\lambda)-\varepsilon\right)\inf_{\mu\in supp\,\theta}\mu^2\left\|J\theta(R)e^{-itR}\psi\right\|^2+O(t^{-\mu})+O(t^{-2}). \qquad (5.20)$$

Then, summing up (5.18) to (5.20), we obtain the lower bound of the mean value of the l.h.s. of (5.16):

$$\left\langle\theta(R)J\left(\mathrm{D}_R\left[iR,\frac{(Q^a)^2}{2t}\right]\right)J\theta(R)\right\rangle_t\gtrapprox$$
$$\gtrapprox\frac{2}{t}\left\{\left(\rho^A_{R_{a_{\min}}}(\lambda)-\varepsilon\right)\inf_{\mu\in supp\,\theta}\mu^2-\sup_{x\in supp\,J}\left(\frac{x_a^2}{2}+\sum_{l=1}^{\dim X^a}\left\|[iR,Q_l]\right\|\,\left|\frac{x_l}{2}\right|\right)\right\}\left\|J\theta(R)e^{-itR}\psi\right\|^2,$$

where $\gtrapprox$ means $\geq$ modulo addition of some terms of an integrable order (whenever this type of *equality* arises, the sign $\approx$ will be used). But since the support of $\hat{J}$ is a dilation of *suppJ*, the hypothesis (5.5) ensures strict positivity of the quantity in the above curly brackets, which proves (5.3).

We pass now to the proof of the implication (5.7) $\Rightarrow$ (5.3) and compute

$$\theta(R)J\left(\mathrm{D}_R\left[iR,\frac{(Q^a)^2}{2t}\right]\right)J\theta(R)=-\frac{1}{t}\theta(R)J\left(\mathrm{Re}\,T^a\cdot\frac{Q^a}{t}+\frac{(Q^a)^2}{t^2}\right)J\theta(R)-$$
$$-\frac{1}{t}\theta(R)J\left(\mathrm{Re}\,T_a\cdot\frac{Q^a}{t}+T_a^2\right)J\theta(R)+$$

$$+\frac{1}{t}JR_{a_{\min}}\theta(R_{a_{\min}})[iR_{a_{\min}},A]\theta(R_{a_{\min}})R_{a_{\min}}J -$$

$$-\frac{1}{t}\theta(R)J\frac{Q_a^2}{t^2}J\theta(R) + O(t^{-\mu}) + O(t^{-2}). \tag{5.21}$$

Estimating the terms of the r.h.s. above like in (5.18) – (5.20), we see that the l.h.s. of (5.21) dominates (modulo some integrable terms)

$$2\left\{\left(\rho^A_{R_{a_{\min}}}(\lambda) - \varepsilon\right)\inf_{\mu\in \operatorname{supp}\theta}\mu^2 - \frac{1}{2}\sup_{x\in \operatorname{supp} J}x^2\right\}\|J\theta(R)e^{-itR}\psi\|^2$$

in which the curly brackets contain a strictly positive number as a consequence of (5.7). Finally,

$$[iR,Q_a]^2 = T_a^2 + \operatorname{Re} T_a\cdot\frac{Q_a}{t} + \left[iR,\frac{Q_a^2}{2t}\right] \tag{5.22}$$

allows us to calculate

$$\theta(R)J\left(\mathrm{D}_R\left[iR,\frac{(Q^a)^2}{2t}\right]\right)J\theta(R) = -\frac{1}{t}\theta(R)J\left[iR,\frac{Q_a^2}{2t}\right]J\theta(R) + O(t^{-\mu}) + O(t^{-2}) -$$

$$-\frac{1}{t}\theta(R)J\left(\operatorname{Re} T_a\cdot\frac{Q^a}{t} + T_a^2\right)J\theta(R) +$$

$$+\frac{1}{t}JR_{a_{\min}}\theta(R_{a_{\min}})[iR_{a_{\min}},A]\theta(R_{a_{\min}})R_{a_{\min}}J, \tag{5.23}$$

which together with the estimate

$$\frac{1}{t}\left\langle JR\theta(R)P\cdot\frac{Q}{t}JR\theta(R)\right\rangle_t \lesssim \sum_{l=1}^{\dim X}\|P_l\hat{\theta}(R)\|\sup_{x\in \operatorname{supp} J}|x_l|\;\|R\theta(R)Je^{-itR}\psi\|^2 \lesssim$$

$$\lesssim \sum_{l=1}^{\dim X}\|P_l\hat{\theta}(R)\|\sup_{x\in \operatorname{supp} J}|x_l|\;\|\sup_{\mu\in \operatorname{supp}\theta}\mu^2\|J\theta(R)e^{-itR}\psi\|^2$$

shows that the l.h.s. of (5.23) is

$$\gtrsim \frac{1}{t}\left\{2\left(\rho^A_{R_{a_{\min}}}(\lambda) - \varepsilon\right)\inf_{\mu\in \operatorname{supp}\theta}\mu^2 -\right.$$

$$\left.-\sum_{l=1}^{\dim X}\|P_l\hat{\theta}(R)\|\sup_{x\in \operatorname{supp} J}|x_l|\;\|\sup_{\mu\in \operatorname{supp}\theta}\mu^2\right\}\|J\theta(R)e^{-itR}\psi\|^2.$$

Using the hypothesis (5.6) (written for $b = a_{\min}$), we see again that the expression in curly brackets is positive, and this completes the proof of the Lemma.

