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Totally umbilical submanifolds of symmetric spaces

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Complete classification of totally umbilical submanifolds of dimension > 2 in symmetric spaces is presented.

1. Introduction

It is well known that totally geodesic submanifolds of symmetric spaces admit simple algebraic description. E. Cartan's theorem [8, Ch. IV, Th. 7.2] states that these submanifolds are the exponents of the Lie triple systems in a space tangent to the symmetric space. Thus, the differential-geometric problem reduces to a linear-algebraic one.

But for a somewhat wider class of submanifolds, namely, for the totally umbilical submanifolds, such simple way of classification does not exist. It is due to the fact maybe that these submanifolds have been classified only for space forms [1, Ch. 11, §3], for compact rank one symmetric spaces [2–4], for the Grassman manifolds $G(2, n)$ [11,12], and for conformally flat symmetric spaces (this class of spaces consists of space forms, products of the line and space form and products of the sphere of curvature $c > 0$ and the hyperbolic space of curvature $(-c)$ [6, Ch. VII]). Classification of the umbilical submanifolds in conformally flat spaces is based on the invariance of these submanifolds under the conformal change of metric. Some results on the umbilical submanifolds of symmetric spaces may be found in the B.-Y. Chen monograph [6, Ch. VII].

The Riemannian manifold M^n is called *locally-symmetric*, if its curvature tensor is parallel: $\widetilde{\nabla}\widetilde{R} = 0$, where $\widetilde{\nabla}$ is a Riemannian connection on M, $\widetilde{R}$ — its curvature tensor. Although most of our statements hold for locally-symmetric spaces, as well we will consider M to be complete and simply-connected, i.e. to be a globally symmetric space.

The *second fundamental form* of a submanifold $N^l \subset M^n$ is defined by the Gauss formula

$$h(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y, \tag{1.1}$$

where X and Y are the vector fields tangent to N and ∇ is the induced connection on N. For the vector field ξ normal to N its covariant derivative according to the Weingarten formula is

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{1.2}$$

where X is the vector field tangent to N, $A_\xi X$ and $D_X \xi$ are the conponents tangent and normal to N, respectively; A_ξ is the bilinear symmetric operator (Weingarten operator) in the space tangent to N

$$g(A_\xi X, Y) = g\left(h(X,Y), \xi\right)$$

for the vector fields X and Y tangent to N; g is the metric tensor on M; D_X is the normal connection derivative. The *mean curvature vector* is $H = (\mathrm{Tr}h)/l$, where Tr denotes the trace according to the metric g induced on N; $\alpha = \|H\| = (g(H,H))^{1/2}$ is the *mean curvature* of a submanifold $N \subset M$.

The submanifold N^l in the Riemannian space M^n is called *totally umbilical* if for arbitrary vector fields X and Y tangent to N

$$h(X,Y) = g(X,Y)H$$

("the first and second fundamental forms are proportional").

In the case of $h = 0$ the submanifold is called *totally geodesic*. If the mean curvature vector of the umbilical submanifold is parallel in the normal bundle ($DH = 0$) and does not vanish ($H \neq 0$ at least at one point), then it is called an *extrinsic sphere*.

The equations of Gauss, Codazzi, and Ricci for the totally umbilical submanifold $N^l \subset M^n$ are given respectively by

$$\widetilde{R}(X,Y,Z,T) = R(X,Y,Z,T) - \alpha^2(g(X,T)g(Y,Z) - g(X,Z)g(Y,T)), \tag{1.3}$$

$$\widetilde{R}(X,Y,Z,\xi) = g(Y,Z)g(D_X H,\xi) - g(X,Z)g(D_Y H,\xi), \tag{1.4}$$

$$\widetilde{R}(X,Y,\xi,\eta) = R^D(X,Y,\xi,\eta), \tag{1.5}$$

where the vectors X,Y,Z,and T tangent to N, ξ and η are normal to N. Here R is the curvature tensor on N, R^D is the normal connection D curvature tensor.

The submanifold $N \subset M$ is *full* if it does not lie in any proper geodesic submanifold in M. We will denote by $M^d(k)$ the d-dimensional space form of the curvature k; we will also use the notation $S^d(k)$, $k > 0$, E^d and L^d, $k < 0$, for the spheres, Euclidean and hyperbolic spaces, respectively. We will denote the Riemannian product as $\Pi_{j=1}^{M} M_j$ or $M_1 \times M_2$ (in the case of two factors). The *warped product* $E^1 \times^f M$ of the line (E^1, dt^2) and the Riemannian manifold (M, ds^2) with the function $f(t) > 0$ is their Cartesian product endowed with the metric $dt^2 + f ds^2$.

Every umbilical submanifold of the space form is an extrinsic sphere or a totally geodesic one [1, Ch. II, §3].They are spheres and subspaces in Euclidean spaces, small and great spheres in a sphere, spheres, subspaces, equidistants, and horospheres in a hyperbolic space. Extrinsic spheres in symmetric spaces have been completely classified. Namely, they are extrinsic hyperspheres in the totally geodesic space form ([5]; a more exact proof was given by the author in [13]).

Our main result is as follows.

Theorem 1. *Let N^l, $l \geq 3$, be a totally umbilical connected submanifold in a globally symmetric space M^n. Then it is either a totally geodesic or totally umbilical and complete in the totally geodesic submanifold $\widetilde{M} \subset M^n$ isometric to the space forms product.*

Let $\widetilde{M} = \Pi_{\gamma=1}^{N} M_\gamma^{d_\gamma}(k_\gamma)$ be the decomposition of $\widetilde{M}$ with no more than one Euclidean factor. Then the pair $(\widetilde{M}, N)$ is of one of the following types:

Type	$\widetilde{M}$	N	α
A	$N \geq 1, \sum_{\gamma=1}^{N} k_\gamma^{-1} \neq 0,$ $d_2 = \ldots = d_N = l+1, d_1 = l,$ or $d_1 = l+1$	$M^l(c)$	const
A_0	E^{l+1}	S^l	
A_1	$N \geq 2$ or $N = 1$ and $d_1 = l+1, k_1 \neq 0$		
B	$N \geq 2, d_1 = 0, \sum_{\gamma=2}^{N} k_\gamma^{-1} \neq 0, \infty.$	$E^1 \times {}^f M^{l-1}(c),$ $N \neq M^l(c)$	
B_1	$N \geq 3, d_2 = l-1,$ $d_3 = \ldots = d_N = l+1.$	$E^1 \times M^{l-1}(c), c \neq 0$	const
B_2	$l \leq d_2, \ldots, d_N \leq l+1$	$f = \text{const}, c \neq 0$ $f \neq \text{const}$	const $\neq$ const
C	$N \geq 2, \sum_{\gamma=2} k_\gamma = 0$ $1 < d_1, \ldots, d_N \leq l+1.$ No more than two factors M_γ have dimension $d_\gamma < l$. For any $1 \leq \gamma, \delta \leq N, d_\gamma + d_\delta \geq l.$	conformally flat	
C_1	$N \geq 3, d_1 + d_2 = l,$ $d_3 = \ldots = d_N = l+1.$	$S^{d_1}(c) \times L^{d_2}(-c),$ $c \neq 0$	const
C_2	$d_2 = \ldots = d_N = l+1, d_1 = l,$ or $d_1 = l+1.$	$M^l(c)$	const
C_3	$d_\gamma + d_\delta > l$ for arbitrary γ, δ	$N \neq M^l(c)$	

Remark 1. It is easy to see that umbilicity is "inherited" in the following sense: if N is umbilical in $\widetilde{M}$ and $\widetilde{M}$ is geodesic in M, then N umbilical in M. The constancy of the mean curvature and the extrinsic sphericity are inherited too.

Remark 2. Type A_1 admits some factor to be Euclidean, i.e. $\sum_{\gamma=2}^{N} k_\gamma^{-1}$ can equal infinity.

R e m a r k 3. In the following we mean everywhere that N is locally isometric (nonisometric, conformally flat).

R e m a r k 4. Conformal flatness of the induced metric of the umbilical non-geodesic submanifold in symmetric space has been proved in [10] for $l \geq 4$. It is true also for $l = 3$ (Proposition 2). We give there the explicit form of such a metric. The constancy of the mean curvature under the symmetricity condition of N has been also proved in [10].

R e m a r k 5. It is important to note that there is *no* one-to-one correspondence between the types of spaces $\widetilde{M}$ and types of submanifolds N. The table (see Theorem 1) gives the classification of *pairs* $(\widetilde{M}, N)$. In particular, N of the type C_3 can be isometric to N of the types B or C_1; N of the type B_2 to N of the type B_1. The extremal types B_2 and C_1 are the only ones in which $\widetilde{M}$ determines completely the intrinsic geometry of N. The type C_1 is defined only with the purpose to use it in Theorem 2; it is similar to A_1 type.

Note that if the curvature of M is of a (not strictly) constant sign, then it has no type C totally geodesic submanifold. In particular we have the following corollary.

C o r o l l a r y. *Let $N^l \subset M^n$, $l \geq 3$, be a connected totally umbilical submanifold in the compact type irreducible symmetric space. Then it is totally geodesic or totally umbilical in the totally geodesic product $\widetilde{M}$ of flat torus and spheres. One of the following cases can take place:*

A_1. Every factor of $\widetilde{M}$ is of the dimension $l + 1$ (except possibly one of dimension l) and N is isometric to a sphere;

B_2. One of the factors is a circle and the other factors are l- or $(l + 1)$-dimensional spheres. N is isometric to the warped (not direct!) product of a line and a sphere.

This corollary follows immediately from Theorem 1 and Gauss equation (1.3): the curvature of an umbilical nongeodesic submanifold will be strictly positive.

It follows from Theorem 1 that for classification of umbilical submanifolds of a dimension ≥ 3 in symmetric spaces it is enough to carry out such a classification for the space forms products of the types mentioned above. To do this we need certain additional constructions. First of all we replace $\widetilde{M}$ by its universal covering space.

Our aim is to construct a suitable embedding of $\widetilde{M}$ into a (pseudo)-Euclidean space E and then to intersect $\widetilde{M} \subset E$ by the osculating space in the (almost arbitrary) point of $N \subset \widetilde{M}$.

We define the standard embedding of the sphere $S^d(k)$, $k > 0$, $d \geq 1$, as the embedding of a hypersphere into E^{d+1} centered at the origin and of radius $k^{-1/2}$ (if $d = 1$, we get the circle in a Euclidean plane). The standard embedding of a hyperbolic space $L^d(k)$, $k < 0$, $d \geq 1$, is the embedding of the upper hemisphere $-(x^1)^2 + (x^2)^2 + \ldots + (x^{d+1})^2 = -R^2$, $x^1 > 0$, centered at the origin and of an imaginary radius $R = k^{-1/2}$, into a Minkowsky space $E^{d+1,1}$ with the metric $ds^2 = -(dx^1)^2 + (dx^2)^2 + \ldots + (dx^{d+1})^2$ (in particular, for $d = 1$ we get the branch of as hyperbola). The standard embedding of the Euclidean space E^d, $d > 1$, is its embedding as the subspace into E^{d+1}.

Let us *define the standard embedding of type A and C manifolds* $\widetilde{M}$ as the product of standard embeddings of its factors. The ambient space E in general is the pseudo-Euclidean space of the corresponding dimension and index.

For the type B of the manifold $\widetilde{M}$ we define the "curvature" k_1 of the one-dimensional factor to be equal to $-\left(\sum_{\gamma=2}^{N} k_\gamma^{-1}\right)^{-1}$. Then $k \neq 0$ and $\sum_{\gamma=1}^{N} k_\gamma^{-1} = 0$. Now embed this

factor $M^1(k_1)$ in a standard way as above: we will get the circle in E^2 or the branch of hyperbola in $E^{2,1}$. The product of this embedding and the standard embeddings of other factors is called the *standard embedding of type B manifold* $\widetilde{M}$. Note that in the case of $k_1 > 0$ we will reject the simple connectedness of $\widetilde{M}$.

The *osculating space* at the point $Q \in N \subset E$ to submanifold in a pseudo-Euclidean space is the direct sum of the tangent space $T_Q N$ and the span of the second fundamental form h at Q (the latter space is called the *first normal space*).

It is well known that one can get the totally umbilical submanifold $N \subset S^n$ (resp. $N \subset L^n$) as the intersection of S^n (resp. L^n) with the osculating space to $N \subset E$ at its arbitrary point.

A similar situation takes place also in the general case.

Theorem 2. *Let $N^l \subset \widetilde{M}$, $l \geq 3$, be complete, connected, totally umbilical non-geodesic submanifold in the space M of types A, B, C. Then:*

1) *if $\widetilde{M}$ is of type A_0, then N is the hypersphere;*

2) *if $\widetilde{M}$ is not of type A_0, then N is the intersection of $\widetilde{M}$ and the osculating space* L *to $N \subset E$ at its general point:*

a) *if N is of constant curvature (types A_1, C_2), then* dim L $= l + 1$. *Osculating space* L *is spanned onto the tangent space and mean curvature vector to $N \subset E$. The submanifold N is umbilical in E;*

b) *if $(\widetilde{M}, N)$ are of the types B_1, B_2, C_1, C_2, then* dim L $= l + 2$. *Osculating space* L *is spanned onto the tangent space, the mean curvature vector and the position vector to $N \subset E$.*

R e m a r k 6. We define the *general point* to be the point of N with maximal number of different principal Ricci curvatures. In the case 2a) (i.e. types A_1, C_2) every point is a general one. As to 2b) (types B_1, B_2, C_1, C_2), we shall show that this concept is well justified because of analyticity of N (Proposition 2).

2. Plan of the proof of Theorem 1

The first step of the proof of Theorem 1 is the following.

Proposition 1. *Let M^n, $n \leq 3$, be the Riemannian space, $Q \in M^n$, L be the l-dimensional subspace in $T_Q M^n$, $2 \leq l < n$, H^* be the vector in $T_Q M^n$ orthogonal to L. Then:*

1) *there exists not more than unique totally umbilical submanifold $N^l \subset M^n$ such that $Q \in N^l$, $T_Q N^l = L$ and the mean curvature vector of N^l at Q equals H^*;*

2) *if this submanifold exists, then it lies as the complete totally umbilical submanifold in the totally geodesic submanifold $\widetilde{M} \subset M$, which passes through the point Q and is tangent to $L \oplus H^*$: $T_Q \widetilde{M} \supset L \oplus H^*$.*

Naturally we mean N at 1) to be maximal by inclusion.

Note that Proposition 1 is the generalization of the well known fact that a totally geodesic submanifold is determined by a point and a tangent space in it.

Now to prove Theorem 1 one has to solve two problems. The *extrinsic problem* is to prove that the minimal (by inclusion) totally geodesic submanifold $\widetilde{M} \subset M$ tangent to the osculating space to N at the point $Q \in N$ is isometric to the product of type A, B, and C space forms . The *intrinsic problem* is to describe the geometry of N for $\widetilde{M}$ of the above mentioned types.

Let $\{X_i\}_{i=1}^{l}$ be the orthonormal frame of the principal Ricci directions of N and $\{r_i\}_{i=1}^{l}$ the corresponding principal Ricci curvatures. Define the following functions ($i = 1,\ldots,l$):

$$\nu_i = -r_i/(l-2),$$

$$\mu_i = X_i(\alpha^2)/2,$$

$$\tau_i = \left(X_iX_i(\alpha^2) - \nabla_{X_i}X_i(\alpha^2)\right)/2.$$

Let us consider the intrinsic problem at first.

Proposition 2. *Let $N^l \subset M^n$, $l \geq 3$, be a totally umbilical non-totally geodesic submanifold in symmetric space. Then*:

1) *the induced metric is conformally flat and analytic. Namely, for any point $Q \in N$ there exists a neighbourhood $U \ni Q$, $U \subset N$ and the local coordinate system $\{x^i\}_{i=1}^{l}$ in U such that the metric of U is*

$$ds^2 = \psi^{-1}\left(\sum_{i=1}^{l}(dx^i)^2\right),$$

where

$$\psi = A\left(\sum_{i=1}^{l}(x^i)^2\right)^2 + \sum_{r=1}^{l}B_rx^r\left(\sum_{i=1}^{l}(x^i)^2\right) +$$

$$+ \sum_{i,r=1}^{l}D_{ir}x^ix^r + \sum_{r=1}^{l}T_rx^r + C,$$

A, B_r, $D_{ir} = D_{ri}$, T_r, and C are constants, $\psi_{|U} > 0$; the mean curvature is of the form

$$\alpha^2 = -4A\sum_{i=1}^{l}(x^i)^2 - 2\sum_{i=1}^{l}B_ix^i + \psi^{-1}\sum_{i=1}^{l}(\partial\psi/\partial x^i)^2 + C_1,$$

$C_1 = \text{const}$;

2) *if all the principal Ricci curvatures except one are equal ($\nu_1 \neq \nu_2 = \ldots = \nu_l$), then N is isometric either to the*

a) *the warped (not direct) product of the line and the space form in the case of $\nu_1 \not\equiv 0$, or to*

b) *the direct product of the line and the non-Euclidean space form in the case of $\nu_1 \equiv 0$.*

It follows from the first statement of the Proposition that the functions α and $\{\nu_i\}_{i=1}^{l}$ are analytic (and the general points in Theorem 2 are "general indeed"); the fields $\{x_i\}_{i=1}^{l}$ and the functions $\{\mu_i, \tau_i\}_{i=1}^{l}$ are piecewise analytic.

Let us consider now the extrinsic problem. By virtue of the Cartan theorem [8, Ch. IV, Th. 7.2], a totally geodesic submanifold of a symmetric space is the exponent of the Lie triple system in the tangent space. Thus, the problem reduces to a purely algebraic one. Namely, let $Q \in M^n$, $T_Q N^l, H_Q \subset T_Q M^n$ be the tangent space and the mean curvature vector of a certain umbilical submanifold $N \subset M$. We are to determine the minimal subspace $s \subset T_Q M$ that contains $T_Q N^l \oplus H_Q$ and is the Lie triple system, i.e. $[[s,s],s] \subset s$ (here $[,]$ is the commutator in the Lie algebra tangent to the Lie group of isometries of M at the point Q).

To find s we need Proposition 3 stated below.

Remind that the curvature tensor $\widetilde{R}$ of symmetric space M is $\widetilde{R}(X,Y)Z = -[[X,Y],Z]$ for X, Y, Z tangent to M [8, Ch. IV, Th. 4.2].

