Математическая физика, анализ, геометрия
1994, т. 1, № 3/4, с. 479–504

УДК 517.5

Generalization of the Darboux transform

Anne Boutet de Monvel

Université Paris 7, Mathématiques, case 7012, 2, Place Jussieu, F-75251, Paris Cedex 05, France

Vladimir Marchenko

B. Verkin Institute for Low Temperature Physics, 47, Lenin Avenue, 310164, Kharkov, Ukraine

Received February 7, 1994

We construct generalization of the Darboux transform depending on functional parameters, which transfer the solutions of a given n-th order differential equation into solutions of another equation whose coefficients depend on chosen functional parameters.

0. Introduction

In 1882 G.Darboux [1] proved the following result.

Theorem (G. Darboux). *Let* (1) *be a differential equation depending on a parameter z*

$$-y'' + q(x)y = zy, \tag{1}$$

and let $y_0(x)$ be an arbitrary solution of this equation with $z = z_0$. We set

$$D = \frac{d}{dx} - \gamma(x) \ , \ with \quad \gamma(x) = \frac{d}{dx} \ln y_0(x) = y_0^{-1}(x) y_0'(x). \tag{2}$$

Then, applying the operator D to any solution $y(z,x)$ of equation (1) *with any $z \in$ C, we obtain the solution $\phi(z,x) = \frac{d}{dx}(y(z,x)) - \gamma(x)y(z,x)$ of the equation*

$$-\phi'' - q_1(x)\phi = z\phi \ , \ with \quad q_1(x) = q(x) - 2\frac{d}{dx}\gamma(x) = q(x) - 2\frac{d^2}{dx^2}\ln\gamma(x). \tag{3}$$

This beautiful Theorem was completely forgotten for 70 years. Meanwhile in the thirties the theory of a generalized shift was initiated by J. Delsarte and developed by him and B. M. Levitan. Some special transformation operators were used by them. A general notion of the transformation operator was introduced by A. Povzner. The transformation operators are of the Volterra type; they can be introduced for each pair of Schrödinger equations. The Darboux operator (2) transfers solutions of a given equation into solutions of some other equation which is determined by two numerical parameters only, namely, z_0 and $h = y_0^{-1}(0) \cdot y_0'(0)$.

Repeated application of the Darboux transform transfers a solution of a given equation into a solution of a new equation depending on a finite set of parameters. M. Crum [2] has shown that the result of repeated application of the Darboux transform can be compactly written as a Wronsky determinant. So in similar cases solutions of given equations can be transferred into solutions of other equations whose coefficients depend only on a finite set of number parameters. Nevertheless, the Darboux transform has found a wide range of applications in obtaining explicit solutions of nonlinear equations.

In this way V. Matveev [3] found interesting solutions of many nonlinear equations. A. Shabat and A. Veselov [4] have recently found a generalization of chains of Darboux transforms which has found interesting applications in the spectral theory. These transforms depend on a finite number of parameters too.

The aim of this paper is to generalize the Darboux transformation which will depend on some functional parameters. This generalization permits us to obtain a wider class of solutions of nonlinear equations.

The paper consists of three parts. In the first part the abstract form of the Darboux transform is given. In the second part we introduce an elementary projection operation, which is of prime importance later, and in the third part, on the basis of this, the Darboux transforms depending on functional parameters are introduced.

1. The abstract Darboux transform

Denote by U an arbitrary associative algebra over the field C of complex numbers with the unity 1. A mapping L of the algebra U into itself is called an *operator* if

$$L(\lambda x + \mu y) = \lambda L(x) + \mu L(y)$$

for all $x,y \in$ U and $\lambda, \mu \in$ C. The set of all operators is denoted by L(U). It forms an algebra with respect to the conventional operations of addition and multiplication. The operator I defined by the equality $I(x) \equiv x$ plays the role of unity in the algebra L(U). An element $\Gamma \in$ U is called *invertible* if there exists an element $\Gamma^{-1} \in$ U such that $\Gamma^{-1}\Gamma = \Gamma\Gamma^{-1} = 1$. Every element $a \in$ U generates the right, a_r, and the left, a_l, multiplication operators defined by the equalities

$$a_r(x) = xa, \qquad a_l(x) = ax.$$

The associativity of the algebra implies that the operators a_r and a_l commute ($a_l b_r = a_r b_l$) for any $a,b \in$ U. It is obvious that $(a_r)^{-1} = (a^{-1})_r$, $(a_l)^{-1} = (a^{-1})_l$ if a is invertible.