We begin now the second step of the inductive argument, namely, we are to prove the implication $(\mathrm{A}_{n+1}) \Rightarrow (\mathrm{A}_n)$ for any $n = 1,\dots,N_a - 1$. Notice that in (5.5) and (5.6) one could substitute $\rho^A_{R_a}(\lambda)$ instead of $\rho^A_{R_b}(\lambda)$, but since $b \in \mathrm{L}_a(n+1)$ implies $\rho^A_{R_a}(\lambda) \le \rho^A_{R_b}(\lambda)$, (A_{n+1}) would weaken (anyway, this would not create any disadvantage from the point of view of the final result). For proving the above implication it will be enough to fix arbitrarily $b \in \mathrm{L}_a(n)$ and show that (A_n) holds for this b. Thus, let J, $\widehat{J}$ and θ, $\widehat{\theta}$ be chosen with respect to this b so that (5.5) or (5.6) be verified. Then Proposition 6.1 (*iv*) shows that for any such J we can construct (as in the proof of Proposition 3.3) a family $\{J_{k_c}\}$ of $C_0^\infty(\Gamma_b(0))$-functions satisfying

$$J = J \sum_{c \in \mathrm{L}_b} \sum_{k_c=1}^{n_c} J^2_{k_c} \tag{5.24}$$

$$supp\, \widehat{J} \supset \bigcup_{c \in \mathrm{L}_b} \bigcup_{k_c=1}^{n_c} supp\, J_{k_c}. \tag{5.25}$$

Notice that (5.25) is always possible to satisfy because of the strict inclusion of $supp\, J$ in $supp\, \widehat{J}$ and that it will be no loss in supposing that $n_b = 1$ (renote J_{k_b} by J_b). Then, since (A_{n+1}) is assumed true, (5.24) allows us to reduce slightly the problem by estimating

$$\begin{aligned} \int_1^\infty \frac{dt}{t} \left\| J\theta(R)e^{-itR}\psi \right\|^2 \le \|J\|^2_\infty \int_1^\infty \frac{dt}{t} \left\| J_b\theta(R)e^{-itR}\psi \right\|^2 + \\ + \|J\|^2_\infty \sum_{c<b} \sum_{k_c=1}^{n_c} \int_1^\infty \frac{dt}{t} \left\| J_{k_c}\theta(R)e^{-itR}\psi \right\|^2 . \end{aligned} \tag{5.26}$$

Then, as a consequence of (5.25) there is a $C_0^\infty(\Gamma_b(0))$- function $\widehat{J}_{k_c}$ such that $\widehat{J}_{k_c} = 1$ on $supp\, J_{k_c}$ and $supp\, \widehat{J}_{k_c} \subset supp\, \widehat{J}$. This shows that

$$\sup_{x \in supp\, \widehat{J}} \ge \sup_{x \in supp\, \widehat{J}_{k_c}}$$

and since, as a consequence of the definition of the function ρ ,

$$\rho^A_{R_c}(\lambda) \ge \rho^A_{R_b}(\lambda) \text{ for all } c \le b \text{ and all } \lambda \in \mathrm{R}, \tag{5.27}$$

then the assumptions similar to (5.5) and (5.6) in (A_n) (written for J) are stronger than the same assumptions in any of the (A_{n+1})'s written for J_{k_c}. This shows that the second line of (5-27) is $\le C \|\psi\|^2$, so it remains only to estimate the first term in the r.h.s. of (5.27). To do so, let us consider two smooth functions, $\widetilde{J}_b$ and f, having the property that for some positive constant

$$J_b \le C\widetilde{J}_b f \tag{5.28}$$

they are true on all X. Observe that such functions always exist: it is enough to choose them so that their product be a C_0^∞-function and that

$$supp\, J_b \subset supp\, \tilde{J}_b \cap supp\, f. \tag{5.29}$$

We will, moreover, impose some supplementary conditions on $\tilde{J}_b$, namely, we suppose that there is a set $coresupp\, \tilde{J}_b$ (which is centered at the same point of $\mathring{X}_b$ as $suppJ_b$) satisfying

$$coresupp\, \tilde{J}_b \subset supp\, \tilde{J}_b, \tag{5.30}$$

$$coresupp\, \tilde{J}_b \cap \mathring{X}_b \neq 0, \tag{5.31}$$

$$\tilde{J}_b \text{ is constant on } coresupp\, \tilde{J}_b, \tag{5.32}$$

$$\tilde{J}_b \in C_0^\infty(\Gamma_b(0)), \tag{5.33}$$

$$supp\, \tilde{J}_b \subset supp\, \hat{J}. \tag{5.34}$$

Notice that (5.34) is true because of the definition of $\hat{J}$ and (5.25). As for f, we suppose that

$$f \in C_0^\infty(X), \tag{5.35}$$

$$supp\, f \cap \mathring{X}_b \subset coresupp\, \tilde{J}_b \cap \mathring{X}_b, \tag{5.36}$$

$$f = f \circ \pi_b \text{ on } supp\, \hat{J}. \tag{5.37}$$

In conclusion, it will be enough to prove that

$$\int_1^\infty \frac{dt}{t} \left\| f\left(\frac{Q_b}{t}\right)\tilde{J}_b\left(\frac{Q}{t}\right)\theta(R)e^{-itR}\psi \right\|^2 \leq C \|\psi\|^2 \tag{5.38}$$

which will be always true since

$$\int_1^\infty \frac{dt}{t} \left\| f([iR,Q_b])\tilde{J}_b\left(\frac{Q}{t}\right)\theta(R)e^{-itR}\psi \right\|^2 \leq C \|\psi\|^2 \tag{5.39}$$

and

$$\int_1^\infty \frac{dt}{t} \left\langle \tilde{J}_b\left(\frac{Q}{t}\right) \left\{f^2([iR,Q_b]) - f^2\left(\frac{Q_b}{t}\right)\right\}\tilde{J}_b\left(\frac{Q}{t}\right) \right\rangle_t \leq C \|\psi\|^2. \tag{5.40}$$

In Appendix 6.4 we prove that estimates of similar to type (5.40) are consequences of the propagation theorems 3.2 and 3.3, so it remains to prove (5.39). To do so, we choose as propagation observable the bounded operator

$$\Phi = \theta(R)f([iR,Q_b])\tilde{J}_b\left(\frac{Q}{t}\right)\left[iR, \frac{(Q^a)^2}{2t}\right]\tilde{J}_b\left(\frac{Q}{t}\right)f([iR,Q_b])\theta(R). \tag{5.41}$$