Proposition 3. *Let $N^l \subset M^n$, $l \geq 3$, be a totally umbilical non-totally geodesic submanifold in symmetric space. Then the following equations hold.*

Extrinsic:

$$\left[[X_i, X_j], X_k\right] = 0, \quad i \neq j \neq k, \tag{2.1}$$

$$\left[[X_i, X_j], X_j\right] = \left(\nu_i + \nu_j - \sum_{k=1}^{l} \nu_k/(l-1) + \alpha^2\right) X_i - D_{X_i} H, \quad i \neq j, \tag{2.2}$$

$$\left[[X_i, X_j], X_j\right] - \left[[X_i, X_k], X_k\right] = \left(\nu_j - \nu_k\right) X_i, \quad i \neq j \neq k, \tag{2.3}$$

$$\left[[H, X_i], X_j\right] = \mu_j X_i, \quad i \neq j, \tag{2.4}$$

$$\left[[H, X_i], X_j\right] - \left[[H, X_j], X_j\right] = \mu_i X_i - \mu_j X_j + \left(\nu_i - \nu_j\right) H, \quad i \neq j, \tag{2.5}$$

$$\left[[X_i, H], H\right] - \alpha^2 \left[[X_i, X_j], X_j\right] = \mu_i H - \tau_j X_i - \left[[X_i, D_{X_j} H], X_j\right], \quad i \neq j. \tag{2.6}$$

Intrinsic:

$$g\left(\nabla_{X_i} X_i, X_j\right)(\nu_i - \nu_j) = -\mu_j, \quad i \neq j \neq k, \tag{2.7}$$

$$X_i(\nu_i) = -4(l-1)/(l-2)\mu_i, \tag{2.8}$$

$$X_i(\nu_j) = -2l/(l-2)\mu_i, \quad i \neq j, \tag{2.9}$$

$$\nu_j\left(\sum_{k=1}^{l} \nu_k/(l-1) - \nu_j\right) - \tau_j = \zeta, \tag{2.10}$$

where ζ is a function on N.

Intrinsic equations are "intrinsic indeed" if we suppose α to be some a priori defined function on N.

Proofs of the Propositions are given in Chapter 3 (except equation (2.10) which is proved in Lemma 4.6 and has not been used earlier).

Further in Chapter 4 we solve the extrinsic problem by constructing the subspace s. Roughly speaking its construction is inductive (Lemmas 4.1, 4.2): first we take the vectors $\{X_i\}_i$ and then we add successively the vectors $\{[[X_i, X_j], X_j]\}_{i,j}$, $\{[[[[X_i, X_j], X_j], X_k], X_k]\}_{i,j,k}$ and so on.

The span of all these vectors is the minimal Lie triple system s′ containing the tangent space $T_Q N$. By (2.1, 2.3) we will find that s′ has quite a simple structure: namely, the submanifold $M' = \exp \mathfrak{s}' \subset M$ is isometric to the product of the space forms (Lemma 4.3). Now one has to "add" the vector H' to s′. To do this we first replace it by the vector H' which is the orthogonal component of H with respect to s′. Then equations (2.4-2.6) will look like (2.1, 2.3) (Lemmas 4.4, 4.6). We get the system s by inductively constructing various recurrent Lie triple brackets containing H' and the vectors of s′ (Lemmas 4.5, 4.7). Lemma 4.8 establishes that $\widetilde{M} = \exp \mathfrak{s}$ is isometric to the product of the space forms. Further it will be shown that $\widetilde{M}$ belongs to one of the mentioned types: A (Lemma 4.11), B (Lemmas 4.12-4.14) or C (Lemmas 4.15-4.17). At the end of Chapter 4 we gather all the results received in Lemma 4.18 which is the algebraic base for Theorem 2.

The intrinsic problem for type A and C submanifolds is solved in Chapter 4 in Lemmas 4.11 and 4.16, respectively; the intrinsic problem for type B is solved by Proposition 2.

Thus, Theorem 1 will be completely proved.

In virtue of case 1) of Proposition 1 to prove Theorem 2 it is enough to verify whether the intersection of $\widetilde{M} \subset E$ and the osculating space to $N \subset E$ is totally umbilical in $\widetilde{M}$. For this purpose we shall use the equations of Lemma 4.18. Theorem 2 is proved in Chapter 5; the explicit description of the submanifolds $N \subset \widetilde{M}$ is given.

Case 1) is trivial. Cases 2a) (Lemmas 5.1, 5.2) and 2b) (Lemmas 5.3, 5.4) are investigated by similar methods. It turns out that the equations of Proposition 3 are not only necessary but also sufficient conditions of existence of the totally umbilical submanifold of dimension ≥ 3 in symmetric space.

3. Proofs of the Propositions

Proof of Proposition 1. 1) Suppose that $N^l \subset M^n$ is a totally umbilical submanifold passing through the point Q with the tangent space $T_Q N = L$ and the mean curvature vector $H_Q = H^*$. Fix the unit vector $X_Q \in T_Q N^l$ and choose the orthonormal frame $X_Q, Y_Q, Y_{1|Q}, \ldots, Y_{l-2|Q}$ in L. Let γ be the geodesic of N passing through Q in the direction of X_Q and X, its unit tangent vector. Translate the vectors $Y_Q, Y_{1|Q}, \ldots, Y_{l-2|Q}$ parallel to γ (according to connection ∇ on N) and denote the corresponding vector fields by $Y, Y_1, \ldots, Y_{l-2}$. They are tangent to N, and $\{X, Y, Y_1, \ldots, Y_{l-2}\}$ is the orthonormal frame. We have $\nabla_X X = 0$, $\nabla_X Y = 0$, $\nabla_X Y_i = 0$, $i = 1, \ldots, l-2$. Then $\widetilde{\nabla}_X X = H$, $\widetilde{\nabla}_X Y = \widetilde{\nabla}_X Y_i = 0$; (1.2) yields $\widetilde{\nabla}_X H = -A_H X + D_X H = \alpha^2 X + D_X H$. By the Codazzi equation (1.4) we obtain $\widetilde{R}(X,Y)Y = D_X H + \sum_{i=1}^{i-2} \widetilde{R}(X,Y,Y,Y_i)Y_i + \widetilde{R}(X,Y,Y,X)X$.

Thus we get an ordinary differential equation system

$$\begin{cases} \widetilde{\nabla}_X X = H\,, \\ \widetilde{\nabla}_X Y = 0\,, \\ \widetilde{\nabla}_X Y_i = 0\,, \qquad i = 1,\ldots,l-2\,, \\ \widetilde{\nabla}_X H = -\,g(H,H)X + \widetilde{R}(X,Y)Y - \widetilde{R}(X,Y,Y,X)X - \sum_{i=1}^{l-2} \widetilde{R}(X,Y,Y,Y_i)Y_i \end{cases} \tag{3.1}$$

with the initial values

$$\begin{cases} X(Q) = X_Q, \quad Y(Q) = Y_Q, \quad Y_i(Q) = Y_{i|Q}, \quad i = 1,\ldots,l-2\,, \\ H(Q) = \widetilde{\nabla}_X X(Q) = H^*\,. \end{cases}$$

This Cauchy problem can be uniquely solved [14], i.e. there exists a unique curve γ in M passing through Q in the direction X_Q and satisfying (3.1). It is important to note that the system (3.1) and the initial values "do not contain" N. On the other hand, every umbilical submanifold $N \subset M$ satisfying the condition has to pass through γ (and what is more, γ is geodesic in N).

If we choose a vector X_Q to be an arbitrary unit vector in L, we will get the set of all geodesics of N passing through Q. Thus N is unique (if it exists of course).

2) To prove this we shall establish that any geodesic γ of N passing through Q lies in $\widetilde{M}$. Let ∇' be the induced connection on $\widetilde{M}$; R' its curvature tensor. Construct the curve $\gamma' \subset \widetilde{M}$ with unit tangent vector field X' and vector fields Y', Y_1',..., Y_{l-2}', H' along it such that $\{X', Y', Y_i'\}_i$ form an orthogonal frame, H' is orthogonal to it and the following system holds:

$$\begin{cases} \nabla'_{X'} X' = H'\,, \\ \nabla'_{X'} Y' = 0, \\ \nabla'_{X'} Y_i' = 0, \qquad i = 1,\ldots,l-2, \\ \nabla'_{X'} H' = -g(H',H')X' - R'(X',Y',Y',X')X' + \\ \quad + R'(X',Y')Y' - \sum_{i=1}^{l-2} R'(X',Y',Y',Y_i')Y_i'. \end{cases} \tag{3.2}$$

We assign to γ' the same initial values as to γ: $Q \in \gamma'$, $X'(Q) = X_Q$, $Y'(Q) = Y_Q$, $Y_i'(Q) = Y_{i|Q}'$, $i = 1,\ldots.l-2$, $H'(Q) = H^*$.

In virtue of the ordinary differential equation theory [14] there exists a curve γ' which is the solution of this Cauchy problem. Notice that $\widetilde{M}$ is totally geodesic in M. Consequently $\widetilde{\nabla}_{X'} X' = \nabla'_{X'} X'$, $\widetilde{\nabla}_{X'} Y' = \nabla'_{X'} Y'$, $\widetilde{\nabla}_{X'} Y_i' = \nabla'_{X'} Y_i'$, $\widetilde{\nabla}_{X'} H' = \nabla'_{X'} H'$ by (1.1, 1.2) and $\widetilde{R}(X',Y')Y' = R'(X',Y')Y'$ by the Gauss and Codazzi equations [1, Ch.II, § 2]. It implies that the fields X', Y', Y_i', $i = 1,\ldots,l-2$, H' and the curve γ' satisfy (3.1) and the initial values of γ. Thus, these curves coincide, i.e. any geodesic γ of N lies in $\widetilde{M}$. Umbilicity of N in $\widetilde{M}$ is obvious by the Gauss formulae. So N is a totally umbilical

submanifold in any geodesic submanifold $\widetilde{M}$ tangent to $L \oplus H^*$ at Q. By mutual intersection of all these submanifolds we will guarantee the fullness of N.

Q. E. D.

Proof of Proposition 2. 1) Symmetricity of M yields that for any fields U, X, Y, Z, and T tangent to N the following holds:

$$(\widetilde{\nabla}_U \widetilde{R})(X,Y,Z,T) = 0.$$

Using the Gauss–Codazzi equations (1.3, 1.4), we get:

$$\begin{aligned}(\nabla_U R)(X,Y,Z,T) = {} & U(\alpha^2)(g(X,T)g(Y,Z) - g(X,Z)g(Y,T)) + \\ & + X(\alpha^2)(g(U,T)g(Y,Z) - g(U,Z)g(Y,T))/2 + \\ & + Y(\alpha^2)(g(X,T)g(U,Z) - g(U,T)g(X,Z))/2 + \\ & + Z(\alpha^2)(g(U,Y)g(T,X) - g(U,X)g(T,Y))/2 + \\ & + T(\alpha^2)(g(U,X)g(Z,Y) - g(U,Y)g(Z,X))/2.\end{aligned}$$

In particular,

$$(\nabla_U R)(X,Y,Z,T) = 0. \tag{3.3}$$

$$(\nabla_U R)(X,Y,X,T) = 0 \tag{3.4}$$

$$(\nabla_X R)(X,Y,X,T) = 0 \tag{3.5}$$

$$(\nabla_X R)(X,Y,Z,Y) = -Z(\alpha^2)\|X\|\|Y\|/2, \tag{3.6}$$

$$(\nabla_X R)(X,Y,X,Y) = -2X(\alpha^2)\|X\|\|Y\|, \tag{3.7}$$

where the fields X, Y, T, Z, and U are mutually orthogonal.

Now we shall prove the conformal flatness of N. As it has been mentioned above, the proof is needed only for the case of $l = 3$. One is to verify [1, Ch. 1, § 5] that for arbitrary vector fields X, Y and Z are tangent to N

$$(\nabla_Z \mathrm{Ric})(X,Y) - Z(\mathrm{scal})g(X,Y)/4 = (\nabla_X \mathrm{Ric})(Z,Y) - X(\mathrm{scal})g(Z,Y)/4,$$

where Ric and scal denote the Ricci tensor and the scalar curvature of N^3, respectively.

Consider the point $Q \in N^3$ and the normal geodesic coordinates $\{x^i\}_{i=1}^3$ centred at Q. It is enough to verify the above equation for the fields $\{\partial/\partial x^i\}_{i=1}^3$. Without loss of generality put $Z = \partial/\partial x^1$, $X = \partial/\partial x^2$ and $Y = \partial/\partial x^i$, $i = 2$ or 3.

We have to verify that at the point Q,

$$\begin{aligned}(\nabla_1 R)_{121i} + (\nabla_1 R)_{323i} - \delta_{2i}\sum\nolimits_{r,q=1}^{3}(\nabla_1 R)_{rqrq} = \\ (\nabla_2 R)_{212i} + (\nabla_1 R)_{313i} - \delta_{1i}\sum\nolimits_{r,q=1}^{3}(\nabla_1 R)_{rqrq},\end{aligned}$$

where

$$(\nabla_i R)_{jkhm} = \left(\nabla_{\partial/\partial x^i} R\right)(\partial/\partial x^j, \partial/\partial x^k, \partial/\partial x^h, \partial/\partial x^m)_{|Q}$$

and

$$R_{rqrq} = R(\partial/\partial x^r, \partial/\partial x^q, \partial/\partial x^r, \partial/\partial x^q)_{|Q}.$$

Now in the case of $i = 3$ both sides of the above equality vanish (by (3.5)). In the case of $i = 2$ we have by (3.6, 3.7)

$$(-1/2)\partial(\alpha^2)/\partial x^1 = (-1/2)\partial(\alpha^2)/\partial x^1.$$

Thus N is conformally flat. Now for any point $Q \in N$ the neighborhood $U \ni Q$ can be chosen in which the conformal coordinate system $\{x^i\}_{i=1}^l$ exists. Let $ds^2 = \psi^{-1}(x^1,\ldots,x^l)\sum_{i=1}^l (dx^i)^2$ be the arc length of the metric on N. The conditions (3.3–3.7) are equivalent to the system of the partial differential equations

$$\begin{cases} \psi_{ijk} = 0, & i,j,k \text{ are distinct}, \\ \left(\psi_{ii} + \alpha^2 - \psi^{-1}\sum_{r=1}^l (\psi_r)^2/4\right)_k = 0, & i \neq k, \\ \left(\psi_{ii}/3 + \alpha^2 - \psi^{-1}\sum_{r=1}^l (\psi_r)^2/4\right)_i = 0, \end{cases}$$

where the subscripts denote the partial derivatives.

It can be easily proved by direct integration that the functions ψ and α are of the form stated in the Proposition.

Let be ψ and α are analytic. In view of this the functions $\{\nu_i\}_{i=1}^l$ are analytic too (because of the analyticity of the Ricci tensor [9, Ch. 7]). In particular, the set of the general points on N is an open dense set.

2) First we make some simplification. Obviously the conformal changes of the coordinates $\{x^i\}$ (i.e. the inversions and the similarity transformations) preserve the form of the metric. In particular, under the inversion $x'^i = x^i\left(\sum_{k=1}^l (x^k)^2\right)^{-1}$ we get the form ds'^2 with $A' = C$, $B'_i = T_i$, $D'_{ir} = D_{ir}$, $T'_i = B_i$, $C' = A$, $i,r = 1,\ldots,l$. Moreover, under the translation $x^i \to x^i + \varepsilon^i$ one can regard the coefficient C to be nonzero. Therefore, we may assume $A \neq 0$ without loss of generality. Now the translation $x'^i = x^i - B/(2A)$ annihilates the cubic terms of ψ and the rotation $x''^i = U^i_j x'^j$ $\left(U = \left(U^i_j\right)_{i,j=1}^l \in O(l)\right)$ diagonalizes the quadratic terms. Preserving the original notation, we get

$$\psi = A\left(\sum_{i=1}^l (x^i)^2\right)^2 + \sum_{i=1}^l D_i (x^i)^2 + \sum_{i=1}^l T_i x^i + C.$$

Under the condition the Ricci tensor matrix must be the sum of the scalar matrix and the rank one matrix. Hence,

$$\psi = A\left(\sum_{i=1}^l (x^i)^2\right)^2 + D\sum_{i=1}^l (x^i)^2 + p(x^1), \tag{3.8}$$

where $p(x^1) = D'(x^1)^2 + Tx^1 + C$ and

$$(4AC - D^2)D' = AT^2. \tag{3.9}$$

In the case of $D' = 0$ we get $T = 0$. To exclude the spaces of the constant curvature ($\nu_1 = \nu_2 = \ldots = \nu_l$) we require $4AC \neq D^2$. Then, by (3.9),

$$ds^2 = \sum_{i=1}^{l} (dx^i)^2 / \left(A \left(\sum_{i=1}^{l} (x^i)^2 \right)^2 + D \sum_{i=1}^{l} (x^i)^2 + C \right).$$

Introducing the spherical coordinate system $\{r, \varphi^2, \ldots, \varphi^l\}$, we obtain $ds^2 = (dr^2 + r^2 ds^2_{l-1})/(Ar^4 + Dr^2 + C)$, where ds^2_{l-1} is the unit $(l-1)$-dimensional sphere arc length. Reparametrizing $dt = dr(Ar^4 + Dr^2 + C)^{-1/2}$, we get $ds^2 = dt^2 + f(t) ds^2_{l-1}$. Thus N is isometric to the warped product $E^1 \times^f S^{l-1}$. It can be seen that $\nu_1 \neq 0$.