An operator $\nu \in$ L(U) is called a *homomorphism* if $\nu(xy) = \nu(x)\nu(y)$ for all $x,y \in$ U. The set of all homomorphisms is denoted by Hom (U). An operator $\partial \in$ L(U) is called a *derivative* if

$$\partial(xy) = \partial(x)y + x\partial(y)$$

for all $x,y \in$ U. The set of all derivatives is denoted by Der(U).

D e f i n i t i o n 1. *An operator* $\delta \in$ L(U) *is called a generalized derivation if there exist such operators* $\alpha,\beta \in$ L(U) *that for all* $x,y \in$ U

$$\delta(xy) = \delta(x)\alpha(y) + x\beta(y).$$

The set of all the *generalized derivations* with given operators α, β is denoted by $\Delta(\alpha, \beta)$.

For example all the operators a_r, a_l, $\nu \in \mathrm{Hom}(U)$, $\partial \in \mathrm{Der}(U)$, $\nu - I$, $a_l + \lambda\partial$, $a_l\nu$ are generalized derivatives with $a_r \in \Delta(0,a_r)$ $a_l \in \Delta(I,0)$, $\nu \in \Delta(\nu,0)$, $\partial \in \Delta(I,\partial)$, $\nu - I \in \Delta(\nu, \nu - I)$, $a_l\nu \in \Delta(\nu,0)$, $a_l + \lambda\partial \in \Delta(I,\lambda\partial)$.

The ordinary Leibnitz formula for $\partial^n(xy)$ in the case of a generalized derivation $\delta \in \Delta(\alpha,\beta)$ is:

$$\partial^n(xy) = \sum_{j=0}^{n} \partial^{n-j}(x)\,[\alpha^{n-j}, \beta^j](y), \tag{1.1}$$

where the operators $[\alpha^{n-j}, \beta^j] \in \mathrm{L(U)}$ are sums of all distinct products containing $n - j$ operators α and j operators β:

$$[\alpha^n, \beta^0] = \alpha^n;\ [\alpha^{n-1}, \beta^1] = \alpha^{n-1}\beta + \alpha^{n-2}\beta^2 + \ldots + \beta\alpha^{n-1};\ \ldots$$

In particular, if $\alpha\beta = \beta\alpha$, then

$$[\alpha^{n-j}, \beta^j] = \frac{n!}{(n-j)!j!}\alpha^{n-j}, \beta^j.$$

D e f i n i t i o n 2. *Let Γ be an invertible element of the algebra* U *and $\delta \in \Delta(\alpha,\beta)$. The element*

$$\gamma = \Gamma^{-1}\delta(\Gamma)$$

is called the logarithmic derivative of Γ with respect to the generalized derivative δ.

R e m a r k. It follows from the definition of the generalized derivative that if $\delta \in \Delta(a,b)$, then

$$\delta(\Gamma) = \delta(\Gamma 1) = \delta(\Gamma)\alpha(1) + \Gamma\beta(1)$$

and

$$\gamma = \gamma\alpha(1) + \beta(1), \qquad \gamma\alpha(1) = \gamma - \beta(1). \tag{1.2}$$

In what follows we denote by Γ the invertible elements of the algebra U and by $\gamma = \Gamma^{-1}\delta(\Gamma)$ their logarithmic derivatives.

The invertible elements $\Gamma \in$ U generate inner homomorphisms in the algebra L(U) that transfer the operators $L \in$ L(U) into the operators

$$\tilde{L} = \Gamma_l^{-1} L \Gamma_l.$$

The above operators obviously satisfy the equality

$$\tilde{L}(x) = \Gamma^{-1} L(\Gamma x).$$

Let now $\delta \in \Delta(\alpha,\beta)$ and $\tilde{\delta} = \Gamma_l^{-1}\delta\Gamma_l$. Then,

$$\tilde{\delta}(x) = \Gamma^{-1}\delta(\Gamma x) = \Gamma^{-1}(\delta(\Gamma)\alpha(x) + \Gamma\beta(x)) = \gamma\alpha(x) + \beta(x) = (\gamma_l\alpha + \beta)(x)$$