As in the proof of Lemma 5.1, an inequality of the same type as (5.10) shows that it will be enough to estimate each of the terms yielded by the derivation of $D_R\Phi$. Starting with the one containing $D_R f([iR,Q_b])$, we will use the spectral Fourier formula to calculate

$$\left[iR,f([iR,Q_b])\right] = \int_X ds\,\hat{f}(s)\int_0^1 d\tau\, e^{is(1-\tau)[iR,Q_b]}\left[iR,[iR,Q_b]\right] e^{is\tau[iR,Q_b]} \quad .(5.42)$$

Then, we commute $\tilde{J}_b$ through $\exp(is\tau[iR,Q_b])$ to obtain products of $\tilde{J}_b$, $\tilde{J}_b'$, with the above double commutator (which gives $O(t^{-\mu})$ contributions). After all these computations have been performed, we obtain (modulo some $O(t^{-2})$ terms)

$$\left[iR,f([iR,Q_b])\right] \approx \int_X ds\,\hat{f}'(s)\int_0^1 i\tau d\tau\int_0^1 d\sigma e^{is(1-\tau)[iR,Q_b]}\left[iR,[iR,Q_b]\right]\times$$

$$\times\left(\tilde{J}_b + \frac{1}{t}\tilde{J}_b' e^{is\tau\sigma[iR,Q_b]}\left[[iR,Q_b]Q\right]e^{-is\tau\sigma[iR,Q_b]}\right)e^{is\tau[iR,Q_b]}, \qquad (5.43)$$

which shows that

$$\int_1^\infty \frac{dt}{t}\left\langle \theta(R)\left(D_R f([iR,Q_b])\right)\tilde{J}_b\left[iR,\frac{(Q^a)^2}{2t}\right]\tilde{J}_b f([iR,Q_b])\theta(R)\right\rangle_t \le C\,\|\psi\|^2 .$$

We pass now to the proof of integrability w.r.t. t of the second term resulting from Φ's Heisenberg derivation, i.e. the one containing

$$D_R\tilde{J}_b = \tilde{J}_b'\cdot T_{a_{\min}} + O(t^{-2}).$$

Let $\{\tilde{j}_{k_c} \mid c\in L_b \text{ and } k_c = 1,\dots,\tilde{n}_c\}$ be a family of $C_0^\infty(\Gamma_b(0))$-functions having supports centered at some points of $\mathring{X}_c$, satisfying

$$\tilde{J}_b' = \sum_{c\in L_b}\sum_{k_c=1}^{\tilde{n}_c}\tilde{j}_{k_c}^2\,\tilde{J}_b' \qquad (5.44)$$

and

$$supp\,\tilde{j}_{k_c}\cap X_b = 0 \quad \text{for all} \quad c<b. \qquad (5.45)$$

Notice also that according to assumptions (5.36) and (5.32), one can choose the subfamily $\{\tilde{j}_{k_b} \mid k_b = 1,\dots,\tilde{n}_b\}$ such that

$$\left(\bigcup_{k_b=1}^{\tilde{n}_b} supp\,\tilde{j}_{k_b}\cap\mathring{X}_b\right)\cap supp\,f = 0. \qquad (5.46)$$

Hence, denoting by h_{k_c} a $C_0^\infty(\Gamma_b(0))$-function equal to one on the support of $\tilde{j}_{k_c}$ and making use of (5.44), we get

$$\left\langle \theta(R)f([iR,Q_b])\left(\mathrm{D}_R \widetilde{J}_b\right)\left[iR,\frac{(Q^a)^2}{2t}\right]\widetilde{J}_b f([iR,Q_b])\theta(R)\right\rangle_t \lessapprox$$

$$\lessapprox \frac{1}{t}\sum_{c\in \mathrm{L}_b}\sum_{k_c=1}^{\widetilde{n}_c}\left\langle \theta(R)\widetilde{j}_{k_c} f([iR,Q_b])\underbrace{\widetilde{J}_b{}'\cdot T_{a_{\min}}\left[iR,\frac{(Q^a)^2}{2t}\right]h_{k_c}}_{O(1)} f([iR,Q_b])\widetilde{j}_{k_c}\theta(R)\right\rangle_t \lessapprox$$

$$\lessapprox \frac{C}{t}\sum_{c\in \mathrm{L}_b}\sum_{k_c=1}^{\widetilde{n}_c}\left\|f([iR,Q_b])\widetilde{j}_{k_c}\theta(R)e^{-itR}\psi\right\|^2 \lessapprox$$

$$\lessapprox \frac{C}{t}\sum_{c\in \mathrm{L}_b}\sum_{k_c=1}^{\widetilde{n}_c}\left\|f\left(\frac{Q_b}{t}\right)\widetilde{j}_{k_c}\left(\frac{Q}{t}\right)\theta(R)e^{-itR}\psi\right\|^2, \tag{5.47}$$

where at the last step Appendix 6.4 has been used. Let us prove that the r.h.s. of (5.47) is $\leq C\,\|\psi\|^2$. For this, observe first that as a consequence of assumption (5.34), we can choose the family $\{\widetilde{j}_{k_c}\}$ such that

$$supp\,\widetilde{j}_{k_c} \subset supp\,\widehat{J} \text{ for all } c\in \mathrm{L}_b. \tag{5.48}$$

Secondly, (5.46) joint with (5.45) tells us that for all $c\in \mathrm{L}_b$,

$$supp\,(f\widetilde{j}_{k_c}) \cap \mathring{X}_b = 0, \tag{5.49}$$

which, together with the inclusion (see (5.48)),

$$supp\,(f\widetilde{j}_{k_c}) \subset \Gamma_b(0) = \coprod_{c\in \mathrm{L}_b}\mathring{X}_c$$

gives

$$supp\,(f\widetilde{j}_{k_c}) \subset \bigcup_{c\in \mathrm{L}_b\setminus\{b\}}\mathring{X}_c \text{ for all } c\in \mathrm{L}_b. \tag{5.50}$$