Now assume $D' \neq 0$. Introduce the local coordinate system $\{x, r, \varphi^3, \ldots, \varphi^l\}$ such that

$$\begin{cases} x^1 = x, \\ x^2 = r \cos \varphi^3, \\ x^3 = r \sin \varphi^3 \cos \varphi^4, \\ \cdots \\ x^{l-1} = r \sin \varphi^3 \ldots \sin \varphi^{l-1} \cos \varphi^l, \\ x^l = r \sin \varphi^3 \ldots \sin \varphi^{l-1} \sin \varphi^l. \end{cases}$$

Then $ds^2 = (dx^2 + dr^2 + r^2 ds^2_{l-2})/(A(x^2 + r^2)^2 + D(x^2 + r^2) + p(x))$, where ds^2_{l-2} is the unit $(l-2)$-dimensional sphere arc length. The principal Ricci direction corresponding to the "exclusive" principal curvature $r_1 = -(l-2)\nu_1$ is collinear to the vector

$$\left(x - (2A(x^2 + r^2) + D)D'/(2Ap') \right) \partial/\partial x + r \partial/\partial r,$$

where $p' = dp/dt = 2D'x + T$. Introduce the coordinates $\{u, t, \varphi^3, \ldots, \varphi^l\}$ such that $\partial/\partial t$ is collinear to the $\left(x - (2A(x^2 + r^2) + D)D'/(2Ap') \right) \partial/\partial x + r \partial/\partial r$, and $\partial/\partial u \perp \partial/\partial t$. We can choose

$$t = (x^2 + r^2 + D/(2A))/p',$$

$$u = (x^2 + r^2 - D/(2A) + xT/D')/r.$$

By direct calculation, we get

$$ds^2 = F_1(t)/(4F_2(t)) dt^2 + F_1(t) F_2(t)/D'^2 (F_3(u)^{-2} du^2 + F_3(u)^{-1} ds^2_{l-2}),$$

where

$$F_1(t) = (At^2 + 1/(4D'))^{-1},$$

$$F_2(t) = t^2 D'^2 + tT - D/(2A),$$

$$F_3(u) = u^2 + T^2/D'^2 + 2D/A.$$

The metric $ds^2_{l-1} = F_3(u)^{-2} du^2 + F_3(u)^{-1} ds^2_{l-2}$ is of the constant curvature $T^2/D'^2 + 2D/A$. Under the reparametrization $d\tau = (F_1/4F_2)^{1/2} dt$, we get

$ds^2 = d\tau^2 + F(\tau)ds^2_{l-1}$. The constancy of F is equivalent to $T = 0$, $D' = -2D$ and therefore $\nu_1 = 0$.

Q. E. D.

Proof of Proposition 3. It is known [10] that the submanifold N^l of dimension $l \geq 4$ is conformally flat. Thus the Weyl conformal tensor vanishes identically [1, Ch. 1, §5]. By the Gauss and Codazzi equations (1.3, 1.4) we obtain

$$\tilde{R}(X_i, X_j)X_k = 0, \quad i \neq j \neq k, \tag{3.10}$$

$$\tilde{R}(X_i, X_j)X_j = \left(-\nu_i - \nu_j + \sum_{k=1}^{l} \nu_k/(l-1) - \alpha^2\right)X_i + D_{X_i}H. \tag{3.11}$$

Equation (3.11) yields

$$\tilde{R}(X_i, X_j)X_j - \tilde{R}(X_i, X_k)X_k = (\nu_k - \nu_j)X_i, \quad i \neq j \neq k. \tag{3.12}$$

Formulae (3.10–3.12) are equivalent to (2.1, 2.3), respectively. Differentiating (3.10) by X_i according to $\tilde{\nabla}$ we get, using (3.10–3.12, 1.1),

$$\tilde{R}(H, X_j)X_k = g(\nabla_{X_i}X_i, X_k)(\nu_i - \nu_k)X_j + g(\nabla_{X_i}X_j, X_k)(\nu_k - \nu_j)X_i.$$

Multiplying this by X_i we obtain that the second term of the right hand side vanishes (by (3.3)) and multiplying by X_j we get (2.7) and the equation

$$\tilde{R}(H, X_j)X_k = -\mu_k X_j, \quad j \neq k, \tag{3.13}$$

which is equivalent to (2.4).

Differentiating (3.12) by X_i and taking into account (3.10-3.12, 1.1, 2.7), we get

$$\tilde{R}(H, X_j)X_j - \tilde{R}(H, X_k)X_k = (X_i(\nu_k - \nu_j))X_i + (\nu_k - \nu_j)H - \mu_j X_j + \mu_k X_k.$$

Multiplying by X_i we get (2.5).

Differentiating (3.13) by X_k and with account for (1.1, 1.2, 2.5, 2.7, 3.13) we obtain (2.6).

All the extrinsic equations have been proved. It remains to verify (2,8, 2.9).

Equation (3.11) yields

$$\tilde{R}(X_i, X_j, X_j, X_i) = -\nu_i - \nu_j + \sum_{k=1}^{l} \nu_k/(l-1) - \alpha^2, \quad i \neq j.$$

Differentiating this by X_i we get

$$X_i(-\nu_i - \nu_j + \sum_{k=1}^{l} \nu_k/(l-1) - 2\alpha^2) = 0.$$

Now (2.5) implies (2.8). Differentiating the same equation by X_k, $k \neq i, j$, we obtain (2.9).

Equation (2.10) will be proved in Lemma 4.6 (and it will not be used before).

Q. E. D.

4. Solution of the Extrinsic Problem. Proof of Theorem 1

The argument of this Chapter is mainly of pure algebraic character (at least the first ten Lemmas). By Proposition 3 one has $l+1$ vectors $\{X_i\}_{i=1}^{l}, H$ at a point $Q \in M$ satisfying (2.1-2.6). Our purpose is to determine the minimal subspace $s \subset T_Q M$ that contains these vectors and is closed under the Lie triple brackets (i.e. curvature–invariant). Roughly speaking, such a subspace is spanned by all "recurrent" brackets of an odd number of the arguments $\{X_i\}_i$ and H. We will construct it in two steps. Firstly we shall get a subspace s' containing $\{X_i\}_i$ and being Lie triple system and then extend it to s "adding" vector H.

Here the following lemma takes place.

Lemma 4.1. *Suppose that in the tangent space* $\mathfrak{m} = T_Q M^n$, $n > 3$, *of the Riemannian symmetric space orthonormal vectors* $\{X_i\}_{i=1}^{l}$, $l \geq 3$, *and* l *numbers* $\{\nu_i\}_{i=1}^{l}$ *satisfying (2.1, 2.3) are given.*

Then there exist l *mutually orthogonal subspaces* $\{L_i\}_{i=1}^{l}$ *such that* $X_i \in L_i$, $i = 1,2,\ldots,l$, *and*

$$\begin{aligned} &[[L_i, L_j], L_k] = 0, \\ &[[L_i, L_j], L_j] \subset L_i, \\ &[[L_i, L_i], L_j] = 0, \\ &[[L_i, L_i], L_i] = 0, \end{aligned} \tag{4.1}$$

for i, j, k *distinct. The direct sum* $\otimes_{i=1}^{l} L_i$ *is the minimal Lie triple system containing the vectors* $\{X_i\}_{i=1}^{l}$.

To prove this Lemma we shall use Lemma 4.2. To state it we consider l operators R_i: $\mathfrak{m} \to \mathfrak{m}$ defined as follows: $R_i X = [[X, X_i], X_i] - \nu_i X, \quad i = 1,2,\ldots,l.$

By equation (2.3) $R_j X_i = R_k X_i$ with i, j, k distinct. Thus we obtain the well-defined operator R acting on the vectors $\{X_i\}_i$ (and on their span). By definition put $RX_i = R_j X_i$ for an arbitrary $j \neq i$ and extend R on the Span (X_i) by linearity.

The idea of Lemma 4.2 is the following. It turns out that not only $R_j X_i = R_k X_i$, (i, j, k are distinct), but also $R_j(RX_i) = R_k(RX_i)$ (for arbitrary distinct i, j, k). Hence, one can define the operator R on the vectors $\{RX_i\}_{i=1}^{l}$ by $R^2 X_i = R(RX_i) = R_j(RX_i)$ for arbitrary $j \neq i$. Moreover, the two-dimensional spaces Span (X_i, RX_i) are mutually orthogonal, therefore one can extend the action of R on the subspace Span $(X_1, X_2, \ldots, X_l, RX_1, \ldots, RX_l)$ by linearity. Further, we obtain that $R_j(R^2 X_i) = R_k(R^2 X_i)$ with i, j, k distrinct, i.e. one can define $R^3 X_i$, $i = 1,\ldots,l$.

The three-dimensional subspaces Span $(X_i, RX_i, R^2 X_i)$, $i = 1,\ldots,l$, are again mutually orthogonal and the operator R can be extended on their direct sum. We will proceed in

this way until every of l spaces Span (X_i, RX_i, R^2X_i), $i = 1,\ldots,l$ is invariant under the action of R. These spaces are just the spaces $\{L_i\}_{i=1}^{l}$.

The principal role in Lemma 4.2 below is played by statement 1) which is (at the same time) the inductive definition and the verification of its correctness. Let R^0 be equal to the identical operator I on the domain of definition.

Lemma 4.2. *In the conditions of Lemma 4.1 the following statements are true for every* $q \geq 0$:

1) *Suppose the vectors* R^sX_i *to be already defined (for* $i = 1,\ldots,l$ *and every* $0 \leq s \leq q$*).*

Then for every $1 \leq i \leq l$ *and for each* $j \neq i$ *and arbitrary integers* $0 \leq s, t, r$, *satisfying* $s + t + r = q$, *all vectors*

$$\left[[R^sX_i, R^tX_j], R^rX_j\right] - \nu_j R^qX_i$$

are equal (in particular, taking $s = q$, $t = r = 0$, *we get* $R_j(R^qX_i)$*). We shall denote by* $R^{q+1}X_i$ *their common value; it is well defined.*

2) *Let us denote* $L_i^q = \text{Span}\,(X_i, RX_i, \ldots, R^qX_i)$ *for every* $i = 1,\ldots,l$. *The* l *spaces* $\{L_i^q\}_{i=1}^{l}$ *are mutually orthogonal, therefore one can extend the action of* R *to* $\oplus_{i=1}^{l} L_i^q$ *by linearity.*

3) $\left[[R^sX_i, R^tX_j], R^rX_k\right] = 0$ *for* i, j, k *distinct and arbitrary* $s, t, r \geq 0$ *with* $s + t + r = q$.

4) $\left[[R^sX_i, R^tX_i], R^rX_j\right] = 0$ *for* $i \neq j$ *and arbitrary* $s, t, r \geq 0$ *with* $s + t + r = q$.

5) $\left[[R^sX_i, R^tX_i], R^rX_i\right] = 0$ *for arbitrary* $s, t, r \geq 0$ *with* $s + t + r = q$.

Proof is carried out by induction on q. The induction base ($q = 0$) is trivial. In fact, (2.3) yields $1)_0$, (2.1) yields $3)_0$, $2)_0$ is true by the condition and $4)_0$, $5)_0$ are obtained by virtue of the Lie brackets skew symmetry.

Now we shall carry out the induction step. Suppose 1)–5) to be true for $0 \leq q \leq m - 1$. Let $q = m \geq 1$.

Before proving statements $1)_m$–$5)_m$, we shall obtain a useful equation. Let X be an arbitrary vector in m, $s > 0$, $t \geq 0$ be integers with $s + t \leq m$. Then, for $i \neq j$ by $1)_{s-1}$, $1)_t$, $3)_{s+t-1}$ and Jacobi identity

$$\left[[R^sX_i, R^tX_j], X\right] = \left[[[R^{s-1}X_i, X_k], X_k] - \nu_k R^{s-1}X_i, R^tX_j], X\right] =$$

$$= -\nu_k\left[[R^{s-1}X_i, R^tX_j], X\right] + \left[[R^{s-1}X_i, [X_k, [X_k, R^tX_j]]], X\right] = \left[[R^{s-1}X_i, R^{t+1}X_j], X\right].$$

Thus, for arbitrary integers s_1, s_2, t_1, $t_2 \geq 0$ with $s_1 + t_1 = s_2 + t_2 \leq m$ the following is true:

$$\left[[R^{s_1}X_1, R^{t_1}X_j],X\right] = \left[[R^{s_2}X_i, R^{t_2}X_j],X\right] = \left[[R^{s_1+t_1}X_i, X_j],X\right], \qquad (4.2)$$
$$i \neq j, \qquad X \in \mathfrak{m}.$$

Now return to the proof of $1)_m - 5)_m$.

The rest of the proof is in fact routine verification of these equations with the help of an inductive hypothesis and the Jacobi identity.

$1)_m$. Fix $1 \leq i \leq l$. From (4.2) it follows immediately that it is sufficient to consider various brackets of the form

$$\left[[R^{m-r}X_i, X_j], R^rX_j\right] - \nu_j R^{m-1}X_j$$

and to prove their equality for all $j \neq i$ and $0 \leq r \leq m$.

Let $j \neq i$. Suppose $r > 0$. Choose $k \neq i,j$ and $0 \leq s < r$. By $1)_{r-1}$, $1)_{m-1}$, $3)_{m-r+s}$, $3)_{m-r}$, $1)_{m-s-1}$, $1)_{m-1}$ and Jacobi identity we obtain:

$$\left[[R^{m-r}X_i, X_j], R^rX_j\right] - \nu_j R^mX_i = -\nu_j R^mX_i +$$
$$+ \left[[R^{m-r}X_i, X_j], [[R^{r-s-1}X_j, X_k], R^sX_k] - \nu_k R^{r-1}X_j\right] = -\nu_j R^mX_i -$$
$$- \nu_k\left(R^mX_i + \nu_j R^{m-1}X_i\right) - \left[R^sX_k, [[R^{m-r}X_i, X_j], [R^{r-s-1}X_j, X_k]]\right] =$$
$$- (\nu_j + \nu_k)R^mX_i - \nu_j\nu_k R^{m-1}X_i + \left[[R^{m-s}X_i + \nu_j R^{m-s-1}X_i, X_k], R^sX_k\right] =$$
$$= \left[[R^{m-s}X_i, X_k], R^sX_k\right] - \nu_k R^mX_i.$$

Thus, every expression of the form $\left[[R^{m-r}X_i, X_j], R^rX_j\right] - \nu_j R^mX_i$ with $j \neq i$ and $0 < r \leq m$ equals every expression $\left[[R^{m-s}X_i, X_k], R^sX_k\right] - \nu_k R^mX_i$ with $k \neq i,j$ and $0 \leq s < r$, therefore the same is true for $0 \leq s < m$. Moreover, all expressions $\left[[R^{m-r}X_i, X_j], R^rX_j\right] - \nu_j R^mX_i$ are equal to one another for $0 < r \leq m$. Replacing j by k we shall get that the same is true for $r = 0$. Since $m \geq 1$, the statement $1)_m$ is proved.

$2)_m$. By the inductive hypothesis we must show that R^mX_i is orthogonal to R^sX_j for $i \neq j$ and $s \leq m$.

By $1)_{m-1}$

$$g\left(R^mX_i, R^sX_j\right) = -g\left([[X_i, X_k], R^sX_j], R^{m-1}X_k\right) - \nu_i g(R^{m-1}X_i, R^sX_j).$$

If $s < m$, then by inductive hypothesis this expression vanishes. Let us consider the case of $s = m$. For the second term we have by $1)_{m-1}$:

$$g\left(R^{m-1}X_i, R^mX_j\right) = -g\left([X_j, X_i], [R^{m-1}X_i, R^{m-1}X_i]\right) - \nu_i g(R^{m-1}X_i, R^{m-1}X_j) = 0.$$

The first term vanishes by $3)_m$. Thus, $2)_m$ follows from $3)_m$.

$3)_m$. Choose $1 \leq i, j, k \leq 1$ to be distinct. Notice that (4.2) yields

$$\left[[R^m X_i, X_j], X_k\right] = -\left[[R^m X_j, X_i], X_k\right]. \tag{4.3}$$

Further, by the inductive hypothesis and the Jacobi identity

$$\left[[R^m X_i, X_j], X_k\right] = \left[[[[X_i, X_k], R^{m-1} X_k], X_j], X_k\right] =$$
$$= \left[[[X_i, X_k], X_k], [R^{m-1} X_k, X_j]\right] + \left[[X_k, [R^{m-1} X_k, X_j]], [X_i, X_k]\right] =$$
$$= \left[RX_i, [R^{m-1} X_k, X_j]\right] + \left[R^m X_j, [X_i, X_k]\right].$$

Thus, by (4.2) and the Jacobi identity

$$\left[[R^m X_i, X_j], X_k\right] + \left[[R^m X_i, X_k], X_j\right] = \left[R^m X_j, [X_i, X_k]\right] +$$
$$+ \left[R^m X_k, [X_i, X_j]\right] = \left[[R^m X_j, X_i], X_k\right] + \left[[R^m X_k, X_i], X_j\right].$$

By virtue of (4.2),

$$\left[[R^m X_i, X_j], X_k\right] = -\left[[R^m X_i, X_k], X_j\right].$$

This equation and (4.3) yield that the expression

$$a_{ijk} = \left[[R^m X_i, X_j], X_k\right]$$

is skew-symmetric by every pair of subscripts.

By the Jacobi identity, $\left[[X_i, X_j], R^m X_k\right] = -2a_{ijk}$. Now $1)_q$, $3)_q$ with $q < m$ and the Jacobi identity imply

$$-2a_{ijk} = \left[[X_i, X_j], [[X_k, X_i], R^{m-1} X_i]\right] =$$
$$= \left[R^{m-1} X_i, [X_k, [X_i, [X_i, X_j]]]\right] - \left[[X_k, X_i], R^m X_j\right] =$$
$$= 2a_{ijk} + \left[[RX_j, X_k], R^{m-1} X_i\right] =$$
$$= 2a_{ijk} + \left[[RX_j, X_k], [[X_i, X_j], R^{m-2} X_j]\right] =$$
$$= 2a_{ijk} + \left[R^{m-2} X_j, [X_i, [X_j, [RX_j, X_k]]]\right] - \left[[X_i, X_j], R^m X_k\right] =$$
$$= 4a_{ijk} + \left[[R^2 X_k, X_i], R^{m-2} X_j\right] = \ldots =$$
$$= 2ma_{ijk} + \left[[R^m X_I, X_J], X_K\right].$$

Here (I,J,K) is the even permutation of the subscripts (i,j,k) which is the m-th power of the permutation $(i,j,k) \to (j,k,i)$.

Because of evenness we get $-2a_{ijk} = 2ma_{ijk} + a_{ijk}$. Thus, $a_{ijk} = 0$. Hence by (4.2) $3)_m$ is proved.

Remind that this implies $2)_m$.

$4)_m$ is the trivial consequence of $1)_m$ and the Jacobi identity.

$5)_m$. First suppose $r > 0$. Fix $1 \le i \le l$ and choose $j \ne i$. Then,

$$\left[[R^s X_i, R^t X_i], R^r X_i\right] = \left[[R^s X_i, R^t X_i], [[X_i, X_j], R^{r-1} X_i]\right] =$$

$$= \left[R^{r-1}X_j, [X_j, [[R^sX_i, R^tX_i], X_i]]\right]$$

by the Jacobi identity and by the inductive hypothesis. Now, if among the integers s, t, r at least two do not equal zero, then the proof is finished by the Jacobi identity and the calculation above. Thus, it is sufficient to prove that $\left[[R^mX_i, X_i], X_i\right] = 0$.