and the operator $\tilde{\delta}$ may be expressed in terms of the logarithmic derivative $\gamma = \Gamma^{-1}\delta(\Gamma)$ as follows:

$$\tilde{\delta} = \Gamma_l^{-1}\delta\Gamma_l = \gamma_l\alpha + \beta.$$

It follows from the obvious equality $\tilde{\delta}^k = \Gamma_l^{-1}\,\delta^k\,\Gamma_l$ that

$$(\gamma_l\alpha+\beta)^k(1) = \tilde{\delta}^k(1) = \Gamma^{-1}\,\delta^k(\Gamma 1) = \Gamma^{-1}\,\delta^k(\Gamma) \tag{1.3}$$

and

$$(\gamma_l\alpha+\beta)(\Gamma^{-1}\,\delta^k(\Gamma)) = (\gamma_l\alpha+\beta)^{k+1}(1) = \Gamma^{-1}\,\delta^{k+1}(\Gamma). \tag{1.4}$$

The operators R of the form

$$R = \sum_{k=0}^{n} a(k)_r\,\delta^k, \qquad \left(R(x) = \sum_{k=0}^{n} \delta^k(x)a(k)\right),$$

where $\delta \in \Delta(\alpha,\beta)$ and $a(k)$ are arbitrary elements of the algebra U, are called the *generalized differential operators.*

Definition 3. *The first order generalized differential operator D defined by the equality*

$$D(x) = \delta(x) - x\Gamma^{-1}\,\delta(\Gamma) = \delta(x) - x\gamma \tag{1.5}$$

is called the Darboux operator generated by the g.d. $\delta \in \Delta(\alpha,\beta)$ and the invertible element $\Gamma \in U$. The element $\phi = D(x)$ is called the Darboux transform of x.

Remark. Let $\delta \in \Delta(\alpha,\beta)$. Then,

$$D(x) = \delta(x) - x\gamma = \delta(x1) - x\gamma = \delta(x)\alpha(1) + x\beta(1) - x\gamma = \delta(x)\alpha(1) - x(\gamma - \beta(1))$$

and, according to (1.2),

$$D(x) = D(x)\alpha(1). \tag{1.6}$$

Lemma 1. *The following equalities are valid for all $k = 0,1,2,\ldots$*

$$\delta^k(x) - x\Gamma^{-1}\,\delta^k(\Gamma) = Q_{k-1}(\phi), \tag{1.7}$$

where the generalized differential operators Q_p are defined by the formulae

$$\begin{cases} Q_{-1}(\phi) = 0; \qquad Q_0(\phi) = \phi; \\ Q_p(\phi) = \delta^p(\phi) + \sum_{j=0}^{p-1} \delta^j(\phi\alpha(\gamma_l\alpha+\beta)^{p-j}(1)) \quad (p \geq 1) \end{cases} \tag{1.8}$$

and satisfy the recurrent equalities

$$Q_p(\phi) = \delta Q_{p-1}(\phi) + \phi\alpha(\gamma_l\alpha+\beta)^p(1).$$

Proof. The equality (1.7) is trivial for $k = 0$ ($x = x$) and is equivalent to (1.5) for $k = 1$. Now, if (1.7) is valid for $k = p$, then

$$\delta^{p+1}(x) - \delta(x\Gamma^{-1}\,\delta^p(\Gamma)) = \delta Q_{p-1}(\phi)$$

and, since $\delta \in \Delta(\alpha,\beta)$, we have

$$\delta(x\Gamma^{-1}\delta^p(\Gamma)) = \delta(x)\alpha(\Gamma^{-1}\delta^p(\Gamma)) + x\beta(\Gamma^{-1}\delta^p(\Gamma)) =$$
$$= \phi\alpha(\Gamma^{-1}\delta^p(\Gamma)) + x(\gamma\alpha(\Gamma^{-1}\delta^p(\Gamma))) + x\beta(\Gamma^{-1}\delta^p(\Gamma)) =$$
$$= \phi\alpha(\Gamma^{-1}\delta^p(\Gamma)) + x(\gamma_l\alpha + \beta)(\Gamma^{-1}\delta^p(\Gamma)).$$