On the other hand, since $b\in \mathrm{L}_a(n)$ is fixed, any $c<b$ will belong to $\bigcup_{i=n+1}^{N_a}\mathrm{L}_a(i)$, or,equivalently, for any $c<b$ there is a $\widetilde{b}\in \mathrm{L}_a(n+1)$ such that $c\leq\widetilde{b}$. This shows that (5.50) can be written as:

$$supp\,(f\widetilde{j}_{k_c}) \subset \bigcup_{\widetilde{b}\in \mathrm{L}_a(n+1)}\coprod_{c\in \mathrm{L}_{\widetilde{b}}}\mathring{X}_c \equiv \bigcup_{\widetilde{b}\in \mathrm{L}_a(n+1)}\Gamma_{\widetilde{b}}(0), \tag{5.51}$$

We would like to apply (A_{n+1}) to any of the functions $f\widetilde{j}_{k_c}$. Observe first that another way of writing the hypothesis (A_{n+1}) (which makes reference to the whole lattice level $\mathrm{L}_a(n+1)$ and not to the elements $\widetilde{b}$ from it) is to demand the support of J to be included in $\bigcup_{\widetilde{b}\in \mathrm{L}_a(n+1)}\Gamma_{\widetilde{b}}(0)$ and to replace in (5.5) or in (5.6) $\rho^A_{R_{\widetilde{b}}}(\lambda)$ by $\min_{\widetilde{b}\in \mathrm{L}_a(n+1)}\rho^A_{R_{\widetilde{b}}}(\lambda)$. Then, (5.48) tells us that for all $c\in \mathrm{L}_b$ there exists a C_0^∞ -

function $\hat{j}_{k_c}$ (which equals one on $supp\, \tilde{j}_{k_c}$ and has support included in $supp\, \hat{J}$), which together with (5.26) and (5.51) allows us to conclude that the hypothesis of (A_{n+1}) written for the product function $f\tilde{j}_{k_c}$ is true, and thus the r.h.s. of (5.47) is integrable.

It remains to estimate the term resulting from $D_R\Phi$ which contains the Heisenberg derivative of $\frac{1}{2t}\left[iR,(Q^a)^2\right]$. It will be computed as follows:

$$\theta(R)f([iR,Q_b])\tilde{J}_b\left(D_R\left[iR,\frac{(Q^a)^2}{2t}\right]\right)\tilde{J}_bf([iR,Q_b])\theta(R) =$$

$$= \theta(R)f([iR,Q_b])\tilde{J}_b\left(\frac{\partial}{\partial t}\left[iR,\frac{(Q^a)^2}{2t}\right]\right)\tilde{J}_bf([iR,Q_b])\theta(R) -$$

$$-\theta(R)f([iR,Q_b])\tilde{J}_b\left[iR,\left[iR,\frac{Q_a^2}{2t}\right]\right]\tilde{J}_bf([iR,Q_b])\theta(R) +$$

$$+f([iR,Q_b])\tilde{J}_b\theta(R)R\left[iR,\frac{A}{t}\right]R\theta(R)\tilde{J}_bf([iR,Q_b]) -$$

$$-\left[f([iR,Q_b]),\theta(R)\right]\tilde{J}_b\left[iR,\left[iR,\frac{Q^2}{2t}\right]\right]\tilde{J}_bf([iR,Q_b])\theta(R) +$$

$$+f([iR,Q_b])\theta(R)\tilde{J}_b\left[iR,\left[iR,\frac{Q^2}{2t}\right]\right]\tilde{J}_b\,[f([iR,Q_b]),\theta(R)] +$$

$$+f([iR,Q_b])[\theta(R),\tilde{J}_b]\left[iR,\left[iR,\frac{Q^2}{2t}\right]\right]\tilde{J}_b\theta(R)f([iR,Q_b]) -$$

$$-f([iR,Q_b])\tilde{J}_b\theta(R)\left[iR,\left[iR,\frac{Q^2}{2t}\right]\right][\theta(R),\tilde{J}_b]\,f([iR,Q_b]). \tag{5.52}$$

Since the commutator $\left[f([iR,Q_b]),\theta(R)\right]$ is of order $O(t^{-\mu})$ on the support of $\tilde{J}_b$ (which is a compact subset of $\Gamma_b(0)$), the fourth and the fifth terms of the r.h.s. of (5.52) are integrable. The same is true for the last two terms in the above equality. Indeed, notice first that up to addition of some $O(t^{-2})$ contributions, they are of the form

$$\frac{1}{t}f([iR,Q_b])\tilde{J}_b'\cdot[\theta(R),Q]\left[iR,[iR,Q]\right]\cdot\frac{Q}{t}\tilde{J}_b\theta(R)f([iR,Q_b]).$$

Remark that there are some components of the above double commutator (namely, those relative to some basis in X^b) which are *not* small (in the sense that they do not confer integrable decay in t) on $supp\,\tilde{J}_b$. Nevertheless, the above term is of the same type as the l.h.s. of (5.47), so we can use the same argument to show its integrability.

It remains to estimate the first three lines of the r.h.s. of (5.52). We will proceed, as in the proof of Lemma 5.1, and minorize the first of these terms by

$$-\frac{1}{t}\sum_{l=1}^{\dim X^a}\left\|[iR,Q_l]\right\|\sup_{x\in supp\,\tilde{J}_b}|x_l|\,\left\|\tilde{J}_bf([iR,Q_b])\theta(R)e^{-itR}\psi\right\|^2+O(t^{-2})\,\|\psi\|^2. \tag{5.53}$$

Using the Mourre estimate, the mean value $\langle \cdot \rangle_t$ of the third one is

$$\gtrsim \frac{2}{t}\left(\rho^A_{R_b}(\lambda) - \varepsilon\right)_{\mu \in supp\,\theta} \inf \mu^2 \left\|f([iR,Q_b])\tilde{J}_b\theta(R)e^{-itR}\psi\right\|^2. \tag{5.54}$$