By virtue of $1)_q$, $4)_q$ with $q \le m$ and $5)_q$ with $q < m$ we get for $i \neq j$

$$\left[[R^mX_i, X_i], X_i\right] = \left[[[[R^{m-1}X_i, X_j], X_j], X_i], X_i\right] =$$

$$= \left[[X_i, [X_j, X_i]], [R^{m-1}X_i, X_j]\right] + \left[[[R^{m-1}X_i, X_j], X_i], [X_j, X_i]\right] -$$

$$- \left[[R^mX_j + \nu_i R^{m-1}X_j, X_j], X_i\right] =$$

$$= -\left[RX_i + \nu_j X_i, [R^{m-1}X_i, X_j]\right] - \left[R^mX_j + \nu_i R^{m-1}X_j, [X_j, X_i]\right].$$

This expression vanishes by $1)_q$ $(q = m-1, m)$.

So $5)_m$ is proved.

Q. E. D.

Proof of Lemma 4.1. We define $L_i = \mathrm{Span}\,(X_i, RX_i, R^2X_i, \ldots)$ for $i = 1,2,\ldots,l$. It is clear that for certain q $L_i = L_i^q$, $i = 1,2,\ldots,l$. Obviously $X_i \in L_i$. By 2) of Lemma 4.2, $\{L_i\}_i$ are mutually orthogonal. Hence the operator R can be extended linearly to the subspace $\mathfrak{s}' = \oplus_{i=1}^{l} L_i$. The expressions (4.1) follow immediately from statements 1), 3), 4), and 5) of Lemma 4.2. Thus, $\mathfrak{s}'$ is the Lie triple system. From its construction one can easily see that $\mathfrak{s}'$ is the minimal Lie triple system containing the vectors $\{X_i\}_{i=1}^{l}$.

Q. E. D.

Since $\mathfrak{s}' \subset \mathfrak{m}$ is a Lie triple system, its exponent has to be a totally geodesic submanifold in M.

Lemma 4.3. $\exp \mathfrak{s}'$ *is isometric to the product of space forms.*

Proof is of a pure "pointwise" character. The operator R: $\mathfrak{s}' \to \mathfrak{s}'$ is symmetric according to the scalar product on $\mathfrak{s}'$ induced from that on $\mathfrak{m}$. Let us denote by $\{\mathfrak{s}'_\gamma\}_{\gamma=1}^{B}$ the (mutually orthogonal) eigenspaces and by $\{\xi_\gamma\}_{\gamma=1}^{B}$ the corresponding (distinct) eigenvalues of R. Let

$$X_i = \sum_{\gamma=1}^{B} X_{\gamma|i}$$

be the decomposition of the vector X_i, $i = 1,2,\ldots,l$, by the eigenvectors of R. Here $X_{\gamma|i}$ is the projection of X_i onto $\mathfrak{s}'_\gamma$. Let us denote $\lambda_{\gamma|i} = \|X_{\gamma|i}\| \ge 0$. Since L_i is invariant under the action of R we get $L_i = \mathrm{Span}_{\gamma=1}^{B}(X_{\gamma|i})$. This, in particular, implies orthogonality of any two vectors $X_{\gamma|i}$ and $X_{\beta|j}$ for $i \neq j$ and arbitrary β, γ. Consider the Lie triple brackets of the vectors $\{X_{\gamma|i}\}$. By Lemma 4.1. for arbitrary β, γ, δ

$$\left[[X_{\beta|i}, X_{\gamma|i}], X_{\delta|i}\right] = 0,$$

$$\left[[X_{\beta|i}, X_{\gamma|i}], X_{\delta|j}\right] = 0, \qquad i \neq j, \tag{4.5}$$

$$\left[[X_{\beta|i}, X_{\gamma|j}], X_{\delta|k}\right] = 0, \tag{4.6}$$

where i, j, k are distinct.

Consider the brackets $\left[[X_{\beta|i}, X_{\gamma|j}], X_{\delta|j}\right]$, $i \neq j$. We have, by (4.2, 4.5),

$$\xi_\beta\left[[X_{\beta|i}, X_{\gamma|j}], X_{\delta|j}\right] = \left[[RX_{\beta|i}, X_{\gamma|j}], X_{\delta|j}\right] =$$
$$= \left[[X_{\beta|i}, RX_{\gamma|j}], X_{\delta|j}\right] = \xi_\gamma\left[[X_{\beta|i}, X_{\gamma|j}], X_{\delta|j}\right].$$

Moreover,

$$\xi_\beta\left[[X_{\beta|i}, X_{\gamma|j}], X_{\delta|j}\right] = -\left[[RX_{\beta|i}, X_{\delta|j}], X_{\gamma|j}\right] =$$
$$= -\left[[X_{\beta|i}, RX_{\delta|j}], X_{\gamma|j}\right] = \xi_\delta\left[[X_{\beta|i}, X_{\gamma|j}], X_{\delta|j}\right].$$

Thus, for $i \neq j$

$$\left[[X_{\beta|i}, X_{\gamma|j}], X_{\delta|j}\right] = 0, \tag{4.7}$$

if at least two of subscripts β, γ, δ are distinct.

Finally, in the case of $\beta = \gamma = \delta$, we obtain

$$\left[[X_{\gamma|i}, X_{\gamma|j}], X_{\gamma|j}\right] = \left[[X_{\gamma|i}, X_j], X_j\right] =$$
$$= RX_{\gamma|i} + \nu_j X_{\gamma|i} = (\xi_\gamma + \nu_j)X_{\gamma|i}. \tag{4.8}$$

Hence, by (4.5–4.8) the following is true for the subspaces $\mathfrak{s}'_\gamma$ $(= \operatorname{Span}_{i=1}^{l}(X_{\gamma|i}))$:

$$\left[[\mathfrak{s}'_\beta, \mathfrak{s}'_\gamma], \mathfrak{s}'_\delta\right] = 0, \quad \beta, \gamma, \delta \text{ are distinct},$$
$$\left[[\mathfrak{s}'_\beta, \mathfrak{s}'_\gamma], \mathfrak{s}'_\gamma\right] = 0, \quad \beta \neq \gamma,$$
$$\left[[\mathfrak{s}'_\gamma, \mathfrak{s}'_\gamma], \mathfrak{s}'_\gamma\right] \subset \mathfrak{s}'_\gamma.$$

The Cartan decomposition theorem [8] implies that the submanifold $\exp \mathfrak{s}'$ is the Riemannian product of the submanifolds $M'_\gamma = \exp \mathfrak{s}'_\gamma \subset \exp \mathfrak{s}' \in M$.

Let us consider now each M'_γ and verify that it is isometric to the space form. The one-dimensional case is trivial. The two-dimensional case is also trivial because M'_γ is totally geodesic in M and thus symmetric (in the intrinsic sense).

Let $\dim \mathfrak{s}'_\gamma \geq 3$. Fix three nonzero vectors $X_{\gamma|i}, X_{\gamma|j}, X_{\gamma|k} \in \mathfrak{s}'_\gamma$. In virtue of (4.8),

$$R\left(X_{\gamma|i}, X_{\gamma|j}, X_{\gamma|j}, X_{\gamma|i}\right) = -g\left([[X_{\gamma|i}, X_{\gamma|j}], X_{\gamma|j}], X_{\gamma|i}\right) = -(\xi_\gamma + \nu_j)\lambda^2_{\gamma|i}.$$

This and (4.6) imply

$$\widetilde{R}(X_{\gamma|i}, X_{\gamma|j})X_{\gamma|k} = 0,$$

$$\widetilde{R}\left(X_{\gamma|i}, X_{\gamma|j}, X_{\gamma|j}, X_{\gamma|i}\right) = \left(-(\xi_\gamma + \nu_j)\lambda^{-2}_{\gamma|i}\right)\lambda^2_{\gamma|i}\lambda^2_{\gamma|j} =$$

$$= \left(-(\xi_\gamma + \nu_i)\lambda_{\gamma|i}^{-2}\right)\lambda_{\gamma|i}^2\lambda_{\gamma|j}^2,$$

$$\widetilde{R}\left(X_{\gamma|i}, X_{\gamma|k}, X_{\gamma|k}, X_{\gamma|i}\right) = \left(-(\xi_\gamma + \nu_k)\lambda_{\gamma|k}^{-2}\right)\lambda_{\gamma|i}^2\lambda_{\gamma|k}^2 =$$

$$= \left(-(\xi_\gamma + \nu_i)\lambda_{\gamma|i}^{-2}\right)\lambda_{\gamma|i}^2\lambda_{\gamma|k}^2.$$

Thus, the sectional curvatures of M along all 2-planes $X_{\gamma|i} \wedge X_{\gamma|j}$ $(i \neq j)$ are equal to one another. Let us denote their common value by k_γ. From the above argument,

$$k_\gamma\lambda_{\gamma|k}^2 + \xi_\gamma + \nu_k = 0. \tag{4.9}$$

Moreover,

$$\widetilde{R}\left(X_{\gamma|i}, X_{\gamma|j}, X_{\gamma|k}, X_{\gamma|t}\right) = k_\gamma\left(g(X_{\gamma|i}, X_{\gamma|t})\, g(X_{\gamma|j}, X_{\gamma|k}) - g(X_{\gamma|i}, X_{\gamma|k})\, g(X_{\gamma|j}, X_{\gamma|t})\right)$$

for arbitrary γ, i, j, k, t. Since M'_γ is totally geodesic in M, one can replace the curvature tensor of M by that of M'_γ in the left-hand side of the latter equation. Thus, M'_γ is the space form of the curvature k_γ.

Q. E. D.

We have also obtained important equation (4.9). One can easily see that it is true always (even if $X_{\gamma|k} = 0$) when k_γ is defined, i.e. dim $s'_\gamma > 1$. In the case of dim $s'_\gamma = 1$, we have

$$\xi_\gamma + \nu_i = 0 \tag{4.10}$$

for every i with $L_i \perp s_\gamma$, i.e. for every i except one.

Our purpose now is to "extend" s' to s "adding" the vector H.

The idea of this extension is as follows. At first we replace the vector H by the vector H' which is orthogonal to the s' component of H. It is clear that the minimal Lie triple systems containing $s' \oplus H$ and $s' \oplus H'$ coincide. It will be seen that equations (2.4–2.6) become especially simple like (2.1,2.3). Therefore it will be possible to extend the operator R to H', then to RH', etc. By the method similar to that of Lemmas 4.1 and 4.2, we will obtain the space L_H, which will satisfy together with $\{L_i\}_{i=1}^l$ all relations (4.1) of Lemma 4.1. The space $s = s' \oplus L_H$ is the minimal Lie triple system containing $\oplus_{i=1}^l X_i \oplus H$. Its exponent will be the product of the space forms.

Lemma 4.4. *Let H' be the orthogonal to s' component of vector H, then for $i \neq j$*

$$\left[[H', X_i], X_j\right] = 0, \tag{4.11}$$

$$\left[[H', X_i], X_i\right] - \left[[H', X_j], X_j\right] = (\nu_i - \nu_j)H'. \tag{4.12}$$

P r o o f. In fact the vector H' will be obtained in quite a different way and only then we will show it to be the component of H orthogonal to s'. Let $H = H' + \sum_{i=1}^l P_i(R)X_i$. Here $P_i(R)$ are the polynomials of the operator R, $i = 1,\ldots,l$, i.e. $P_i(R)X_i$ is some vector

of L_i (notice that any vector of L_i can be obtained in this way). Our purpose now is to select the polynomials P_i to satisfy (4.11, 4.12) with H' as above. By Lemma 4.2 and (2.4), we have

$$\begin{aligned}\left[[H', X_i], X_j\right] &= \mu_j X_i - \sum_k \left[[P_k(R)X_k, X_i], X_j\right] = \\ &= \mu_j X_i - \left[[P_j(R)X_j, X_i], X_j\right] = \\ &= \mu_j X_i - P_j(R)\left([[X_j, X_i], X_j]\right) = \\ &= \left(P_j(R)(R + \nu_j I) + \mu_j I\right) X_i\end{aligned}$$

with I is an identical operator.

By Lemma 4.2 and (2.5),

$$\begin{aligned}&\left[[H', X_i], X_i\right] - \left[[H', X_j], X_j\right] = \\ &= \mu_i X_i - \mu_j X_j + (\nu_i - \nu_j)H' + (\nu_i - \nu_j)\sum_k P_k(R)X_k - \sum_k \left(\left[[P_k(R)X_k, X_i], X_i\right] - \right. \\ &\left. - \left[[P_k(R)X_k, X_j], X_j\right]\right) = (\nu_i - \nu_j)H' + \sum_{k \neq i,j}\left((\nu_i - \nu_j)P_k(R) + \right. \\ &\left. + P_k(R)(R + \nu_i I) + P_k(R)(R + \nu_j I)\right)X_k + \left(\mu_i I + (\nu_i - \nu_j)P_i(R) + \right. \\ &\left. + P_i(R)(R + \nu_j I)\right)X_i - \left(\mu_j I + (\nu_j - \nu_i)P_j(R) + P_j(R)(R + \nu_i I)\right)X_j = \\ &= (\nu_i - \nu_j)H' + \left(P_i(R)(R + \nu_i I) + \mu_i I\right)X_i - \left(P_j(R)(R + \nu_j I) + \mu_j I\right)X_j.\end{aligned}$$

Thus, to satisfy (4.11,4.12), it is necessary and sufficient to find l polynomials P_i, $i = 1,\ldots,l$, such that all operators $P_i(R)(R + \nu_i I) + \mu_i I$ vanish on the vectors $\{x_j\}_{j=1}^{l}$ (this implies that the operators $P_i(R)(R + \nu_i I) + \mu_i I$ vanish identically on s'). We shall proceed as follows. Let $\chi(\xi) = \Pi_{\gamma=1}^{B}(\xi - \xi_\gamma)$ be the minimal polynomial of the operator R: $s' \to s'$. Let $\chi_i(\xi)$ and a_i denote the quotient and the remainder of the division of $\chi(\xi)$ by $(\xi + \nu_i)$, respectively.

First suppose that $a_i \neq 0$. Put $P_i = \mu_i a_i^{-1}\chi_i$. Then,

$$P_i(R)(R + \nu_i I) + \mu_i I = \mu_i a_i^{-1}\chi(R) = 0$$

according to the Hamilton-Cayley theorem [9, Ch. IV, §§ 3, 4].

Now let us consider the case of $a_i = 0$. This means that there exists $1 \leq \gamma \leq B$ such that $-\xi_\gamma = \nu_i$ (for a given i). We shall show that in this case $\mu_i = 0$, i.e. one can take $P_i = 0$.

If $\dim s'_\gamma = 1$, then by (4.10) there exist $l - 1 \geq 2$ of $\{\nu_j\}_j$ with $\xi_\gamma + \nu_j = 0$, i.e. there exist $j \neq i$ such that $\nu_j = \nu_i$. Then (2.7) implies $\mu_i = 0$. If $\dim s'_\gamma > 1$, then by (4.9) $k_\gamma \lambda_{\gamma|i} = 0$. Now if $\lambda_{\gamma|i} \neq 0$, then $k_\gamma = 0$ and (4.9) implies $\xi_\gamma + \nu_j = 0$ for arbitrary $j = i,\ldots,l$. Therefore, $\mu_i = 0$ by (2.7). If $\lambda_{\gamma|j} = 0$, then find j with $\lambda_{\gamma|j} \neq 0$ and multiply both sides of (2.4) by $X_{\gamma|j}$. The equations of Lemma 4.3 yield

$$\mu_i \lambda^2_{\gamma|j} = \mu_i g(X_j, X_{\gamma|j}) = g\left([[H, X_j], X_i], X_{\gamma|j}\right) =$$

$$= g\Big([[X_{\gamma|j}, X_i], X_j], H\Big) = 0.$$

Hence, $\mu_i = 0$.

Thus, (4.11, 4.12) are satisfield. Now let us prove the vector $H' = H - \sum_{i=1}^{l} P_i(R)X_i$ to be orthogonal to s' .

At first we notice that for $X_i^* \in L_i$ orthogonality of H' and X_i^* implies orthogonality of H' and RX_i^* . Really, multiplying (4.12) by X_i^* and taking into account 1) and 5) of Lemma 4.2 we obtain

$$0 = g((\nu_i - \nu_j)H', X_i^*) = g\Big([[H', X_i], X_i] - [[H', X_j], X_j], X_i^*\Big) =$$

$$= -g\Big([[X_i^*, X_j], X_j], H'\Big) = -g(RX_i^* + \nu_j X_i^*, H') = -g(RX_i^*, H').$$

Hence, it is sufficient to verify that H' is orthogonal to X_i for every $i = 1,2,\ldots,l$. Notice that $g(H', X_i) = g\Big(H - \sum_{j=1}^{l} P_j(R)X_j, X_i\Big) = -g(P_i(R)X_i, X_i)$, since L_i and L_j are orthogonal $(i \neq j)$. If $\mu_j = 0$, then $P_i = 0$ and therefore $g(H', X_i) = 0$. Let $\mu_i \neq 0$. Multiplying (4.12) by X_i, we obtain

$$-g\Big([[H', X_j], X_j], X_i\Big) = g((\nu_i - \nu_j)H', X_i).$$

Thus, $g\Big(RX_i + \nu_i X_i, H'\Big) = 0$. By the above remark $g\Big(R^q(R + \nu_i I)X_i, H'\Big) = 0$ for any $q > 0$. Hence $g\Big(P_i(R)(R + \nu_i I)X_i, H'\Big) = 0$, and therefore $(-\mu_i X_i)$ is orthogonal to H' . This yields $g(X_i, H') = 0$.

Q. E. D.

In the case of $H' = 0$ the system s has been already constructed: it coincides with s'. Suppose that $H' \neq 0$. It should be interesting, of course, to get the equality like $[[X_i, H'], H'] - [[X_i, X_j], X_j] = (\nu_H - \nu_j)X_i$, where ν_H is a certain number similar to the implication (2.4, 2.5) → (2.8, 2.9). If it were so, we should find ourselves in fact in the conditions of Lemmas 4.1 and 4.2. Unfortunately it is impossible. Therefore we will have to prove the Lemmas "repeating" Lemmas 4.1 and 4.2 but with participation of H' .

By (4.12), for arbitrary $i \neq j$, $R_i H' = R_j H'$. Thus, the action of the operator R is well defined on the vector H' , namely, $RH' = R_i H'$ for arbitrary i.