It follows from here, according to (1.3), (1.4), that

$$\delta(x\Gamma^{-1}\delta^p(\Gamma)) = \phi\alpha(\gamma_l\alpha + \beta)^p(1) + x\Gamma^{-1}\delta^{p+1}(\Gamma)$$

and

$$\delta^{p+1}(x) - x\Gamma^{-1}\delta^{p+1}(\Gamma) = \delta Q_{p-1}(\phi) + \phi\alpha(\gamma_l\alpha + \beta)^p(1) = Q_p(\phi).$$

Here

$$Q_p(\phi) = \delta Q_{p-1}(\phi)\phi\alpha(\gamma_l\alpha + \beta)^p(1) =$$
$$= \delta^p(\phi) + \sum_{j=0}^{p-2}\delta^{j+1}(\phi\alpha(\gamma_l\alpha + \beta)^{p-j-1}(1)) + \delta^0(\phi\alpha(\gamma_l\alpha + \beta)^p(1)) =$$
$$= \delta^p(\phi) + \sum_{i=0}^{p-1}\delta^i(\phi\alpha(\gamma_l\alpha + \beta)^{p-i}(1)).$$

Corollary. *The identities*

$$R(x) - x\Gamma^{-1}R(\Gamma) = \sum_{k=0}^{n} Q_{k-1}(\phi)a(k) \tag{1.9}$$

are valid for all the generalized differential operators R.

Indeed, we get (1.9) just by multiplying (1.7) by $a(k)$ and summing up the results.

Now we will derive the explicit form of the operators Q_p for the most interesting generalized derivations.

Let $\delta \in \Delta(\alpha, 0)$. Then, $\gamma_l\alpha + \beta = \gamma_l\alpha$, and since in this case

$$\delta(xy) = \delta(x)\alpha(y), \ldots, \delta^k(xy) = \delta^k(x)\alpha^k(y), \tag{1.10}$$

then we have

$$\delta^k(\phi\alpha(\gamma_l\alpha + \beta)^{p-k}(1)) = \delta^k(\phi)\alpha^{k+1}(\gamma_l\alpha)^{p-k}(1)$$

and

$$Q_p(\phi) = \delta^p(\phi) + \sum_{i=0}^{p-1}\delta^i(\phi)\alpha^{i+1}(\gamma_l\alpha)^{p-i}(1).$$

It follows from (1.6) and (1.10) that

$$\delta^p(\phi) = \delta^p(\phi\alpha(1)) = \delta^p(\phi)\alpha^{p+1}(1).$$

This allows us to transform the previous formula into

$$Q_p(\phi) = \sum_{i=0}^{p} \delta^i(\phi)\alpha^{i+1}(\gamma_l\alpha)^{p-i}(1).$$

Let $\delta \in \Delta(I,\beta)$. Then,

$$\gamma_l\alpha + \beta = \gamma_l + \beta, \qquad \alpha\beta = \beta\alpha, \qquad [\alpha^{n-j}, \beta^j] = \beta^j C_n^j$$

and, according to (1.1), we have

$$\delta^i(\phi\alpha(\gamma_l\alpha + \beta)^{p-i}(1)) = \sum_{k=0}^{i} \delta^{i-k}(\phi)C_i^k\beta^k(\gamma_l + \beta)^{p-i}(1) =$$

$$= \sum_{k=0}^{i} \delta^k(\phi)C_i^{i-k}\beta^{i-k}(\gamma_l + \beta)^{p-i}(1).$$

Hence,

$$Q_p(\phi) = \sum_{i=0}^{p}\sum_{k=0}^{i} \delta^k(\phi)C_i^{i-k}\beta^{i-k}(\gamma_l + \beta)^{p-i}(1) =$$

$$= \sum_{k=0}^{p} \delta^k(\phi)\sum_{i\geq k}^{p} C_i^{i-k}\beta^{i-k}(\gamma_l + \beta)^{p-i}(1) =$$

$$= \sum_{k=0}^{p} \delta^k(\phi)\sum_{m=0}^{p-k} C_{m+k}^m\beta^m(\gamma_l + \beta)^{p-k-m}(1).$$

And finally for $\delta \in \Delta(I,\beta)$,

$$Q_p(\phi) = \sum_{j=0}^{p} \delta^j(\phi)\sum_{k=0}^{p-j} \frac{(k+j)!}{k!j!}\beta^k(\gamma_l + \beta)^{p-j-k}(1).$$