Finally, the second term of the r.h.s. of (5.52) is

$$\approx -\mathrm{Re}\left\langle \theta(R)f([iR,Q_b])\tilde{J}_b[iR,[iR,Q_a]]\cdot\frac{Q_a}{t}\tilde{J}_b f([iR,Q_b])\theta(R)\right\rangle_t -$$
$$-\frac{1}{t}\left\langle \theta(R)\tilde{J}_b f([iR,Q_b])[iR,Q_a]^2 f([iR,Q_b])\tilde{J}_b\theta(R)\right\rangle_t, \tag{5.55}$$

and since $\Gamma_b(0) \subseteq \Gamma_a(0)$ for any $b \in L_a$, the above double commutator will be of the order of $O(t^{-\mu})$ on $supp\,\tilde{J}_b$. The second term of the above sum can be estimated by means of

$$[iR,Q_a]^2 = -T_a^2 + 2\mathrm{Re}\,T_a\cdot[iR,Q_a] + \frac{Q_a^2}{t^2} \tag{5.56}$$

as being (for $\beta > 0$ sufficiently small)

$$\gtrsim -\frac{1}{t}\sup_{x \in supp\,\tilde{J}_b} x_a^2 \left\|f([iR,Q_b])\tilde{J}_b\theta(R)e^{-itR}\psi\right\|^2 -$$

$$-\frac{2}{t}\sum_{l=1}^{\dim X_a}\mathrm{Re}\left\langle \theta(R)\tilde{J}_b\langle T_l\rangle_\beta^{1/2} f([iR,Q_b])\underbrace{\langle T_l\rangle_\beta^{-1}T_l}_{O(1)}\underbrace{[iR,Q_l]f([iR,Q_b])}_{O(1)}\langle T_l\rangle_\beta^{1/2}\tilde{J}_b\theta(R)\right\rangle_t +$$

$$+\frac{1}{t}\|f\|_\infty^2\left\||T_a|\tilde{J}_b\theta(R)e^{-itR}\psi\right\|^2. \tag{5.57}$$

Notice that in order to obtain the second term above one has shown (in the same manner as in Appendix 6.4) that $\left[f([iR,Q_b]),\langle T_l\rangle_\beta^{1/2}\right]$ brings an integrable contribution on the support of $\tilde{J}_b$. Then, the last two lines above are integrable as a consequence of propagation theorems 3.2 and 3.3, so summing up (5.53), (5.54) and (5.57) we conclude that the l.h.s. of (5.52) dominates (in the sense of $\gtrsim$)

$$\frac{1}{t}\left\{2\left(\rho^A_{R_b}(\lambda) - \varepsilon\right)_{\mu \in supp\,\theta}\inf \mu^2 - \right.$$
$$\left. -\sup_{x \in supp\,\tilde{J}_b} x_a^2 - \sum_{l=1}^{\dim X^a}\left\|[iR,Q_l]\right\|\sup_{x \in supp\,\tilde{J}_b}|x_l|\right\}\left\|f([iR,Q_b])\tilde{J}_b\theta(R)e^{-itR}\psi\right\|^2.$$

Since the assumption (5.34) together with the hypothesis of (A_n) ensures strict positivity for the quantity in the above curly brackets, the estimate (5.39) is proven and hence the first implication of the theorem as well.

It remains to prove that (5.3) is a consequence of (5.2). This will be done in the same manner as above, the only difference being that we have to invoke relation (5.22) instead of (5.56) when wanting to estimate the second term in (5.55). The comments we made after (5.55) show that, up to some integrable terms, the first two lines in the r.h.s. of (5.52) dominate

$$-\frac{1}{t}\left\langle \theta(R)f([iR,Q_b])\tilde{J}_b\left[iR,\frac{Q^2}{2t}\right]\tilde{J}_b f([iR,Q_b])\theta(R)\right\rangle_t$$

which, in turn, can be computed as in the proof of Lemma 5.1 (see the estimate following (5.23)) and thus minorized (modulo $O(t^{-\mu})$) by

$$-\sum_{l=1}^{\dim X}\left\|P_l\hat{\theta}(R)\right\| \sup_{x\in supp\,\tilde{J}_b}|x_l| \sup_{\mu\in supp\,\theta}\mu^2\left\|f([iR,Q_b])\tilde{J}_b\theta(R)e^{-itR}\psi\right\|^2.$$

This shows, as before, that the l.h.s. of (5.52) dominates

$$\frac{1}{t}\left\{2\left(\rho^A_{R_b}(\lambda)-\varepsilon\right)\cdot\inf_{\mu\in supp\,\theta}\mu^2-\right.$$

$$\left.-\sup_{\mu\in supp\,\theta}\mu^2\sum_{l=1}^{\dim X}\left\|P_l\hat{\theta}(R)\right\|\sup_{x\in supp\,\tilde{J}_b}|x_l|\right\}\left\|f([iR,Q_b])\tilde{J}_b\theta(R)e^{-itR}\psi\right\|^2$$

where the quantity in the curly brackets is strictly positive as a consequence of relation (5.6) corresponding to (A_n).

The rest of this section will be devoted to the proof of statement (1.10). We will use a standard induction argument (see [33,2,22,26]), performed at the levels of some arbitrary lattice L. Let us first denote, for any $a,b\in L$, $a\le b$, by $\Omega^{\pm}_{a,b}$ the wave operators $\Omega^{\pm}(R_a,R_b;E_b)$ and notice that they exist owing to the existence of $W^{\pm}(R_a,R_b;J)$ with $supp\,J\subset\Gamma_b(0)$. Then, since for the rank one lattice statement (1.10), written for $\Omega^{\pm}(R_a,R_b;E_b)$, is trivial, we will suppose it to be true for any lattice of rank N, and prove it for all lattices L of the rank $N+1$. Actually, it is enough to prove it for any state ψ localized via a smooth cutoff θ in a compact of $\mathbb{R}\setminus\xi(R)$, because, due to the fact that the set of the critical values of R is countable, a covering argument (see Proposition 4.2.6 in [2]) will allow us to extend the result to all R. Let us begin by computing, using a conveniently chosen partition of unity in X (e.g. the smooth one constructed in Section 2):

$$e^{-itR}\psi = J_{a_{\max}}e^{-itR}\psi+\sum_{n=2}^{N+1}\sum_{a\in L(n)}J_a e^{-itR}\psi \simeq \sum_{n=2}^{N+1}\sum_{a\in L(n)}e^{-itR_a}\sum_{b\in L_a}\underbrace{\Omega^{\pm}_{a,b}(\Omega^{\pm}_{a,b})^* W_a^{\pm}\psi}_{\psi_{a,b}}.$$