The following Lemma is similar to Lemma 4.2.

Lemma 4.5. 1) *Suppose that for some $q \geq 0$ the vectors $R^s H'$, $0 \leq s \leq q$, have been already defined. Then for any $1 \leq i \leq l$ and for arbitrary integers s, t, $r \geq 0$ with $s + t + r = q$ all the vectors*

$$\Big[[R^s H', R^t X_i], R^r X_i\Big] - \nu_i R^q H'$$

are equal to one another (in particular, in the case of $s = q$, $t = r = 0$ we will get $R_i(R^q H')$). We denote their common value by $R^{q+1} H'$: such a definition is correct.

2) *The space $L_H{}^q = \mathrm{Span}(H', RH', \ldots, R^q H')$ is orthogonal to s'.*

3) $\Big[[R^s H', R^t X_i], R^r X_j\Big] = 0$ *for $i \neq j$ and arbitrary integers $s, t, r \geq 0$ with $s + t + r = q$.*

4) $\left[[R^s X_i, R^t X_i], R^r H'\right] = 0$ *under the same conditions.*

Proof is carried out by induction on q. The base follows immediately from the previous Lemma.

We have to verify the inductive step. Suppose 1)–4) to be proved for $0 \le q \le m-1$. Similarly to (4.2) we get the equation

$$\left[[R^s H', R^t X_i], X\right] = \left[[R^r H', R^u X_i], X\right] \tag{4.13}$$

for arbitrary $X \in \mathfrak{m}$ and $s + t = r + u \le m$.

$1)_m$. Equation (4.13) yields that it is enough to prove this proposition only for the brackets

$$\left[[R^{m-r} H', X_i], R^r X_i\right] - \nu_i R^m H' .$$

Repeating in our case the calculations of $1)_m$ (Lemma 4.2), we obtain that for $i \ne j$ and $m \ge r > s \ge 0$

$$\left[[R^{m-r} H', X_i], R^r X_i\right] - \nu_i R^m H' = \left[[R^{m-s} H', X_i], R^s X_i\right] - \nu_i R^m H'.$$

This implies, similarly to $1)_m$ of Lemma 4.2, that the above equation is true without restrictions on r and s, i.e. for arbitrary $1 \le i \ne j \le l$ and $0 \le s,\ r \le m$.

$2)_m$ can be proved in a much simpler way than that of Lemma 4.2. It is enough to show that $R^m H'$ is orthogonal to RX_i^r with arbitrary $i = 1, 2, \ldots, l$ and $0 \le r \le m$. For $j \ne i$, by $2)_0$, $2)_{m-1}$, and Lemma 4.2,

$$\begin{aligned} g(R^m H', R^r X_i) &= g\left([[H', X_j], R^{m-1} X_j], R^r X_i\right) = \\ &= g\left([[R^r X_i, R^{m-1} X_j], X_j], H'\right) = \\ &= g\left(R^{m+r} X_i + \nu_i R^{m+r-1} X_i, H'\right) = 0. \end{aligned}$$

$3)_m$ admits simple verification too (notice that the same method is valid for the third point of Lemma 4.2 in the case $l \ge 4$).

By virtue of (4.13), it is sufficient to show that

$$\left[[H', R^{m-r} X_i], R^r X_j\right] = 0,$$

where $i \ne j$, $0 \le r \le m$.

If $m > r$, then by Lemma 4.2 and the inductive hypothesis for $k \ne i, j$ we obtain

$$\begin{aligned} \left[[H', R^{m-r} X_i], R^r X_j\right] &= \left[[H', [[X_i, X_k], R^{m-r-1} X_k]], R^r X_j\right] = \\ &= -\left[[[X_i, X_k], [R^{m-r-1} X_k, H']], R^r X_j\right] = \\ &= \left[[[R^{m-r-1} X_k, H'], R^r X_j], [X_i, X_k]\right] + \\ &+ \left[[R^r X_j, [X_i, X_k]], R^{m-r-1} X_k, H']\right] = 0. \end{aligned}$$

It remains to prove $\left[[H', X_i], R^m X_j\right] = 0$. We have

$$\left[[H', X_i], R^m X_j\right] = \left[[H', X_i], [[X_j, X_k], R^{m-1} X_k]\right] =$$
$$= -\left[R^{m-1} X_k, [[H', X_i], [X_j, X_k]]\right] =$$
$$= \left[R^{m-1} X_k, [X_k, [[H', X_i], X_j]]\right] +$$
$$+ \left[R^{m-1} X_k, [X_j, [X_k, [H', X_i]]]\right] = 0.$$

Notice that by the Jacobi identity $\left[[R^s H', R^t X_i], R^r X_j\right] = 0$ for $i \neq j$ and $s, t, r \geq 0$ with $s + t + r = m$.

$4)_m$ follows from $1)_m$ and the Jacobi identity.

Q. E. D.

Denote $L_H = \mathrm{Span}\,(H', RH', R^2H', \ldots)$ and $\mathfrak{s} = \mathfrak{s}' \oplus L_H$. Then L_H is orthogonal to $\mathfrak{s}'$ and

$$\left[[L_H, L_i], L_j\right] = 0, \quad i \neq j,$$
$$\left[[L_H, L_i], L_i\right] \subset L_H,$$
$$\left[[L_i, L_j], L_H\right] = 0, \quad i \neq j,$$
$$\left[[L_i, L_i], L_H\right] = 0.$$

Let us consider now the brackets of the forms $\left[[L_H, L_H], L_i\right]$ and $\left[[L_H, L_H], L_H\right]$. According to this purpose we need equation (2.6) with H to be replaced by $H' + \sum_{k=1}^{l} P_k(R) X_k$.

Lemma 4.6. $\left[[X_i, H'], H'\right] = \left(\sum_k \mu_k P_k(R) + R^2 + \left(\sum_k \nu_k/(l-1)\right) R + \zeta I\right) X_i$.

Here $\zeta = \nu_j \left(\sum_k \nu_k/(l-1) - \nu_j\right) - \tau_j$; *these expressions are equal to one another for arbitrary* $j = 1, 2, \ldots, l$, *thus (2.10) is fulfilled.*

Proof. By (2.6) we get

$$\left[[X_i, H'], H'\right] = \alpha^2 \left[[X_i, X_j], X_j\right] + \mu_i H - \tau_i X_i + \left[[D_{X_j} H, X_i], X_j\right] -$$
$$- \sum_{k \neq i} \left[[X_i, P_k(R) X_k], P_k(R) X_k\right] -$$
$$- \sum_{k \neq i} \left[[X_i, P_k(R) X_k], P_i(R) X_i\right] - \left[[X_i, H'], P_i(R) X_i\right].$$

After routine calculations using Lemmas 4.2, 4.5, and (2.2) one can obtain

$$\left[[X_i, H'], H'\right] = \left(\left(P_i(R)(R + \nu_i) I + \mu_i I\right) H' + \sum_k \mu_k P_k(R) + R^2 + \right.$$
$$\left. + \sum_k \nu_k/(l-1) R + \left(\nu_j \left(\sum_k \nu_k/(l-1) - \nu_j\right) - \tau_j\right) I\right) X_i.$$

Since $j \neq i$ is arbitrary, the coefficient of the identity operator in the latter parentheses does not depend on j. Thus (2.10) is proved. We denote this coefficient by ζ. The operator

$P_i(R)(R+\nu_i)I+\mu_i I$ acting on H' in the right-hand side of the equation vanishes on s'. It turns out that it vanishes on $s = s' \oplus L_H$ too.

We have the symmetric operator R acting on s. Strictly speaking "the new" operator R and the "old" one are different because they have different domains of definition (s and s', respectively), though they coincide on s'. But we will mean everywhere R to be an operator on s.

Consider the eigenspaces of the operator R. We can divide them into 2 classes. The first is the class of the spaces $\{s_\gamma\}_{\gamma=1}^{B}$ with the eigenvalues $\{\xi_\gamma\}_{\gamma=1}^{B}$. One can easily see that for every $1 \le \gamma \le B$, s' is either the hyperspace of s_γ or coincides with it. The second class consists of the spaces $\{s_\gamma\}_{\gamma=B+1}^{C}$ whose eigenvalues are different from any of $\{\xi_\gamma\}_{\gamma=1}^{B}$. These subspaces are the one-dimensional subspaces of L_H, because they are orthogonal to s'. Decompose the vector H' by the eigenvectors of R:

$$H' = \sum_{\gamma=1}^{C} H_\gamma .$$

Here H_γ is the projection of H' onto s. Denote $\lambda_{\gamma|H} = \|H_\gamma\| \ge 0$. It can be easily seen that

$$s_\gamma = \mathrm{Span}_i (X_{\gamma|i}, H_\gamma), \quad \gamma \le B,$$

$$s_\gamma = \mathrm{Span}\,(H_\gamma), \quad \gamma > B.$$

Let $\beta > B \ge \gamma, \delta$ and $1 \le i \le l$. By (4.13).

$$\xi_\beta \left[[H_\beta, X_{\gamma|i}], X_{\delta|i}\right] = \xi_\gamma \left[[H_\beta, X_{\gamma|i}], X_{\delta|i}\right].$$

Thus, $\left[[H_\beta, X_{\gamma|i}], X_{\delta|i}\right] = 0$ and therefore $\left[[H_\beta, X_i], X_i\right] = 0$ for arbitrary $1 \le i \le l$ and $\beta > B$. Multiplying now (4.12) by H_β with $\beta > B$, we obtain

$$0 = g\left(\left[[H_\beta, X_i], X_i\right] - \left[[H_\beta, X_j], X_j\right], H'\right) = (\nu_i - \nu_j)\lambda^2_{\beta|H}.$$

If for any $i \ne j$, $\nu_i \ne \nu_j$, then by (2.7) $\mu_i = 0$ identically and by Lemma 4.4 $P_i = 0$ for every $1 \le i \le l$.

If there exists $i \ne j$ with $\nu_i \ne \nu_j$, then $\lambda^2_{\beta|H} = 0$. Hence $B = C$ and the spectrum of R on s coincides with that on s'. Thus, the minimal polynomials coincide too.

Therefore $\left(P_i(R)(R+\nu_i)I + \mu_i I\right)H' = 0$.

Q. E. D.

Denote for brevity

$$P(R) = \sum_{k=1}^{l} \mu^l_{k=1} \mu_k P_k(R) + R^2 + \sum_{k=1}^{l} \nu_k/(l-1)R + \zeta I .$$

Then by the previous Lemma

$$\left[[X_i, H'], H'\right] = P(R)X_i, \quad i = 1,2,\ldots,l.$$

Lemma 4.7. *The following equations are true*:

$$\left[[R^s X_i, R^t H'], R^r H'\right] = P(R)\, R^{s+t+r} X_i, \tag{4.15}$$

$$\left[[R^t H', R^r H'], R^s X_i\right] = 0, \tag{4.16}$$

$$\left[[R^t H', R^s H'], R^r H'\right] = 0 \tag{4.17}$$

for any integers $s, t, r \geq 0$ *and for* $1 \leq i \leq l$.

Proof. At first we prove (4.15). By (4.13) and the previous Lemmas, one can obtain

$$\left[[R^s X_i, R^t H'], R^r H'\right] = P(R) R^{s+t+r} X_i + \nu_i \left(P(R) R^{s+t+r-1} X_i - \right.$$
$$\left. - \left[[R^s X_i, R^t H'], R^{r-1} H'\right]\right).$$

Induction by $q = s + t + r$ implies (4.15).

From (4.15) and the Jacobi identity immediately follows (4.16).

Now verify (4.17). Suppose $t > 0$. One can see that

$$\left[[R^t H', R^s H'], R^r H'\right] = -\nu_i \left[[R^{t-1} H', R^s H'], R^r H'\right]$$

with an arbitrary i. Induction by $q = s + t + r$ yields (4.17).

Q. E. D.

By (4.15-4.17),

$$\left[[L_i, L_H], L_H\right] \subset L_i,$$

$$\left[[L_H, L_H], L_i\right] = 0,$$

$$\left[[L_H, L_H], L_H\right] = 0.$$

Thus, the space $\mathfrak{s} = \oplus_{i=1}^{l} L_i \oplus L_H$ is the Lie triple system in $\mathfrak{m}$. It remains to prove that its exponent $\widetilde{M} = \exp \mathfrak{s}$ is isometric to the space forms product of the types A, B or C. The following construction is similar to that in the proof of Lemma 4.3.

Lemma 4.8. *The totally geodesic submanifold* $\widetilde{M} = \exp \mathfrak{s} \subset M$ *is isometric to the product of the space forms.*

Proof. From Lemma 4.6 one can easily see that the eigenspaces of the operator $R\colon \mathfrak{s} \to \mathfrak{s}$ are

$$\mathfrak{s}_\gamma = \mathrm{Span}_i (X_{\gamma|i}, H_\gamma), \quad \gamma \leq B,$$

$$\mathfrak{s}_\gamma = \mathrm{Span}\, (H_\gamma), \quad \gamma > B.$$

In the way similar to Lemma 4.3 one can establish that all the triple brackets containing the vectors $X_{\gamma|i}$ and H_γ vanish except maybe the brackets of the form

$$\left[[X_{\gamma|i}, X_{\gamma|j}], X_{\gamma|j}\right] (i \neq j), \quad \left[[H_\gamma, X_{\gamma|i}], X_{\gamma|i}\right] (\gamma \leq B),$$

and

$$\left[[X_{\gamma|i}, H_\gamma], H_\gamma\right] \quad (\gamma \le B).$$

These exclusive brackets by Lemmas 2.3, 2.5–2.7 lie in the corresponding s_γ.

Thus, every of s_γ is the Lie triple system. Moreover, these spaces are mutually orthogonal and all of their triple brackets vanish except $[[s_\gamma, s_\gamma], s_\gamma] \subset s_\gamma$ $(\gamma \le B)$. By virtue of E. Cartan's decomposition theorem [8] $\widetilde{M}$ is isometric to the product of the totally geodesic submanifolds $M_\gamma = \exp s_\gamma$. It remains to show that $\{M_\gamma\}_\gamma$ are of a constant curvature.

The one-dimensional and the two-dimensional cases are trivial. Suppose $\dim s_\gamma \ge 3$. If s_γ is orthogonal to H', then it is sufficient to refer to Lemma 4.3.

Let $s_\gamma = s' \oplus H_\gamma$ with $H_\gamma \ne 0$. For any two nonzero vectors $X_{\gamma|i}$ and $X_{\gamma|j}$ $(i \ne j)$, by Lemma 4.3, $\widetilde{R}(X_{\gamma|i}, X_{\gamma|j}, X_{\gamma|j}, X_{\gamma|i}) = k_\gamma \lambda^2_{\gamma|i} \lambda^2_{\gamma|j}$.

By Lemma 4.5, $\widetilde{R}(H_\gamma, X_{\gamma|j}, X_{\gamma|i}, H_\gamma) = k_\gamma \lambda^2_{\gamma|i} \lambda^2_{\gamma|H}$. Moreover, $\widetilde{R}(H_\gamma, X_{\gamma|i}) X_{\gamma|j} = 0$ and $\widetilde{R}(X_{\gamma|i}, X_{\gamma|j}) X_{\gamma|k} = 0$ $(k \ne i,j)$. Hence, the sectional curvature of the ambient space M along the two-dimensional planes lying in s_γ is constant. By virtue of geodesicity this holds for M_γ too.

Q. E. D.

Now let us investigate the structure of s in detail to show that $M = \exp s$ is of type A, B or C. Notice that the completeness of N in $\widetilde{M}$ has been already established by virtue of Proposition 1 and the minimality of the system s (as the Lie triple system containing $\{X_i\}_{i=1}^{l}$ and H).

One can easily see that equation (4.9) holds for the system s. The orthogonality of L_i and s_γ implies, in particular,

$$\nu_i + \xi_\gamma = 0. \tag{4.18}$$

First we shall prove the following simple

Lemma 4.9. *The system* s *contains no more than one Euclidean factor (including one-dimensional).*

Proof. If s_γ is a Euclidean factor of dimension ≥ 2, then (4.9) implies that every of ν_i $(i = 1,2,\ldots,l)$ equals $(-\xi_\gamma)$.

If $\dim s_\gamma = 1$ and $s_\gamma \subset L_H$, then the same is true by (4.18). After all, if $\dim s_\gamma = 1$, $s_\gamma \subset L_j$ (for some $1 \le i \le l$), then by (4.10) each of ν_j with $j \ne i$ is $(-\xi_\gamma)$. But $\{\xi_\gamma\}_\gamma$ are mutual distinct and $l \ge 3$.

Q. E. D.

This Lemma in fact justifies the notation $M_\gamma = M^\gamma(k_\gamma)$ of Theorem 1 for the submanifolds $\exp s_\gamma$. In particular, $C = N$, $\dim s_\gamma = d_\gamma$.

The system s is of the same form for all points $Q \in N$ because of its minimality and by Proposition 1. However the vector field H' can be discontinuous and thus the systems

s' constructed at different points of N can be non-isomorphic. To avoid this trouble we will restrict our consideration to the general points:

Lemma 4.10. *In a general point there is no subspace* s_γ *orthogonal to* s'.

Proof. Lemma 4.8 implies that any such space is one-dimensional. From Lemma 4.9 it follows that it is unique if it exists; in particular, $B = N - 1$ or $B = N$.

Suppose s_N to be orthogonal to s' ($\dim s_N = 1$). Then (4.18) yields immediately that all $\{\nu_i\}_i$ at the point Q are equal. By the condition the same holds all over N. Then (2.7) yields $\alpha = \text{const}$, $\mu_i = \tau_i = 0$ $(i = 1,2,\ldots,l)$, $H' = H$ (by Lemma 4.4). Now, by (2.6), for $i \neq j$,

$$\left[[X_i, H], H\right] = \alpha^2 \left[[X_i, X_j], X_j\right] + \left[[D_{X_j} H, X_i], X_i\right].$$

Multiplying this by $X_{\gamma|i}$ $(\gamma \leq B)$, we obtain:

$$k_\gamma \lambda_\gamma \lambda_{\gamma|H}^2 = \alpha^2 k_\gamma \lambda_\gamma^3 - k_\gamma \lambda_\gamma^2 \left(k_\gamma \lambda_\gamma^3 - \lambda_\gamma \sum_{\rho=1}^{B} \lambda_\rho^4 k_\rho\right).$$

Dividing this by $k_\gamma \lambda_\gamma \neq 0$ and summing up with $\gamma = 1,2,\ldots,B$, we get $\alpha^2 - \lambda_{N|H}^2 = \alpha^2$. Hence $H_N = 0$.