Thus,

$$Q_p(\phi) = \sum_{j=0}^{p} \delta^j(\phi)q_p(j), \tag{1.11}$$

where

$$q_p(j) = \begin{cases} \alpha^{j+1}(\gamma_l\alpha)^{p-j}(1) & \text{if } \delta \in \Delta(\alpha,0), \\ \sum_{k=0}^{p-j} \frac{(k+j)!}{k!j!}\beta^k(\gamma_l + \beta)^{p-j-k}(1) & \text{if } \delta \in \Delta(I,\beta). \end{cases} \tag{1.12}$$

Theorem 1. *If x, $\Gamma \in$ U satisfy the equations*

$$R(x) = \delta_1(x), \qquad R(\Gamma) = \delta_2(\Gamma), \tag{1.13}$$

with

$$R(x) = \sum_{k=0}^{n} \delta^k(x) a(k),$$

$\delta \in \Delta(\alpha, \beta)$, $\delta_1 \in \Delta(\alpha_0, \beta_0)$, $\delta_2 \in \Delta(\alpha_0, \beta_0)$, *and* $\delta_1 \delta = \delta\delta_1$, $\delta_2 \delta = \delta\delta_2$,

then the Darboux transform

$$\phi = \delta(x) - x\Gamma^{-1}\delta(\Gamma) = \delta(x) - x\gamma$$

of x satisfies the equation

$$\delta_1(\phi) = \sum_{k=0}^{n} \{ Q_k(\phi)\alpha(a(k)) + Q_{k-1}(\phi)(\beta(a(k)) - a(k)\alpha_0(\gamma)) \}.$$

P r o o f. Applying the operator δ to equations (1.13), we get

$$\delta R(x) = \delta_1(\delta(x)) = \delta_1(x\gamma + \phi) =$$

$$= \delta_1(x)\alpha_0(\gamma) + x\beta_0(\gamma) + \delta_1(\phi) = R(x)\alpha_0(\gamma) + x\beta_0(\gamma) + \delta_1(\phi),$$

$$\delta R(\Gamma) = \delta_2(\delta(\Gamma)) = \delta_2(\Gamma\gamma) = \delta_2(\Gamma)\alpha_0(\gamma) + \Gamma\beta_0(\gamma) = R(\Gamma)\alpha_0(\gamma) + \Gamma\beta_0(\gamma).$$

Thus,

$$\delta R(x) - R(x)\alpha_0(\gamma) - x\beta_0(\gamma) = \delta_1(\phi),$$

$$\delta R(\Gamma) - R(\Gamma)\alpha_0(\gamma) - \Gamma\beta_0(\gamma) = 0,$$

and

$$R_1(x) - x\Gamma^{-1} R_1(\Gamma) = \delta_1(\phi),$$

where R_1 is the generalized differential operator

$$R_1(y) = \delta R(y) - R(y)\alpha_0(\gamma) - y\beta_0(\gamma) =$$

$$= \sum_{k=0}^{n} \left\{\delta(\delta^k(y)a(k)) - \delta^k(y)a(k)\alpha_0(\gamma)\right\} - y\beta_0(\gamma) =$$

$$= \sum_{k=0}^{n} \left\{\delta^{k+1}(y)\alpha(a(k)) + \delta^k(y)(\beta(a(k)) - a(k)\alpha_0(\gamma))\right\} - \delta^0(y)\beta_0(\gamma).$$

It follows from here due to (1.9) that

$$\delta_1(\phi) = \sum_{k=0}^{n} \{ Q_k(\phi)\alpha(a(k)) + Q_{k-1}(\phi)(\beta(a(k)) - a(k)\alpha_0(\gamma)) \}.$$

We denote by

$$(x;k) = (x_1, x_2, \ldots, x_m; k_1, k_2, \ldots, k_m), \quad x_i \in \mathbf{R},\ k_i \in \mathbf{Z},$$

the points of the direct product $\mathbf{R}^m \times \mathbf{Z}^m$ of the m-dimensional Euclidean space and the m-dimensional lattice.