In the above modulo $o(1)$ equality $\simeq$, $W^{\pm}_a$ stands for the limits in (4.2), and Corollary 5.2 and the induction hypothesis were used. Thus we have:

$$\psi = \sum_{n=2}^{N+1} \sum_{a \in \mathrm{L}(n)} e^{-itR} e^{-itR_a} \sum_{b \in \mathrm{L}_a} \Omega_{a,b}^{\pm} \psi_{a,b} =$$

$$= \sum_{a \in \mathrm{L} \setminus \{a_{\max}\}} \sum_{b \in \mathrm{L}_a} e^{-itR} e^{itR_b} \mathrm{E}_b \psi_{a,b} = \sum_{a \in \mathrm{L} \setminus \{a_{\max}\}} \sum_{b \in \mathrm{L}_a} \Omega_b^{\pm} \psi_{a,b},$$

which proves $\psi \in \oplus_{b \in \mathrm{L} \setminus \{a_{\max}\}} \Omega_b^{\pm} \mathrm{H}$.

6. Appendices

6.1. Appendix

Definition 6.1. *Let us define for all* $a \in \mathrm{L}$ *the following sets*:

i) $\mathrm{L}_a = \{ b \in \mathrm{L} \mid b \leq a \}$,

ii) $\mathrm{L}^a = \{ b \in \mathrm{L} \mid b \geq a \}$,

iii) $\overset{\circ}{X}_a = X_a \setminus \bigcup_{b \not\leq a} X_b$,

iv) $\Gamma_a(0) = X \setminus \bigcup_{b \not\leq a} X_b$.

Notice also the particular cases of $a = a_{\min}$, for which $\overset{\circ}{X}_{a_{\min}} = \Gamma_{a_{\min}}(0) X \setminus \bigcup_{b \not\leq a_{\min}} X_b$ and $a = a_{\max}$, for which $\overset{\circ}{X}_{a_{\max}} = \{ 0 \}$ and $\Gamma_{a_{\max}}(0) = X$. Denoting by the sign $\bigsqcup$ the disjoint union of sets, we have:

Proposition 6.1. *For an arbitrary* $a \in \mathrm{L}$,

i) $\overset{\circ}{X}_a = X_a \setminus \bigcup_{b > a} X_b$,

ii) $\overset{\circ}{X}_a \cap \overset{\circ}{X}_b = 0$ *for all* $a, b \in \mathrm{L}$ *with* $a \neq b$,

iii) $X_a = \bigsqcup_{b \in \mathrm{L}^a} \overset{\circ}{X}_b$,

iv) $\Gamma_a(0) = \bigsqcup_{b \in \mathrm{L}_a} \overset{\circ}{X}_b$.

Remark also that for all $a \in \mathrm{L}$, $\mathrm{L}_a \cap \mathrm{L}^a = \{ a \}$ yields $\overset{\circ}{X}_a = \Gamma_a(0) \cap X_a$.

Proof: Remember that we denoted by $\not\sim$ the "incomparability" between two elements of the lattice L. Then, on one hand $X_a \setminus \bigcup_{b \not\sim a} X_b = X_a \setminus \bigcup_{b \not\sim a} \{ X_b \cap X_a \}$, and on the other hand, since L is sup-stable, there is a $c \in \mathrm{L}$ with $c > a$ and $c > b$ such that $X_b \cap X_a = (X^b + X^a)^{\perp} = (X^c)^{\perp} = X_c$. This shows the inclusion

$$X_a \setminus \bigcup_{b \not\sim a} X_b \subseteq X_a \setminus \bigcup_{c > a} X_c$$

and thus (*i*) is proved. Let now $a \in \mathrm{L}$ be arbitrarily fixed, and take first $b \in \mathrm{L} \setminus \mathrm{L}_a$. Then, by Definition 6.1 (*i*), $X_b \cap \mathring{X}_a = 0$ and $\mathring{X}_b \subset X_b$, so (*ii*) is true for these b. For those b belonging to $\mathrm{L}_a \setminus \{a\}$, notice that according to (*i*), for all $c > b$, we have $X_c \cap \mathring{X}_b = 0$. But a is one of these c, and thus (*ii*) is proved. In order to prove (*iii*), suppose as usual a arbitrarily fixed and observe that the rank of an element $b \in \mathrm{L}^a$, denoted by $|b|_{\mathrm{L}^a}$, is generally different from $|b|_{\mathrm{L}}$. Denote also for $1 \le j \le \operatorname{rank} \mathrm{L}^a$ the j-th *level* in L^a as $\mathrm{L}^a(j) \equiv \{ b \in \mathrm{L}^a \mid |b|_{\mathrm{L}^a} = j \}$, and the union of $\mathrm{L}^a(j)$ with all the levels in L^a which are below it by $\mathrm{L}^a_j \equiv \{ b \in \mathrm{L}^a \mid |b|_{\mathrm{L}^a} \ge j \}$. Denote $n \equiv \operatorname{rank} \mathrm{L}^a = |a|_{\mathrm{L}^a}$ and take as an induction hypothesis the statement

$$X_a = \bigsqcup_{b \in \mathrm{L}^a_{n-j}} \mathring{X}_b \cup \bigcup_{l=1}^{n-j-1} \bigcup_{b \in \mathrm{L}^a(l)} X_b . \tag{6.1}$$