Q. E.D.

Thus, if there exists the one-dimensional subspace $s_\gamma \subset s$, then it is unique and it lies in s'. Notice that this is false without the point Q generality condition.

Now let us prove that $\widetilde{M}$ is of the types A, B or C. It is convenient to "return" from H' to H. Let

$$H = \sum_{\gamma=1}^{N} \left(\sum_{i=1}^{l} X'_{\gamma|i} + H_\gamma\right)$$

be the decomposition of H by $\{s_\gamma\}_{\gamma=1}^{N}$; here $X'_{\gamma|i}$ is collinear to $X_{\gamma|i}$. Put $\theta_{\gamma|i} = \|X'_{\gamma|i}\| \geq 0$. Lemma 4.4 yields that if $\lambda_{\gamma|i} = 0$, then $\theta_{\gamma|i} = 0$ too. Moreover,

$$\sum_{\gamma=1}^{N} \left(\sum_{i=1}^{l} \theta_{\gamma|i}^2 + \lambda_{\gamma|H}^2\right) = \alpha^2 > 0, \tag{4.19}$$

$$\sum_{\gamma=1}^{N} \theta_{\gamma|i} \lambda_{\gamma|i} = 0, \quad i = 1,2,\ldots,l, \tag{4.20}$$

$$\sum_{\gamma=1}^{N} \lambda_{\gamma|i}^2 = 1, \quad i = 1,2,\ldots,l. \tag{4.21}$$

Let us consider now the class of spaces $\widetilde{M}$, which will lead us to the type A of Theorem 1.

Lemma 4.11. *Suppose* $\widetilde{M}$ *contains no one-dimensional factors and* $\sum_{\gamma=1}^{N} k_\gamma^{-1} \neq 0$. *Then*:

1) *All* $\{\nu_i\}_i$ *are equal; denote their common value by* ν *and let* N^l *be the space form of a curvature* $c = -(l-2)/(l-1)\nu$.

2) *For an arbitrary* $\gamma = 1,\ldots,N$, *all* $\{\lambda_{\gamma|i}\}_{i=1}^{l}$ *are equal. Denote their common value by* $\lambda_\gamma \geq 0$.

3) $\alpha = \text{const}, \mu_i = \tau_i = 0$ *for any* $i = 1,2,\ldots,l$.

4) $H = H'$, *i.e.* $\theta_{\gamma|i} = 0$ *for any* i, γ.

5) *For an arbitrary* $\gamma = 1,\ldots,N$, *holds the equation*

$$\lambda_{\gamma|H}^2 = \lambda_\gamma^2 (c - k_\gamma \lambda_\gamma^2). \tag{4.22}$$

6) *For every* $\gamma = 1,\ldots,N$ $d_\gamma = l + 1$ *except maybe that with* $d_\gamma = l$. *This corresponds to the cases of* $\lambda_{\gamma|H} \neq 0$ *and* $\lambda_{\gamma|H} = 0$, *respectively.*

Proof. To prove 1) and 2) let us consider two cases:

a) one of the factors is Euclidean, i.e. $k_\gamma = 0$ for some $1 \leq \gamma \leq N$. Then (4.9) yields that all $\{\nu_i\}$ are equal in value. Denote their common value by $\nu = -\xi_\gamma$. The submanifold N turns out to be Einsteinian. It is well known that the Einsteinian conformally flat manifold of dimension $l \geq 3$ is a space form [7, Ch. 11, § 28]. It can be easily seen that $c = -(l-2)/(l-1)\nu$. Thus, 1) has been established.

If $N = 1$, i.e. $\widetilde{M}$ is a Euclidean space, then by (4.21) all $\{\lambda_{1|i}\}_{i=1}^{l}$ are equal to 1 and therefore 2) is proved.

Suppose $s_\delta \subset s$ to be a subspace distinct to s_γ. If there exists the vector X_j orthogonal to s_δ, then (4.18) implies $\nu_i = -\xi_\delta$; this is impossible because $\xi_\gamma \neq \xi_\delta$. Thus, $\lambda_{\delta|i} > 0$ for $\delta \neq \gamma$ and $1 \leq i \leq l$. Now by (4.9) $\lambda_{\delta|i} = -(\xi_\delta + \nu)/k_\delta$ does not depend on i, i.e. $\lambda_{\delta|i} = \lambda_\delta$ for $1 \leq i \leq l$ and $\delta \neq \gamma$. By (4.21), $\{\lambda_{\gamma|i}\}_{i=1}^{l}$ are equal in value too. They are nonzero in virtue of Lemma 4.10. So 2) is proved.

b) $\widetilde{M}$ contains no Euclidean factor. If there exist $1 \leq i \leq l$ and $1 \leq \gamma \leq N$ with $\lambda_{\gamma|i} = 0$, then $\xi_\gamma = -\nu_i$ and therefore by (4.9) $k_\gamma = 0$. Hence $\lambda_{\gamma|i} > 0$ for any γ, i (4.9) implies $\lambda_{\gamma|i}^2 = -\xi_\gamma k_\gamma^{-1} - \nu_i k_\gamma^{-1}$. Summing up over γ and taking it into account that $\sum_{\gamma=1}^{N} k_\gamma^{-1} \neq 0$, we obtain

$$\nu_i = \left(-1 - \sum_{\gamma=1}^{N} \xi_\gamma k_\gamma^{-1}\right)\left(\sum_{\gamma=1}^{N} k_\gamma^{-1}\right)^{-1},$$

i.e. all $\{\nu_i\}_{i=1}^{l}$ are the same. Denote their common value by ν. The submanifold N is isometric to the space form of the curvature $c = -(l-2)/(l-1)\nu$. Further, (4.9) and Lemma 4.10 yield 2). Case 3) follows immediately from 1) and (2.7). Case 4) follows from 3) and from the method of construction of H'. Dividing

$$k_\gamma \lambda_\gamma \lambda_{\gamma|H}^2 = \alpha^2 \lambda_\gamma^3 k_\gamma - k_\gamma \lambda_\gamma^3 \left(k_\gamma \lambda_\gamma^2 - \sum_{\rho=1}^{N} k_\rho \lambda_\rho^4\right)$$

by $k_\gamma \lambda_\gamma$ (if $k_\gamma \neq 0$) we obtain

$$\lambda_{\gamma|H}^2 = \lambda_\gamma^2 \left(\alpha^2 + \sum_{\rho=1}^{N} k_\rho \lambda_\rho^4 - k_\gamma \lambda_\gamma^2\right).$$

If $k_\delta = 0$ for some $1 \le \delta \le N$, then summing up the above equations over $\gamma \ne \delta$ and taking into account 4) and (4.19), we obtain the same equation:

$$\lambda^2_{\delta|H} = \lambda^2_\delta \left(\alpha^2 + \sum_{\rho=1}^{N} k_\rho \lambda^4_\rho - k_\delta \lambda^2_\delta\right).$$

Now, by virtue of the Gauss equation (2.2) $c = \alpha^2 + \sum_{\rho=1}^{N} k_\rho \lambda^4_\rho$. So 5) is proved.

One can easily get by 2) that $d_\gamma = \dim s_\gamma \ge l$ since $X_{\gamma|l} \ne 0$. If $H_\gamma \ne 0$, then $d_\gamma = l + 1$ and if $H_\gamma = 0$, then $d_\gamma = l$. So 6) is proved.

Suppose that $d_\gamma = 1$ for some $1 \le \gamma \le N$. Then $\lambda_{\gamma|H} = 0$ and $k_\gamma \lambda^2_\gamma = c$ by (4.22). Now (4.9) implies $\xi_\gamma = -\nu - c$. Since $\{\xi_\gamma\}$ are distinct, there is no more than unique γ with $d_\gamma = l$.

Q. E. D.

Thus, under the conditions of Lemma 4.11 we get the type *A* of Theorem 1.

Consider now the case when $\widetilde{M}$ contains the one-dimensional factor. We assume the point $Q \in N^l$ to be the general one. Lemma 4.10 implies that the factor is orthogonal to H'. Let $d_1 = 1$, $s_1 = T_Q M_1 \subset L_1$.

We have the following decompositions:

$$X_1 = X_{1|1} + \sum_{\gamma=2}^{N} X_{\gamma|1}, \qquad X_{1|1} \ne 0,$$

$$X_i = \sum_{\gamma=2}^{N} X_{\gamma|i}, \qquad i = 2,\ldots,l,$$

$$H = X'_{1|1} + \sum_{\gamma=2}^{N} \left(\sum_{i=2}^{l} X'_{\gamma|i} + H_\gamma\right).$$

Lemma 4.12. *Let $\widetilde{M}$ contain some one-dimensional factor $M_1 = \exp s_1$. Let $Q \in N^l$ be the general point. Assume without loss of generality that $s_1 \subset L_1$. Then, at Q:*

1) *All $\{\nu_i\}_i$ with $i > 1$ are the same and are not equal to ν_1.*

2) *For an arbitrary $\gamma \ge 2$ all $\{\lambda_{\gamma|i}\}_{i=1}^{l}$ are the same (denote their common value by $\lambda_\gamma > 0$).*

3) $\sum_{\gamma=2}^{N} k_\gamma^{-1} \ne 0$.

4) $\mu_i = 0$ *for any* $i > 1$.

5) $\theta_{\gamma|i} = 0$ *for* $i > 1$ *and* γ *arbitrary.*

6) *The following equations are true:*

$$k_\gamma \theta_{\gamma|1} \lambda_{\gamma|1} = \mu_1, \quad \gamma > 1, \tag{4.23}$$

$$\sum_{\gamma=2}^{N} k_\gamma \lambda^4_\gamma = -\nu + \nu_1/(l-1) - \alpha^2, \quad \gamma > 1, \tag{4.24}$$

$$-k_\gamma\left(\lambda^2_{\gamma|H} + \theta^2_{\gamma|1}\right) = \zeta + \xi^2_\gamma + \xi_\gamma \sum_{i=1}^{l} \nu_i/(l-1) =$$

$$= -\tau - k_\gamma \lambda_\gamma^2 \left(\xi_\gamma + \nu_1/(l-1)\right), \quad \gamma > 1, \tag{4.25}$$

$$\tau_i = \tau = \mu_1^2/(\nu - \nu_1), \quad i > 1, \tag{4.26}$$

$$\tau_1 = \tau + \nu_1 (\nu - \nu_1)(l-2)/(l-1). \tag{4.27}$$

Proof. 1) By (4.18) $\nu_i = -\xi_1$ for $i > 1$. We denote their common value by ν. The inequality $\nu \neq \nu_1$ will be proved below.

2) From (4.18) it can be easily seen that $\lambda_{\gamma|i} > 0$ for $i > 1$, $\gamma > 1$. Now (4.9) yields $\lambda_{\gamma|i}^2 = -(\xi_\gamma + \nu) k_\gamma^{-1}$, $\gamma > 1$, $i > 1$. Note that $k_\gamma \neq 0$ ($\gamma > 1$) in view of Lemma 4.9.

3) For $\gamma > 1$, $\lambda_{\gamma|1}^2 - \lambda_\gamma^2 = (\nu - \nu_1) k_\gamma^{-1}$ by (4.9). Summing up over $\gamma = 2, \ldots, N$ and taking into account (4.21), we obtain $\lambda_{1|1}^2 = (\nu - \nu_1) \sum_{\gamma=2}^{N} k_\gamma^{-1}$. Thus, $\sum_{\gamma=2}^{N} k_\gamma^{-1} \neq 0$ and $\nu \neq \nu_1$.

4) Case 4) follows trivially from (2.7).

5) Cases 5) and 6) can be obtained from (2.1-2.6) by routine calculations using (4.9) and (2.10). In particular, to establish (4.27), we have to use the generality of Q. For $i \geq 2$,

$$\tau_i = \left(X_i X_i(\alpha^2) - \nabla_{X_i} X_i(\alpha^2)\right)/2 = -g\left(\nabla_{X_i} X_i, X_1\right) X_1(\alpha^2)/2.$$

This and (2.7) imply the first of two equations (4.27). The second equation follows from (2.10) and 1).

Q. E. D.

It is important to note that all equations (2.1-2.6) are fulfilled by Lemma 4.12 and (4.9, 4.10, 4.19-4.21).

It follows from the previous Lemma that for any $\gamma > 1$, $d_\gamma \geq l - 1$. On the other hand $d_\gamma \leq l + 1$ trivially.

The following Lemma characterizes the type B_1.

Lemma 4.13. *Let $\tilde{M}$ contain a one-dimensional factor and an $(l-1)$-dimensional factor: $d_1 = 1$, $d_2 = l - 1$. Then:*

1) *the mean curvature α is a constant;*

2) *$N \geq 3$ and $d_\gamma = l + 1$ for $\gamma > 2$;*

3) *every point of N^l is the general one. The space form N^l is isometric to the direct product of the line and the space form.*

Proof. 1) By 2) of Lemma 4.12, $\lambda_{2|1} = \theta_{2|1} = \lambda_{2|H} = 0$. Then (4.23) yields $\mu_1 = 0$ on the open dense set of the general points on N. Thus, μ_1 vanishes identically and α is constant.

2) It follows from $\mu_1 = 0$ that $\lambda_{\gamma|1}\, \theta_{\gamma|1} = 0$ for $\gamma > 1$. Thus, $\theta_{\gamma|1} = 0$. By (4.20) $\theta_{1|1} = 0$. Therefore, $H = H' = \sum_{\gamma=3}^{N} H_\gamma$. Now (4.18) yields $\nu_1 = -\xi_2$ and, in

particular, $\lambda_{\gamma|1} > 0$ for $\gamma > 2$. By 1) τ vanishes identically; therefore by virtue of (4.26) $\nu_1/(l-1) + \xi_2 = 0$. Hence $\nu_1 = \xi_2 = 0$. Equation (4.26) for $\gamma > 2$ yields $\lambda^2_{\gamma|H} = \lambda^2_\gamma \xi_\gamma \neq 0$ from which it follows that $\lambda_{\gamma|H} > 0$. Case 2) is proved.

Case 3) follows from Proposition 2.

Q. E. D.

Now suppose that $\tilde{M}$ contains a one-dimensional factor but no $(l-1)$-dimensional ones. We get the type B_2.

Lemma 4.14. *Let $\tilde{M}$ contain the one-dimensional factor $d_1 = 1$ and let $d_\gamma \geq l$ for $\gamma > 1$. Then*:

1) *for every $\gamma > 1$ $d_\gamma = 1$ or $l+1$ depending on $\lambda_{\gamma|H} = 0$ or $\lambda_{\gamma|H} \neq 0$, respectively.*

2) *N^l is isometric to the warped (possibly direct) product of the line and the space form and is not isometric to the space form.*

P r o o f. 1) From Lemma 4.12 and the condition it can be easily seen that $d_\gamma = 1$ or $l+1$ for $\gamma > 1$.

Let $d_2 = l$. There are two possible cases: either $\lambda_{211} = \theta_{211} = 0$ or $\lambda_{21H} = 0$.

In the first case we get in a way similar to Lemma 4.13 that $\nu_1 = -\xi_2, \mu_1 = 0, \tau = 0$. Then $\alpha = \text{const}$ and $\tau_1 = 0$. By (4.27), $\nu_1 = 0$ and therefore $\xi_2 = 0$. Applying (4.26), we obtain $\lambda^2_{21H} = 0$, i.e. $d_2 = \dim s_2 = l - 1$. Hence this case is impossible.

2) The case follows from Proposition 2.

Q. E. D.

Thus, we have studied completely the types A and B. It follows from Lemmas 4.11, 4.12 that it remains to consider the case where $\tilde{M}$ has no one-dimensional factors and $\sum_{\gamma=1}^{N} k_\gamma^{-1} = 0$.

Lemma 4.15. *Suppose $\tilde{M}$ contains no one-dimensional factors and $\sum_{\gamma=1}^{N} k_\gamma^{-1} = 0$. Then*

$$\sum_{\gamma=1}^{N} \xi_\gamma k_\gamma^{-1} = -1, \tag{4.28}$$

$$k_\gamma \theta_{\gamma|i} \lambda_{\gamma|i} = \mu_i, \quad 1 \leq i \leq l, \quad 1 \leq \gamma \leq N, \tag{4.29}$$

$$\sum_{\gamma=1}^{N} \xi_\gamma^2 k_\gamma^{-1} = \sum_{i=1}^{l} \nu_i/(l-1) - \alpha^2, \tag{4.30}$$

$$-k_\gamma \left(\lambda^2_{\gamma|H} + \sum_{i=1}^{l} \theta^2_{\gamma|i}\right) = \zeta + \xi_\gamma^2 + \xi_\gamma \sum_{i=1}^{l} \nu_i/(l-1), \quad 1 \leq \gamma \leq N. \tag{4.31}$$

P r o o f consists in substituting of the decompositions of the vectors $\{X_j\}_{j=1}^{l}$ and H into equations (2.1-2.6). Equations (2.1-2.3) are satisfied. To get (4.28), divide (4.9) by

k_γ and sum it by $\gamma = 1,\ldots,N$. It also follows from (4.9) that for any $1 \le i \le l$ there is no more than one $\{\lambda_{\gamma|i}\}_{\gamma=1}^{N}$ equal to zero.

Equation (2.4) implies $\lambda_{\gamma|j}\left(k_\gamma \theta_{\gamma|k}\lambda_{\gamma|k} - \mu_k\right) = 0$ for arbitrary γ and $j \ne k$. If there exist γ and k such that $k_\gamma \theta_{\gamma|k}\lambda_{\gamma|k} - \mu_k \ne 0$, then $\lambda_{\gamma|k} = 0$ for $j \ne k$. Therefore for every $1 \le k \le l$ there exists no more than the unique $1 \le \gamma \le N$ that $k_\gamma \theta_{\gamma|k}\lambda_{\gamma|k} \ne \mu_k$. Taking $\delta \ne \gamma$, we obtain $\theta_{\delta|k}\lambda_{\delta|k} = \mu_k k_\delta^{-1}$. Summing this by $\delta \ne \gamma$, we get (4.29). Equation (2.2) implies $\sum_\gamma k_\gamma \lambda_{\gamma|j}^2 \lambda_{\gamma|i}^2 = -\nu_i - \nu_j - \alpha^2 + \sum_{r=1}^{l} \nu_r/(l-1)$ with $i \ne j$. Now (4.9) and (4.28) yield (4.30).

Equation (2.5) follows from (4.9, 4.29), i.e. it is useless in this case.