Let $O \subset \mathbf{R}^m$ be an open domain. The set $C^\infty B(H)$ of all the infinitely differentiable in the variables x_i mappings $a(x;k)$ of the space $O \subset \mathbf{Z}^m$ into the algebra $B(H)$ of bounded linear operators acting in the Hilbert space H form an algebra. The operations of addition and multiplication, and the differentiations ∂_i are defined in this algebra by the formulae:

$$(\lambda a_1 + \mu a_2)(x;k) = \lambda a_1(x;k) + \mu a_2(x;k),$$

$$(a_1 a_2)(x;k) = a_1(x;k) a_2(x;k),$$

$$\partial_i(a)(x;k) = \frac{\partial a(x;k)}{\partial x_i},$$

where λ and μ are arbitrary complex numbers.

The subalgebra of all constant mappings coincides with the algebra $B(H)$.

An arbitrary (nonlinear in the general case) mapping $\delta(k)$ of the space $\mathbf{Z}^m$ into itself induces the homomorphism δ of $C^\infty B(H)$ into itself:

$$\delta(a)(x;k) = a(x;\delta(k)).$$

An important special case form the homomorphisms δ_i induced by the mappings

$$\delta_i(k) = (k_1, k_2, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_m).$$

The operators of the differentiation ∂_i, homomorphisms δ_i, and the operators of the right and left multiplication by the elements of $B(H)$ commute with each other. If the space H is of a finite dimension r, the algebra $C^\infty B(H)$ coincides with that of the square $(r\times r)$ matrix-valued functions. In particular, if $r = 1$, then it is the algebra of scalar functions.

E x a m p l e. Let the functions $\Gamma, \Psi \in C^\infty B(H)$ satisfy the equations

$$\sum_{i=0}^{n} \partial_1^i(\Gamma) a(i) = z^{(0)}\Gamma + \lambda \partial_2(\Gamma),$$

$$\sum_{i=0}^{n} \partial_1^i(\Psi) a(i) = z\Psi + \lambda \partial_2(\Gamma),$$

where

$$a(i) \in C^\infty(B(H)), \quad z, z^{(0)} \in B(H), \quad \lambda \in \mathbf{C}.$$

Since $z_l^{(0)} + \lambda\partial_2$ and $z_l + \lambda\partial_2$ belong to the same set $\Delta(I, \lambda\partial_2)$ and commute with ∂_1, it follows from Theorem 1 that the Darboux transform

$$\Phi = \partial_1(\Psi) - \Gamma^{-1}\partial_1(\Gamma)$$

satisfies the equation

$$\sum_{i=0}^{n} \{ Q_i(\Phi) a(i) + Q_{i-1}(\Phi) D_1(a(i)) \} = z\Phi + \lambda\partial_2(\Phi),$$

where $Q_i(\Phi)$ are defined by formulae (1.11), (1.12) with

$$\gamma = \Gamma^{-1}\partial_1(\Gamma), \qquad \beta = \partial_1,$$

and

$$D_1(a(i)) = \partial_1(a(i)) - a(i)\gamma.$$

In particular, for $\lambda = 0$, $n = 2$ we get the Darboux Theorem.

If we take the homomorphism δ_1 instead of the differentiation ∂_1, we will obtain the same kind of results but for the finite difference equations.

2. The extension of the algebra U and the projection operation

The generalization of the Darboux transform is based on extension of the initial algebra U to some algebra $U_1 \supset U$ and applying the so-called projection operation of the algebra U_1 onto its subalgebra U.

Let $P \in U_1$ be an idempotent ($P = P^2$). The set of all the elements of the form PaP ($a \in U_1$) is a subalgebra of the algebra U_1. We denote it by $U = PU_1P$. Its unity is the idempotent P. The mapping $a \to PaP$ is called the projection operation of the algebra U_1 onto its subalgebra $U = PU_1P$. The use of this operation is based on the following simple lemma:

Lemma 2. *Assume that the equation*

$$\sum_{i=0}^{n} L_i(u)a(i) = 0, \tag{2.1}$$

where $L_i \in L(U_1)$, $a(i) \in U_1$, *has the solution* $u \in U_1$ *satisfying the condition*

$$Pu = PuP, \qquad P = P^2.$$

If the operators P_r *and* P_l *commute with* L_i $(1 \le i \le n)$, *then the element* $v = PuP$ *is the solution of the equation*

$$\sum_{i=0}^{n} L_i(v)Pa(i)P = 0 \tag{2.2}$$

in the subalgebra $U = PU_1P$.