The first step of the induction is given by the equation

$$X_a = \mathring{X}_a \cup \bigcup_{b \in \mathrm{L}^a - \{a\}} X_b \tag{6.2}$$

in which the set inclusion $\supseteq$ is ensured by (*i*). Further, let us suppose that (6.1) is true for some $j < n$. Then, using (*ii*), one gets

$$X_a = \bigsqcup_{b \in \mathrm{L}^a_{n-j}} \mathring{X}_b \cup \bigcup_{b \in \mathrm{L}^a(n-j-1)} \Big(\mathring{X}_b \cup \bigcup_{c > b} X_c\Big) \cup \bigcup_{l=1}^{n-j-1} \bigcup_{b \in \mathrm{L}^a(l)} X_b =$$

$$= \bigsqcup_{b \in \mathrm{L}^a_{n-j-1}} \mathring{X}_b \cup \bigcup_{c \in \mathrm{M}_j} X_c \cup \bigcup_{l=1}^{n-j-2} \bigcup_{b \in \mathrm{L}^a(l)} X_b ,$$

where the set $\mathrm{M}_j \equiv \{ c \in \mathrm{L} \mid \exists\, b \in \mathrm{L}^a(n-j-1), \text{ such that } c > b \}$ obviously satisfies $\mathrm{M}_j \subseteq \mathrm{L}^a \setminus \bigcup_{l=n-j-1}^{n} \mathrm{L}^a(l)$. Thus (*iii*) is proved.

Finally, (*iii*) and the definition of $\Gamma_a(0)$ give

$$\Gamma_a(0) = \bigsqcup_{b \in \mathrm{L}} \mathring{X}_b \setminus \bigcup_{b \in \mathrm{L} \setminus \mathrm{L}_a} \bigsqcup_{c \in \mathrm{L}^b} \mathring{X}_c .$$

Thus, to prove (*iv*), it will be enough to establish the set inclusion $\mathrm{L}_a \subseteq \mathrm{L} \setminus \{ c \in \mathrm{L}^b \mid b \in \mathrm{L} \setminus \mathrm{L}_a \}$, i.e. $\mathrm{L}_a \cap \{ c \in \mathrm{L}^b \mid b \in \mathrm{L} \setminus \mathrm{L}_a \} = 0$. Assume the existence of some $c_0 \in \mathrm{L}$ in this set intersection. Then, there is a $b_0 \in \mathrm{L} \setminus \mathrm{L}_a$ such that $c_0 > b_0$, and on the other hand $c_0 \le a$, i.e. $b_0 \in \mathrm{L}_a$. Contradiction.

6.2. Appendix. We have to check (see [5]) that the existence of the Abelian operators Ω_a, which is equivalent to

$$\lim_{T\to\infty}\int_1^T \left\| \left(E_H(\mathrm{R}) - \Omega_a\right) e^{-itR_a} \mathrm{E}_a\psi \right\| = 0,$$

implies the convergence

$$\lim_{T\to\infty}\int_1^T \left\| \mathrm{W}_a(t)\psi - \Omega_a\psi \right\| = 0.$$

But, using twice the intertwining relation for Ω_a, we get for all $\psi \in \mathrm{H}(X)$:

$$\left\| \mathrm{W}_a(t)\psi - \Omega_a\psi \right\| = \left\| E_H(\mathrm{R}) e^{-itH_a} \mathrm{E}_a\psi - e^{-itH} E_H(\mathrm{R})\,\Omega_a\psi \right\| =$$
$$= \left\| \left(E_H(\mathrm{R}) - \Omega_a\right) e^{-itH_a} \mathrm{E}_a\psi \right\|.$$

Finally, Lemma 1 of [28] can be used to get the desired result.

6.3. Appendix. The following Lemma puts in evidence some estimates, *uniform* in $\alpha \geq 0$, concerning the approximating family of the Hamiltonians $\{ H_\alpha \}$. Notice that for any $a \in \mathrm{L}$, (1.2) can be written as $H_\alpha = H_{a,\alpha} + \mathrm{I}_{a,\alpha}$, where $\mathrm{I}_{a,\alpha}$ denotes the sum from the first equality in (1.2), performed only on the set $\mathrm{L} \setminus \mathrm{L}_a$.

Lemma 6.1. *For any* $z \ll \inf \sigma\left(H_\alpha \big|_{\alpha=0}\right)$ *and for any* $a \in \mathrm{L}$, *if* $R_{a,\alpha}$ *denotes* $\left(H_{a,\alpha} - z\right)^{-1}$, *then there is a constant* $C > 0$, *independent of* α, *such that for any* $\alpha \geq 0$:

$$\alpha \left\| R_\alpha^{1/2} \chi(a) R_\alpha^{1/2} \right\| + \left\| R_\alpha^{1/2}\, \mathrm{I}_{a,\alpha}\, R_\alpha^{1/2} \right\| \leq C, \tag{6.3}$$

$$\alpha \left\| R_{a,\alpha} \sum_{b \in \mathrm{L}\setminus \mathrm{L}_a} \chi(b) R_\alpha \right\| + \left\| R_\alpha\, \mathrm{I}_{a,\alpha}\, R_{a,\alpha} \right\| \leq C. \tag{6.4}$$

For the proof, notice that the first inequality is a consequence of the hypothesis made on z, and of the obvious identity:

$$1 = R_\alpha^{1/2} (H_\alpha - z) R_\alpha^{1/2} = \alpha R_\alpha^{1/2} \chi(a) R_\alpha^{1/2} + R_\alpha^{1/2} \left(H_\alpha \big|_{\alpha=0} + \sum_{b \neq a} \alpha\chi(b) - z\right) R_\alpha^{1/2}.$$

Then, using

$$R_\alpha - R_{a,\alpha} = - R_\alpha\, \mathrm{I}_{a,\alpha}\, R_{a,\alpha} = - R_{a,\alpha}\, \mathrm{I}_{a,\alpha}\, R_\alpha,$$

we compute

$$\left\| R_{a,\alpha}\, \mathrm{I}_{a,\alpha}\, R_\alpha \right\|^2 = \left\| (R_\alpha - R_{a,\alpha})^2 \right\| \leq 2 \left\| R_{a,\alpha} \right\| \left\| R_\alpha \right\| + \left\| R_{a,\alpha} \right\|^2 + \left\| R_\alpha \right\|^2 \leq C,$$

where all the norms are in $B(\mathrm{H})$ and where the uniform boundedness (w.r.t. α) of the family $\{ \left\| R_{a,\alpha} \right\|_{-1,1} \}$ tells us that C does not depend on α. Finally, we have

$$\alpha \left\| R_{a,\alpha} \sum_{b \in \mathbf{L} \setminus \mathbf{L}_a} \chi(b) R_\alpha \right\| \le \left\| R_{a,\alpha} \mathbf{I}_{a,\alpha} R_\alpha \right\| + \sum_{b \in \mathbf{L} \setminus \mathbf{L}_a} R_{a,\alpha} \langle P \rangle \left\| \left\| V(b) \right\|_{1,-1} \left\| \langle P \rangle R_\alpha \right\|$$

which completes the proof of (6.4).