Equation (2.6) implies that for $1 \le i \le l$ and $1 \le \gamma \le N$ the following holds:

$$\lambda_{\gamma|j}^2 \left(k_\gamma\left(\lambda_{\gamma|H}^2 + \sum_{i=1}^{l} \theta_{\gamma|i}^2\right) + \zeta + \xi_\gamma^2 + \xi_\gamma \sum_{i=1}^{l} \nu_i/(l-1)\right) = 0.$$

If the expression in the parentheses is nonzero, then $\lambda_{\gamma|j} = 0$ for every j and thus $\dim s_\gamma = 1$.

Q. E. D.

Using the equations received we shall investigate the structure of $\widetilde{M}$.

Lemma 4.16. *Let $\widetilde{M}$ contain no one-dimensional factors and $\sum_{\gamma=1}^{N} k_\gamma^{-1} = 0$. Then:*

1) *$N \ge 2$ and there exist no more than two factors of dimension $< l$;*

2) *the sum of the dimensions of any two factors is $\ge l$. If there exist the factors M_γ and M_δ with $d_\gamma + d_\delta = l$, then N^l is of constant mean curvature and isometric to the Riemannian product $S^{d_\gamma}(c)\times L^{d_\delta}(-c)$ (or $S^{d_\delta}(c)\times L^{d_\gamma}(-c)$).*

Proof. 1) Suppose $d_\gamma, d_\delta, d_\beta < l$ for some distinct $1 \le \gamma, \delta, \beta \le N$. Then there exist i, j, k such that $\lambda_{\gamma|i} = \lambda_{\delta|j} = \lambda_{\beta|k} = 0$. The equation (4.9) implies $\nu_i = -\xi_\gamma$, $\nu_j = -\xi_\delta$, $\nu_k = -\xi_\beta$ (in particular, i, j, k are distinct). Applying (4.9) again, we get if that for any $1 \le r \le l$, $(\nu_r - \nu_i)k_\gamma^{-1} \ge 0$, $(\nu_r - \nu_j)k_\delta^{-1} \ge 0$, $(\nu_r - \nu_k)k_\beta^{-1} \ge 0$. Assume that the curvatures k_γ and k_δ are of the same sign. Then $(\nu_j - \nu_i)(\nu_i - \nu_j) \ge 0$, which is impossible.

2) If $d_\gamma + d_\delta < l$ for some $\gamma \ne \delta$, then there exist $1 \le i \le l$ such that $\lambda_{\gamma|i} = \lambda_{\delta|i} = 0$, which is impossible by (4.9).

Suppose now $d_1 + d_2 = l$. Then $\lambda_{1|H} = \lambda_{2|H} = 0$. Renumerate the vectors $\{X_j\}_{j=1}^{l}$ to get $\lambda_{1|i} = 0$ for $i > d_1$ and $\lambda_{2|i} = 0$ for $i \le d_1$. Equation (4.29) yields if that $\mu_i = 0$ for $1 \le i \le l$, i.e. $\alpha = \text{const}$. Moreover, (4.18) implies $\nu_1 = \nu_2 = \ldots = n_{d_1} = -\xi_2$, $\nu_{d_1+1} = \ldots = \nu_l = -\xi_1$. Now [6, Ch. VII] completes the proof.

Q. E. D.

So all the possible cases for $\widetilde{M}$ have been already considered. Now we will separate the type C_2 from the general type C. It will be similar to A_1.

Lemma 4.17. *Let $\tilde{M}$ be of the type C and N^l has the constant curvature c. Then:*
1) *for every $\gamma = 1,2,\ldots,N$ all $\{\lambda_{\gamma|i}\}_{i=1}^{l}$ are equal to some $\lambda_\gamma > 0$;*
2) *$\theta_{\gamma|i} = 0$ for i and γ arbitrary;*
3) *equation (4.22) holds;*
4) *every d_γ equals $l+1$ except maybe one that equal to l.*

P r o o f. 1) Constancy of curvature implies that all $\{\nu_i\}_i$ are equal to $\nu = -(l-1)/(l-2)c$. Now 1) follows from (4.9). If $\lambda_\gamma = 0$, then $d_\gamma = 1$.

2) By (2.7) $\mu_i = 0$, therefore $\alpha = \text{const}$ and (4.29) yields $k_\gamma \lambda_\gamma \theta_{\gamma|i} = 0$.

3) Constancy of α implies $\tau_i = 0$ $(i = 1,2,\ldots,l)$. By (2.10) $\zeta = \nu^2/(l-1)$. Then (4.22) follows from (4.9) and (4.31).

4) By virtue of 3) $\lambda_{\gamma|H}^2 = -\lambda_\gamma^2\left(\xi_\gamma + \nu/(l-1)\right)$. If $\lambda_{\gamma|H} = 0$, then $\xi_\gamma = -\nu/(l-1)$. Since $\{\xi_\gamma\}_{\gamma=1}^{N}$ are mutual distinct, we get that all $\{\lambda_{\gamma|H}\}_{\gamma=1}^{N}$ are positive except may be one.

Q. E. D.

The extrinsic problem is solved.

Theorem 1 is proved.

Now gather the results of this Chapter for the later use in the proof of Theorem 2 (Chapter 5).

Lemma 4.18. *Let $N^l \subset \tilde{M} = \Pi_{\gamma=1}^{N} M_\gamma^{d_\gamma}(k_\gamma)$ be one of the pairs $(\tilde{M}, N)$ of the Table in the Theorem 1 except the type A_0.*

One of the two following cases is possible:

1) *N^l is the space form of the curvature c (types A_1, C_2). Then $1 \le d_1 \le l+1$, $d_2 = \ldots = d_N = l+1$. At an arbitrary point $Q \in N^l$ we have the decompositions*

$$X_i = \sum_{\gamma=1}^{N} X_{\gamma|i}, \quad i = 2,\ldots,l,$$

$$H = \sum_{\gamma=1}^{N} H_\gamma,$$

where H_γ, $X_{\gamma|i} \in T_Q M_\gamma$ and the vectors $\{H_\gamma, X_{\gamma|i}\}_{\gamma,i}$ are mutually orthogonal; $\|X_{\gamma|i}\| = \lambda_\gamma > 0$, $\|H_\gamma\| = \lambda_{\gamma|H}$ $(> 0$ for $\gamma > 1$ and ≥ 0 for $\gamma = 1)$. The following equations hold:

$$\sum_{\gamma=1}^{N} \lambda_\gamma^2 = 1, \tag{4.21}$$

$$\sum_{\gamma=1}^{N} \lambda_{\gamma|H}^2 = \alpha^2 > 0, \tag{4.19}$$

$$\lambda_{\gamma|H}^2 = \lambda_\gamma^2 (c - k_\gamma \lambda_\gamma^2), \quad \gamma = 1,\ldots,N, \tag{4.22}$$

$$k_\gamma \lambda_\gamma^2 + \xi_\gamma + \nu = 0, \tag{4.9}$$

$$c = -(l-2)/(l-1)\nu\,.$$

2) *N^l is not the space form (types B, C_1, C_3). If $\widetilde{M}$ contains the one-dimensional factor ($d_1 = 1$), then define* $k_1 = -\left(\sum_{\gamma=2}^{N} k_\gamma^{-1}\right)^{-1}$. *At the general point $Q \in N^l$, we have the decompositions*

$$X_i = \sum_{\gamma=1}^{N} X_{\gamma|i}, \quad i = 2,\ldots,l,$$

$$H = \sum_{\gamma=1}^{N}\left(\sum_{i=1}^{l} X'_{\gamma|i} + H_\gamma\right),$$

where H_γ, $X_{\gamma|i}$, $X'_{\gamma|i} \in T_Q M_\gamma$; $X'_{\gamma|i}$ is collinear to $X_{\gamma|i}$ and $X_{\gamma|i} = 0$ implies $X'_{\gamma|i} = 0$; the vectors $\{H_\gamma, X_{\gamma|i}\}_{\gamma,i}$ are mutually orthogonal; $\left\|X_{\gamma|i}\right\| = \lambda_{\gamma|i} \geq 0$, $\left\|X'_{\gamma|i}\right\| = \theta_{\gamma|i} \geq 0$, $\left\|H_\gamma\right\| = \lambda_{\gamma|H} \geq 0$. *The following equations hold:*

$$\sum_{\gamma=1}^{N} \lambda_{\gamma|i}^2 = 1, \quad i = 1,2,\ldots,l, \tag{4.21}$$

$$\sum_{\gamma=1}^{N} \theta_{\gamma|i}\lambda_{\gamma|i} = 0, \quad i = 1,2,\ldots,l, \tag{4.20}$$

$$\sum_{\gamma=1}^{N}\left(\sum_{i=1}^{l} \theta_{\gamma|i}^2 + \lambda_{\gamma|H}^2\right) = \alpha^2 > 0, \tag{4.19}$$

$$k_\gamma \lambda_{\gamma|i}^2 + \xi_\gamma + \nu_i = 0, \tag{4.9}$$

$$k_\gamma \theta_{\gamma|i}\lambda_{\gamma|i} = \mu_i, \quad 1 \leq i \leq l, \quad 1 \leq \gamma \leq N, \tag{4.29}$$

$$\sum_{\gamma=1}^{N} k_\gamma^{-1} = 0,$$

$$\sum_{\gamma=1}^{N} \xi_\gamma k_\gamma^{-1} = -1, \tag{4.28}$$

$$\sum_{\gamma=1}^{N} \xi_\gamma^2 k_\gamma^{-1} = \sum_{i=1}^{l} \nu_i/(l-1) - \alpha^2, \tag{4.30}$$

$$-k_\gamma\left(\lambda_{\gamma|H}^2 + \sum_{i=1}^{l} \theta_{\gamma|i}^2\right) = \zeta + \xi_\gamma^2 + \xi_\gamma \sum_{i=1}^{l} \nu_i/(l-1), \quad 1 \leq \gamma \leq N. \tag{4.31}$$

P r o o f. To prove the Lemma we have only to verify that in case 2) equations (4.9, 4.28-4.31) hold for the space $\widetilde{M}$ of the type B, i.e. for $d_1 = 1$.

Assume that $s_1 \subset L_1$, as in Lemma 4.12.

Equation (4.29) for $\gamma > 1$, $i = 1$ follows from (4.23); for $i > 1$ it follows from 4), 5) of Lemma 4.12; for $\gamma = 1$, $i = 1$ by virtue of (4.20) we obtain $0 = \sum_{\gamma=1}^{N} \lambda_{\gamma|1}\theta_{\gamma|1} = \lambda_{1|1}\theta_{1|1} - \mu_1 k_1^{-1}$.

Dividing (4.9) with $i > 1$ by k_γ and summing up by $\gamma = 1,\ldots,N$, we get (4.28).

Equation (4.9) is true for $\gamma > 1$. Let $\gamma = 1$. Then $\lambda_{1|1} = 0$ and $\xi_1 = -\nu_i$ for $\gamma > 1$ (look at the proof of the first case of Lemma 4.12). Thus, by (4.9, 4.21, 4.28) with $\gamma > 2$, $\lambda_{1|1}^2 = 1 - \sum_{\gamma=2}^{N} \lambda_{\gamma|1}^2 = -\xi_1 k_1^{-1} - \nu_1 k_1^{-1}$.

By (4.24) and (4.9) for $i > 1$

$$\sum_{i=1}^{l} \nu_i/(l-1) - \alpha^2 = 2\nu_i + \sum_{\gamma=1}^{N} k_\gamma \lambda_{\gamma|i}^4 =$$

$$= 2\nu_i + \sum_{\gamma=1}^{N} k_\gamma^{-1}\xi_\gamma^2 + 2\nu_i \sum_{\gamma=1}^{N} k_\gamma^{-1}\xi_\gamma + \nu_i \sum_{\gamma=1}^{N} k_\gamma^{-1}.$$

This proves (4.30).

Equation (4.31) with $\gamma > 1$ follows from (4.25). Let $\gamma = 1$. Then $\lambda_{1|H} = 0$. By (4.25) and above mentioned equations,

$$\theta_{1|1}^2 = \alpha^2 - \sum_{\gamma=2}^{N} \left(\theta_{\gamma|1}^2 + \lambda_{\gamma|H}^2\right) = \alpha^2 + \sum_{\gamma=2}^{N} k_\gamma^{-1}\left(\zeta + \xi_\gamma^2 + \xi_\gamma \sum_{i=1}^{l} \nu_i/(l-1)\right) =$$

$$= \alpha^2 + k_1^{-1}\zeta + \sum_{i=1}^{l} \nu_i/(l-1) - \alpha^2 - k_1^{-1}\xi_1^2 + \sum_{i=1}^{l} \nu_i/(l-1)(-1 - k_1^{-1}\xi_1).$$

Q. E. D.

Notice finally that all equations of this Lemma are the consequences of Proposition 3.

5. Proof of Theorem 2

This proof is verificative by nature. Really, by virtue of Proposition 1 the umbilical submanifold can be determined completely by one of its points, the tangent space and the mean curvature vector in this point. The possible types of tangent spaces and mean curvature vectors for $N \subset \widetilde{M}$ have been considered in detail in Chapter 4 (lemma 4.18). Therefore, the proof of Theorem 2 can be constructed in the following way: choose a point $Q \in \widetilde{M}$, a tangent space $T_Q N$, and a mean curvature vector H to $N \subset \widetilde{M}$ at this point. Compute the osculating space L_Q to $N \subset E$ (it can be determined by $T_Q N$ and H). Verify that $L_Q \cap \widetilde{M}$ is umbilical in $\widetilde{M}$ and is of the dimension l. Then the intersection coincides with N^l.

Type A_0 is trivial. Before considering the other types reduce them to the common form, i.e. embed every factor $M_\gamma = M^{d_\gamma}(k_\gamma)$ of $\widetilde{M}$ as the totally geodesic submanifold into $\overline{M}_\gamma = M^{l+1}(k_\gamma)$. Then, considering the standard embeddings into the (pseudo) Euclidean spaces, we get

$$M_\gamma^{d_\gamma}(k_\gamma) \subset E^{d_\gamma + 1(,1)}$$
$$\cap \qquad \cap$$
$$\overline{M}_\gamma^{l+1}(k_\gamma) \subset E^{l+2(,1)}.$$

In this diagram the "vertical" embeddings are totally geodesic ($E^{d_\gamma + 1(,i)}$ passes through the origin in $E^{l+2(,1)}$), and the "horizontal" ones are totally umbilical (and standard). Obviously $M_\gamma = \overline{M}_\gamma \cap E^{d_\gamma + 1(,1)}$ (the intersection in $E^{l+2(,1)}$).

Multiplying these constructions for $\gamma = 1,\ldots,N$, we get the diagram

$$\widetilde{M} = \Pi_{\gamma=1}^N M_\gamma^{d_\gamma}(k_\gamma) \subset E = \Pi_{\gamma=1}^N E^{d_\gamma + 1(,1)},$$
$$\cap \qquad\qquad\qquad \cap$$
$$\overline{M} = \Pi_{\gamma=1}^N M_\gamma^{l+1}(k_\gamma) \subset \overline{E} = \Pi_{\gamma=1}^N E^{l+2(,1)}.$$

The "vertical" embeddings are totally geodesic and the "horizontal" ones are standard. Notice that $\widetilde{M} = \overline{M} \cap E$.

It is convenient to prove Theorem 2 replacing the ambient space $\widetilde{M}$ by $\overline{M}$. It can be easily seen that

a) if N is umbilical in $\widetilde{M}$, then it is also umbilical in $\overline{M}$ with the same tangent space and the mean curvature vector. In particular,

b) the projection of the principal Ricci directions $\{X_j\}_{j=1}^l$ and the vector H of the submanifold $N \subset \overline{M}$ onto the factors $\{\overline{M}_\gamma\}_{\gamma=1}^N$ are equal to those of $N \subset \widetilde{M}$. Moreover, the relations of Lemma 4.18 hold (with the same ξ_γ, k_γ, μ_i, τ_i, α, and ζ);

c) the osculating spaces, the mean curvature vectors, and the position vectors at the point $Q \in N$ to $N \subset E$ and to $N \subset \overline{E}$ coincide.

(Roughly speaking, the differentials of the corresponding embeddings transform the mean curvature vector to $N \subset \widetilde{M}$ into one to $N \subset \overline{M}$ etc.)

Thus, it is enough to prove statement 2) of Theorem 2, replacing $\widetilde{M}$ by $\overline{M}$ and E by $\overline{E}$. This replacement preserves everything except the fullness of $N \subset \overline{M}$.

Denote by $e_\gamma = \pm 1, 0$ the sign of the curvature k_γ, $\gamma = 1,\ldots,N$. Introduce the Cartesian coordinates $\{x^{\gamma i}\}_{\gamma=1}^N{}_{i=1}^{l+2}$ in $\overline{E}$, in which the metric g on $\overline{E}$ of the form

$$ds^2 = \sum_{\gamma=1}^N \left(\sum_{i=1}^{l+1} (dx^{\gamma i})^2 + e_\gamma (dx^{\gamma l+2})^2 \right)$$

(replacing $e_\gamma = 0$ by 1 in the case of $k_\gamma = 0$).

The submanifold $\overline{M} \subset \overline{E}$ can be defined by the system of N equations of the form

$$\sum_{i=1}^{l+1} (x^{\gamma i})^2 + e_\gamma (x^{\gamma l+2},)^2 = k_\gamma^{-1}$$

for $k_\gamma \neq 0$ (in the case of $k_\gamma < 0$ we add the condition $x^{\gamma l+2} > 0$)

or

$$x^{\gamma l+2} = 0$$

for $k_\gamma = 0$.

Choose the orthonormal normal frame $\{ n_\gamma \}_{\gamma=1}^{N}$ on $\overline{M}$ putting

$$n_\gamma^{\delta j} = \begin{cases} 0 & \delta \neq \gamma \\ & \left(\| n_\gamma \|^2 = e_\gamma \right) \text{ for } k_\gamma \neq 0 \text{ and} \\ x^{\gamma j} \mid k_\gamma \mid^{1/2} & \delta = \gamma \end{cases}$$

$n_\gamma^{\gamma l+2} = 1$, the other coordinates to be zero for $k_\gamma = 0$. Denote by $\{ X_\gamma \}_{\gamma=1}^{N}$ the projections of the vector X tangent to $\overline{M}$ onto the spaces tangent to the factors $\{ \overline{M}_\gamma \}_{\gamma=1}^{N}$, respectively.