P r o o f. As the operators L_i commute with P_r, P_l and $Pu = PuP$, we have

$$PL_i(u) = L_i(Pu) = L_i(PuP) = L_i(PuPP) = L_i(PuP)P = L_i(v)P.$$

Multiplying equation (2.1) by P on both sides, we obtain the equality

$$0 = P\left(\sum_{i=0}^{n} L_i(u)a(i)\right)P = \sum_{i=0}^{n} PL_i(u)a(i)P = \sum_{i=0}^{n} L_i(v)Pa(i)P,$$

from where it follows that the element $v = PuP \in U = PU_1P$ satisfies equation (2.2).

The simplest extension of the algebra U is the algebra $\mathrm{Mat}_N(U)$ of the N-th order matrices $A = (a_{ij})$ over U (with the elements $a_{ij} \in U$). The initial algebra U may be identified by the correspondence

$$a \longleftrightarrow \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

with the subalgebra $P\mathrm{Mat}_N(\mathrm{U})P$ of $\mathrm{Mat}_N(\mathrm{U})$, where

$$P = P^2 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.3}$$

Any matrix $L = (L_{ij})$ of the order N with elements $L_{ij} \in \mathrm{L(U)}$ generate the operator $L \in \mathrm{L}(\mathrm{Mat}_N(\mathrm{U}))$ defined by

$$L(A) = (L(A)_{ij}) = \left(\sum_{s=0}^{n} L_{is}(a_{sj}) \right).$$

We will denote by M_N (respectively A_N) the diagonal matrix from $\mathrm{L}(\mathrm{Mat}_N(\mathrm{U}))$ (respectively $\mathrm{Mat}_N(\mathrm{U})$) with one and the same element $M \in \mathrm{L}(U)$ (respectively $A \in \mathrm{U}$) on the diagonal. One can easily check that the diagonal matrix $\delta \in \mathrm{L}(\mathrm{Mat}_N(\mathrm{U}))$ with the generalized derivations $\delta_1, \delta_2, \dots, \delta_N \in \Delta(\alpha, \beta)$ (with the same α, β) on the diagonal is a generalized derivation in the algebra $\mathrm{Mat}_N(\mathrm{U})$ and belongs to the set $\Delta(\alpha_N, \beta_N)$.

Let now $\delta_i \in \Delta(\alpha, \beta) \subset \mathrm{L(U)}$, $f_i \in \mathrm{U}$ $(1 \le i \le N)$ and

$$\delta = \begin{pmatrix} \delta_1 & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \delta_N \end{pmatrix}, \qquad F = \begin{pmatrix} f_1 & 0 & \dots & 0 \\ f_2 & 0 & \dots & 0 \\ . & . & \dots & . \\ f_N & 0 & \dots & 0 \end{pmatrix}. \tag{2.4}$$

D e f i n i t i o n 4. *The matrices* $W \in \mathrm{Mat}_N(\mathrm{U})$ *of the form*

$$F = \begin{pmatrix} \delta_1^{N-1}(f_1) & \delta_1^{N-2}(f_1) & \dots & f_1 \\ . & . & \dots & . \\ . & . & \dots & . \\ \delta_N^{N-1}(f_N) & \delta_N^{N-2}(f_N) & \dots & f_N \end{pmatrix} \tag{2.5}$$

are called the Wronsky matrices and are denoted by $W(\delta \mid F)$.

Lemma 3. *All the Wronsky matrices satisfy the equality*

$$\delta(W(\delta \mid F)) = \delta^N(F)P + W(\delta \mid F)S(1_N - P) = \delta^N(W(\delta \mid F))SP + W(\delta \mid F)S(1_N - P), \tag{2.6}$$

where 1_N *is the unity of the algebra* $\mathrm{Mat}_N(\mathrm{U})$, *while P is defined by equality (2.3) and*

$$S = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

P r o o f. Since multiplication by the matrix S from the right hand side shifts cyclically the columns of a matrix from the algebra $\mathrm{Mat}_N(\mathrm{U})$ by one step to the right, the Wronsky matrices may be represented in the form

$$W(\delta \,|\, F) = \sum_{j=1}^{N} \delta^{N-j}(F)S^{j-1} = \sum_{j=1}^{N} \delta^{N-j}(FS^{j-1}).$$