Note that uniform estimates similar to type (6.4) are useful when it is desirable to obtain decay of the difference $R_\alpha - R_{a,\alpha}$ on the support of some time-dependent cutoff $J \in C_0^\infty(\Gamma_a(0))$ (as in the proof of Proposition 4.1). As far as it concerns inequality (6.3), notice that it is of quadratic type, i.e. if χ is of the form $\tilde{\chi}\tilde{\chi}^*$, then it can be written as $\sqrt{\alpha} \left\| R_\alpha^{1/2} \tilde{\chi} \right\| \le C$, which is not enough for showing that the double commutator

$$[iR_\alpha, [iR_\alpha, Q_a]] = \sum_{b \in \mathbf{L} \setminus \mathbf{L}_a} \alpha R_\alpha^2 [\chi(b), iP_a] R_\alpha^2 = \sum_{b \in \mathbf{L} \setminus \mathbf{L}_a} i\alpha R_\alpha^2 \chi(b) P_a R_\alpha^2 + \text{h.c.}$$

is bounded uniformly w.r.t. α. This is one of numerous reasons which makes the algebraic framework (introduced in Section 2) indispensable.

6.4. Appendix. We have to prove that for any $a \in \mathbf{L}$ and any $b \in \mathbf{L}^a$, given two functions $J \in C_0^\infty(\Gamma_a(0))$ and $g \in C_0^\infty(X)$ satisfying $g = g \circ \pi_b$ on a neighborhood of the intersection of *supp* J with X_b, the estimate

$$\int_1^\infty \frac{dt}{t} \left\langle J\left(\frac{Q}{t}\right) \left\{ g([iR, Q_b]) - g\left(\frac{Q_b}{t}\right) \right\} J\left(\frac{Q}{t}\right) \right\rangle_t \le C \left\| \psi \right\|^2 \tag{6.5}$$

is true for all $\psi \in \mathbf{H}(X)$.

To show this, we use the Fourier spectral formula to compute the difference

$$g([iR, Q_b]) - g\left(\frac{Q_b}{t}\right) = \int_X ds\, \hat{g}'(s) \int_0^1 d\tau\, e^{is\tau[iR, Q_b]} T_b e^{is(1-\tau)\frac{Q_b}{t}}, \tag{6.6}$$

where $\hat{g}' \in \mathbf{J}(X)$. Thus, we have to look, for any $k = 1, \ldots, \dim X_b$, at:

$$\left\langle J\, e^{is\tau[iR, Q_b]}\, T_k\, e^{is(1-\tau)\frac{Q_b}{t}} J \right\rangle_t =$$

$$= \left\langle J \langle T_k \rangle_\beta^{1/2} \underbrace{e^{is\tau[iR, Q_b]} \langle T_k \rangle_\beta^{-1} T_k e^{is(1-\tau)\frac{Q_b}{t}}}_{O(1)} \langle T_k \rangle_\beta^{1/2} J \right\rangle_t +$$

$$+ \left\langle J \langle T_k \rangle_\beta^{1/2} e^{is\tau[iR, Q_b]} \underbrace{\langle T_k \rangle_\beta^{-1} T_k}_{O(1)} [\langle T_k \rangle_\beta^{1/2}, e^{is(1-\tau)\frac{Q_b}{t}}] J \right\rangle_t +$$

$$+ \left\langle J \left[e^{is\tau[iR, Q_b]}, \langle T_k \rangle_\beta^{1/2} \right] \langle T_k \rangle_\beta^{-1/2} T_k e^{is(1-\tau)\frac{Q_b}{t}} J \right\rangle_t. \tag{6.7}$$

It is clear that the first line above is integrable w.r.t. t as a consequence of Proposition 3.3 (the integrabilities in s and τ are trivial). Further, using formula (3.33), we compute (as in the proof of Proposition 3.3):

$$\left[\langle T_k \rangle_\beta^{1/2}, e^{is\tau[iR,Q_b]}\right] = 2s\tau \mathrm{Re}\, T_k \int_0^\infty d\omega\, \omega^{\frac{1}{4}} (\omega + \langle T_k \rangle_\beta^2)^{-1} \times$$

$$\times \int_0^1 d\sigma\, e^{is\tau(1-\sigma)[iR,Q_b]} \, [[iR,Q_b],T_k]\, e^{is\tau[iR,Q_b]} (\omega + \langle T_k \rangle_\beta^2)^{-1} .$$

Hence, we essentially have to decide if for all $k,l = 1,\dots,\dim X_b$, the commutator $[[iR,Q_l],T_k]$ confers on the support of J enough decay in order to ensure integrability with respect to both t and ω. But this has been already shown in the proof of Lemma 4.1 by relation (5.14) (see also the discussion following it), the only difference being that the role of a was played there by $a_{\min}$, so we could take k, l to run from 1 to dim X. Notice also that because of the rapid decay of $\hat{g}'$, the polynomials in s resulted from the various commutations of J with the unitary groups of the type exp $(is\tau\sigma[iR,Q_b])$ do not influence integrability in s. Finally, the second term of the r.h.s. of (6.7) will be treated in the same way as the previous one. Actually it is even simpler, because we will not have to use the good decay along certain directions of the connected components of R, the basically $O(t^{-\mu})$ decay being replaced by the better $O(t^{-2})$.

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