The second fundamental form of the submanifold $\overline{M} \subset \overline{E}$ is

$$h^{ME}(X, Y) = - \sum_{\gamma=1}^{N} e_\gamma \mid k_\gamma \mid^{1/2} n_\gamma g(X_\gamma, Y_\gamma) \tag{5.1}$$

for the vector fields X, Y tangent to $\overline{M}$.

Consider the case a) of Theorem 2: assume N to be of constant curvature c.

In the notation of Lemma 4.18 the following is true.

Lemma 5.1. 1) *The second fundamental form of the submanifold $N \subset \overline{E}$ is*

$$h^{ME}(X, Y) = \left(H - \sum_{\gamma=1}^{N} \lambda_\gamma^2 e_\gamma \mid k_\gamma \mid^{1/2} n_\gamma \right) g(X, Y)$$

for X, Y tangent to N.

2) *N is a totally umbilical submanifold in $\overline{E}$; its mean curvature vector equals* $\mathbf{H} = H - \sum_{\gamma=1}^{N} \lambda_\gamma^2 e_\gamma \mid k_\gamma \mid^{1/2} n_\gamma$.

3) *The osculating space $\mathbf{L}_Q$ at the point $Q \in N$ to $N \subset E$ is of the dimension $l+1$:* $\mathbf{L}_Q = T_Q N \oplus \mathbf{H}_Q$.

Proof. 1) Let $Q \in N$ and $\{ X_j \}_{j=1}^{l}$ be the principal Ricci directions of N at Q. By (5.1) and 1) of Lemma 4.18, we obtain

$$h^{ME}(X_i, X_j) = Hg(X_i, X_j) - \sum_{\gamma=1}^{N} \lambda_\gamma^2 e_\gamma \mid k_\gamma \mid^{1/2} n_\gamma g(X_{\gamma|i}, X_{\gamma|j}) =$$

$$= \left(H - \sum_{\gamma=1}^{N} \lambda_\gamma^2 e_\gamma \mid k_\gamma \mid^{1/2} n_\gamma \right) g(X_i, X_j).$$

Case 1) has been proved. Case 2) is trivial. Case 3) follows from 1): we only need to verify that $\mathbf{H} \neq 0$. It is true since $\lambda_\gamma > 0$ for every $\gamma = 1, \ldots, N$.

Q. E. D.

The following Lemma concludes the proof of statement 2 a) of Theorem 2.

Lemma 5.2. $N = \overline{M} \cap \mathrm{L}_Q$.

P r o o f. Choose a Cartesian coordinate system $\{x^{\gamma i}\}_{\gamma=1\,i=1}^{N\quad l+2}$ in $\overline{E}$ such that the coordinates of the point Q equal $x^{\gamma i}(Q) = 0$ for $i \le l+1$, $x^{\gamma l+2}(Q) = Q_\gamma$, where either $Q_\gamma = |k_\gamma|^{1/2}$ $(k_\gamma \neq 0)$ or $Q_\gamma = 0$ $(k_\gamma = 0)$; the coordinates of the vectors X_i and H are:

$$(X_i)^{\gamma j} = \begin{cases} \lambda_\gamma, & j = i, \\ 0, & j \neq i, \end{cases} \qquad H^{\gamma j} = \begin{cases} \lambda_{\gamma|H}, & j = l+1, \\ 0, & j \neq l+1, \end{cases} \qquad \gamma = 1,\ldots,N.$$

Then,

$$\mathrm{H}^{\gamma j} = \begin{cases} \lambda_{\gamma|H}, & j = l+1, \\ -e_\gamma \lambda_\gamma^2 |k_\gamma|^{1/2}, & j = l+2, \quad \gamma = 1,\ldots,N \\ 0, & 1 \le j \le l, \end{cases}$$

(here and in the sequel we regard X_i, H, H, λ_γ, $\lambda_{\gamma|H}$, α and so on to be calculated for the point Q).

The osculating space L_Q is defined by the system of $N \times (l+2)$ equations:

$$\left.\begin{cases} x^{\gamma 1} = \lambda_\gamma u^1 \\ x^{\gamma 2} = \lambda_\gamma u^2 \\ \cdot \\ \cdot \\ \cdot \\ x^{\gamma l} = \lambda_\gamma u^l \\ x^{\gamma l+1} = \lambda_{\gamma|H} u^{l+1} \\ x^{\gamma l+2} = Q_\gamma - \lambda_\gamma^2 c_\gamma |k_\gamma|^{1/2} u^{l+1} \end{cases}\right\}_{\gamma=1}^{N}$$

To find the intersection $\overline{M} \cap \mathrm{L}_Q$ we have to solve the system of N equations of the form

$$\sum_{i=1}^{l+1} (x^{\gamma i})^2 + e_\gamma (x^{\gamma l+2})^2 = k_\gamma^{-1}, \qquad k_\gamma \neq 0,$$

or

$$x^{\gamma l+2} = 0, \qquad k_\gamma = 0.$$

The latter equation is obviously true (if $\overline{M}$ has the Euclidean factor). In the case of $k_\gamma \neq 0$ we get for every $\gamma = 1,\ldots,N$:

$$k_\gamma^{-1} = \lambda_\gamma^2 \sum_{i=1}^{l} (u^i)^2 + k_\gamma^{-1} - 2\lambda_\gamma^2 u^{l+1} + (u^{l+1})^2 \left(\lambda_{\gamma|H}^2 + k_\gamma \lambda_\gamma^4\right).$$

Now, by (4.22) of Lemma 4.18, we obtain

$$\sum_{i=1}^{l} (u^i)^2 - 2u^{l+1} + (u^{l+1})^2 c = 0.$$

Let $f(t^1,\ldots,t^l)$ be the continuous function defined as follows:

$$\begin{cases} \sum_{i=1}^{l} (t^i)^2 - 2f + cf^2 = 0, \\ f(0) = 0. \end{cases} \tag{5.2}$$

The l-dimensional submanifold $N' = M \cap L_Q \subset E$ is defined by the system of $N\times(l+2)$ equations of the parameters $\{t^i\}_{i=1}^{l}$

$$\left.\begin{cases} x^{\gamma 1} = \lambda_\gamma t^1 \\ x^{\gamma 2} = \lambda_\gamma t^2 \\ \cdot \\ \cdot \\ \cdot \\ x^{\gamma l} = \lambda_\gamma t^l \\ x^{\gamma l+1} = \lambda_{\gamma|H} f \\ x^{\gamma l+2} = Q_\gamma - \lambda_\gamma^2 e_\gamma \mid k_\gamma \mid^{1/2} f \end{cases}\right\}_{\gamma=1}^{N}$$

By routine calculations one gets the first and the second fundamental forms of the submanifold $N' \subset \overline{M}$

$$g_{ij} = \delta_{ij} + cf_i f_j,$$

$$h_{ij}^{N'M} = (\alpha(1 - cf)f_{ij})\eta,$$

where $f_i = \partial f/\partial t^i$, $f_{ij} = \partial^2 f/\partial t^i \partial t^j$ and η is the unit vector with the coordinates

$$\eta^{\gamma i} = -\lambda_\gamma t^i (c - k_\gamma \lambda_\gamma^2)/\alpha, \quad 1 \le i \le l,$$

$$\eta^{\gamma l+1} = \lambda_{\gamma|H}\left(1 - f(c - k_\gamma \lambda_\gamma^2)\right)/\alpha,$$

$$\eta^{\gamma l+2} = \lambda_\gamma^2 e_\gamma \mid k_\gamma \mid^{1/2} f(c - k_\gamma \lambda_\gamma^2)/\alpha$$

for $\gamma = 1,\ldots,N$. Differentiating (5.2) twice, we obtain

$$\delta_{ij} + cf_i f_j = f_{ij}(1 - cf).$$

Therefore, $h^{N'M} = \alpha\eta g$, i.e. N' is totally umbilical in $\overline{M}$. The submanifold N' passes through the point Q, $T_Q N = T_Q N'$ and the mean curvature vectors of N and N' at the point Q are equal.

The proof is completed by applying Proposition 1.

Q. E. D.

Now consider case 2b). There exist the nonequal principal Ricci curvatures at the general point of N. Compute as above the second fundamental form of the submanifold $N \subset \overline{E}$. Notice that $\overline{M}$ contains no Euclidean factors and the relation of statement 2) of Lemma 4.18 holds.

Lemma 5.3. 1) *The second fundamental form of the submanifold $N \subset \overline{E}$ at the general point is*

$$h^{ME}(X_i, X_j) = \delta_{ij}(\mathbf{H} + \nu_i r),$$

where

$$\mathbf{H} = H + \sum\nolimits_\gamma \xi_\gamma \mid k_\gamma \mid^{-1/2} n_\gamma,$$

$$r = \sum\nolimits_\gamma \mid k_\gamma \mid^{-1/2} n_\gamma \text{ is the position vector,}$$

$$\{X_i\}_{i=1}^{l} \text{ are the principal Ricci directions of } N.$$

2) *The mean curvature vector of $N \subset \overline{E}$ is* $\mathbf{H} + \left(\sum_{k=1}^{l} \nu_k / l\right) r$.

3) *The osculating space $\mathbf{L}_Q$ at the general point $Q \in N$ is of the dimension $l + 2$; it is spanned by the tangent space, the mean curvature vector and the position vector at the point Q: $\mathbf{L}_Q = T_Q N \oplus \mathbf{H}_Q \oplus r_Q$.*

Proof. 1) By (4.9) of Lemma 4.18 and (5.1), we obtain:

$$h^{ME}(X_i, X_j) = Hg(X_i, X_j) - \sum\nolimits_\gamma e_\gamma \mid k_\gamma \mid^{-1/2} n_\gamma\, g(X_{\gamma|i}, X_{\gamma|j}) =$$

$$= \delta_{ij}\left(H + \sum\nolimits'_\gamma \xi_\gamma \mid k_\gamma \mid^{-1/2} n_\gamma + \nu_i \sum\nolimits_\gamma \mid k_\gamma \mid^{-1/2} n_\gamma\right).$$

Case 2) can be proved by direct computation.

By 1), $h^{NE} \subset \mathbf{H} \oplus r$. It can be easily seen that $\mathbf{H}$ and r are not collinear since Q is the general point.

Q. E. D.

To complete the proof of Theorem 2 it remains to prove the following Lemma:

Lemma 5.4. $N = \overline{M} \cap \mathbf{L}_Q$ *where Q is the general point.*

Proof. The plan of proof is similar to that of Lemma 5.2. Choose the coordinate system $\{x^{\gamma i}\}$ in $\overline{E}$ such that the coordinates of the point Q are

$$x^{\gamma i}(Q) = 0, \quad 1 \le i \le l + 1,$$

$$x^{\gamma l+2}(Q) = |k_\gamma|^{-1/2}, \quad \gamma = 1,\ldots,N;$$

those of the principal Ricci directions are

$$(X_i)^{\gamma j} = \begin{cases} 0, & j \neq i, \\ \lambda_{\gamma|i}, & j = i, \end{cases} \qquad \gamma = 1,\ldots,N;$$

those of the mean curvature vector H for $N \subset \overline{M}$ are

$$H^{\gamma j} = \begin{cases} \theta_{\gamma|i}, & 1 \leq i \leq l, \\ \lambda_{\gamma|H}, & i = l+1, \\ 0, & i \neq l+2, \end{cases} \qquad \gamma = 1,\ldots,N.$$

Then the coordinates of the vector $\mathbf{H}$ are

$$\mathbf{H}^{\gamma j} = \begin{cases} \theta_{\gamma|i}, & 1 \leq i \leq l, \\ \lambda_{\gamma|H}, & i = l+1, \\ \xi_\gamma |k_\gamma|^{-1/2}, & i \neq l+2, \end{cases} \qquad \gamma = 1,\ldots,N$$

(all vectors and functions are computed for the point Q).

The osculating space is defined by the system of $N\times(l+2)$ equations

$$\left.\begin{cases} x^{\gamma 1} = \lambda_{\gamma|1} u^1 + \theta_{\gamma|1} u^{l+1} \\ \cdot \\ \cdot \\ \cdot \\ x^{\gamma l} = \lambda_{\gamma|l} u^l + \theta_{\gamma|l} u^{l+1} \\ x^{\gamma l+1} = \lambda_{\gamma|H} u^{l+1} \\ x^{\gamma l+2} = |k_\gamma|^{-1/2} \left(\xi_\gamma u^{l+1} + u^{l+2} + 1\right) \end{cases}\right\}_{\gamma=1}^{N}$$

To find the intersection $L_Q \cap \overline{M}$, one has to solve N equations of the form

$$\sum_{i=1}^{l+1} (x^{\gamma i})^2 + e_\gamma (x^{\gamma l+2})^2 = k_\gamma^{-1}, \quad \gamma = 1,\ldots,N \tag{5.3}$$

(with the additional condition $x^{\gamma i+2} > 0$ in the case of $e_\gamma = -1$).

By the relations of statement 2) of Lemma 4.18,

$$k_\gamma^{-1} = k_\gamma^{-1} \left(\sum_{i=1}^{l} (u^i)^2 (-\xi_\gamma - \nu_i) + 2u^{l+1} \sum_{i=1}^{l} \mu_i u^i + \right.$$

$$\left. + \left(-\zeta - \xi_\gamma \sum_{k=1}^{l} \nu_k/(l-1)\right) (u^{l+1})^2 + 2\xi_\gamma u^{l+1} (u^{l+2}+1) + (u^{l+2}+1)^2 \right).$$

Summing up over $\gamma = 1,\ldots,N$, we obtain

$$0 = \sum_{i=1}^{l} (u^i)^2 + \left(\sum_{k=1}^{l} \nu_k/(l-1)\right)(u^{l+1})^2 - 2u^{l+1}(u^{l+2}+1).$$

Multiplying this by ξ_γ and adding the previous equation, we get

$$1 = -\sum_{i=1}^{l} \nu_i (u^i)^2 + 2u^{l+1}\sum_{i=1}^{l} \mu_i u^i - \zeta(u^{l+1})^2 + (u^{l+2}+1)^2.$$

It can be easily seen that the system of the latter two equations is equivalent to (5.3). Define the continuous functions $\Phi(t^1,\ldots,t^l)$ and $\Psi(t^1,\ldots,t^l)$ by the conditions

$$\begin{cases} \sum_{i=1}^{l} (t^i)^2 + \left(\sum_{k=1}^{l} \nu_k/(l-1)\right)\Phi^2 - 2\Phi\Psi = 0, & (5.4) \\ \sum_{i=1}^{l} \nu_i (t^i)^2 - 2\Phi\sum_{i=1}^{l} \mu_i t^i + \zeta\Phi^2 - \Psi^2 = -1, & (5.5) \\ \Phi(0) = 0, \\ \Psi(0) = 1. \end{cases}$$

Then the l-dimensional submanifold $N' = \overline{M} \cap L_Q \subset \overline{E}$ can be defined by the following system of $N\times(l+2)$ equations of the parameters $t^1,\ldots,t^l$:

$$\left.\begin{cases} x^{\gamma 1} = \lambda_{\gamma|1} t^1 + \theta_{\gamma|1}\Phi \\ \cdot \\ \cdot \\ \cdot \\ x^{\gamma l} = \lambda_{\gamma|l} t^l + \theta_{\gamma|l}\Phi \\ x^{\gamma l+1} = \lambda_{\gamma|II}\Phi \\ x^{\gamma l+2} = |k_\gamma|^{-1/2}(\xi_\gamma\Phi + \Psi) \end{cases}\right\}_{\gamma=1}^{N}$$

The coefficients of the metric tensor of N' are

$$g_{ij} = \left(\sum_{k=1}^{l} \nu_k/(l-1)\right)\Phi_i\Phi_j + \delta_{ij} - \Phi_i\Psi_j - \Phi_j\Psi_i,$$

where $\Phi_i = \partial\Phi/\partial t^i$, $\Psi_i = \partial\Psi/\partial t^i$.

One can compute that the first normal space of the submanifold $N' \subset \overline{M}$ is one-dimensional. It is collinear to the vector η with the coordinates

$$\eta^{\gamma i} = B_\gamma\left(\lambda_{\gamma|i} t^i + \theta_{\gamma|i}\Phi\right) - \Phi_i\lambda_{\gamma|i}, \quad 1 \le i \le l,$$

$$\eta^{\gamma l+1} = B_\gamma\lambda_{\gamma|II}\Phi,$$

$$\eta^{\gamma l+2} = B_\gamma|k_\gamma|^{-1/2}(\xi_\gamma\Phi + \Psi) - |k_\gamma|^{-1/2},$$

where $B_\gamma = \xi_\gamma \left(\Phi - \sum_{i=1}^{l} t^i \Phi_i\right) + \Psi - \sum_{i=1}^{l} \left(\nu_i t^i - \mu_i \Phi\right)\Phi_i$ for $\gamma = 1,\ldots,N$.

This vector is spacelike (because it is tangent to $\overline{M}$) and

$$\|\eta\|^2 = \sum_{i=1}^{l} (\Phi_i)^2 - \sum_{\gamma=1}^{N} k_\gamma^{-1} B_\gamma^2 .$$

Now one can obtain

$$h_{ij}^{N'M} = \left(\sum_{k=1}^{l} \Phi_k t^k - \Phi\right) \|\eta\|^{-2} \eta\, g_{ij} .$$

Thus, N' is totally umbilical in $\overline{M}$. The tangent spaces and the mean curvature vectors of the submanifolds N and N' coincide at the point Q. Hence, by virtue of Proposition 1 $N = N'$.

Q. E. D.

Theorem 2 is proved.

It follows from Theorem 2 that if one defines $l \geq 3$ vectors $\{X_i\}_{i=1}^{l}$ and the vector H satisfying the *necessary* conditions of Lemma 4.18 at some point Q of the space $\widetilde{M}$ (of Theorem 1), then there *exists* a totally umbilical submanifold $N \subset \widetilde{M}$ such that $Q \in N$, $T_Q N = \mathrm{Span}\,(X_1, X_2, \ldots, X_l)$, and the mean curvature vector of N at the point Q equals H.

Thus, to find the umbilical submanifold of the space $\widetilde{M}$ (one of the types of Theorem 1) it is enough to choose the numbers $\lambda_{\gamma|i}$, $\theta_{\gamma|i}$, $\lambda_{\gamma|H}$, ν_i, ξ_γ, μ_i, α, and ζ satisfying the relations of Lemma 4.18 (which are in fact the consequences of Proposition 3 equations). From simple algebraic argument one can see that it is always possible.

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