So,

$$\delta(W(\delta \,|\, F)) = \sum_{j=1}^{N} \delta^{N-j+1}(FS^{j-1}) =$$

$$= \delta^N(F) + \sum_{j=1}^{N-1} \delta^{N-j}(FS^j) = \delta^N(F) - F + \sum_{j=1}^{N} \delta^{N-j}(FS^j) =$$

$$= \delta^N(F) - F + \sum_{j=1}^{N} \delta^{N-j}(FS^{j-1})S = \delta^N(F) - F + W(\delta \,|\, F)S$$

and since (2.3), (2.4), (2.5) yield

$$F = FP = W(\delta \,|\, F)SP,$$

then we get

$$\delta(W(\delta \,|\, F)) = \delta^N(F)P + W(\delta \,|\, F)S(1_N - P) = \delta^N(W(\delta \,|\, F))SP + W(\delta \,|\, F)S(1_N - P).$$

Corollary. *The Darboux transform*

$$\phi = \delta(W(\delta \,|\, G)) + W(\delta \,|\, G)\Gamma^{-1}\delta(\Gamma)$$

of the Wronsky matrix $W(\delta \,|\, G)$ *generated by the generalized derivation* δ *and the invertible Wronsky matrix* $\Gamma = W(\delta \,|\, F)$ *satisfy the equality*

$$\phi = \phi P.$$

In fact, since the Wronsky matrices $W(\delta \,|\, G)$ and $W(\delta \,|\, F)$ satisfy equality (2.6), we have

$$\phi = \delta^N(G)P + W(\delta \,|\, G)S(1_N - P) - W(\delta \,|\, G)\{\Gamma^{-1}\delta^N(F)P + S(1_N - P)\} =$$

$$= \{\delta^N(G) - W(\delta \,|\, G)\Gamma^{-1}\delta^N(F)\}P = \phi P. \tag{2.7}$$

R e m a r k. The following formula for the logarithmic derivative $\gamma = \Gamma^{-1}\delta(\Gamma)$ of the invertible Wronsky matrix $\Gamma = W(\delta \,|\, F)$ may be derived from (2.6):

$$\gamma = \Gamma^{-1}\delta(\Gamma) = \Gamma^{-1}\delta^N(F)P + S(1_N - P).$$

This means that only the first column of the matrix γ is unknown. Its elements $\gamma_1, \gamma_2, \dots, \gamma_N$ may be found from the linear system

$$\sum_{j=1}^{N} \delta_i^{N-j}(f_i)\gamma_j = \delta_i^N(F_i), \qquad 1 \le i \le N. \tag{2.8}$$

From equality (2.7) it follows that the Darboux transform ϕ has only one non-zero column, i.e. the first one, and its elements $\phi_1, \phi_2, \dots, \phi_N$ may be expressed in terms of γ_i via the formula

$$\phi_p = \delta_p^N(g_p) - \sum_{j=1}^{N} \delta_i^{N-j}(g_p)\gamma_j. \tag{2.9}$$

Finally it follows from (2.8) and (2.9) that the homogeneous system of $N + 1$ equations

$$\begin{cases} (\delta^N(g_p) - \phi_p)y_0 + \sum_{j=1}^{N} \delta^{N-j}(g_p)y_j = 0, \\ (\delta_i^N(f_i)y_0 + \sum_{j=1}^{N} \delta_i^{N-j}(f_i)y_j = 0, \qquad 1 \le i \le N, \end{cases} \tag{2.10}$$

has the nonzero solution

$$y_0 = 1,\ y_1 = -\gamma_1,\ y_2 = -\gamma_2\ ,\ \dots\ ,\ y_N = -\gamma_N.$$

So, if $\mathrm{U} = C^\infty(B(H))$ and the dimension of the space H is equal to 1 (which means that U consists of the scalar functions with the ordinary arithmetic operations), the solution of the system (2.8) may be found by the Cramer formula:

$$\gamma_p = (\mathrm{Det}\Gamma)^{-1}\mathrm{Det}(\Gamma_p),$$

where $\Gamma = W(\delta \,|\, F)$, Γ_p is obtained from Γ by substitution of the elements of the p-th column by $\delta_i^N(f_i)$, and the determinant of the system (2.10) is equal to zero:

$$\mathrm{Det}\begin{pmatrix} \delta_p^N(g_p) - \phi_p & \delta_p^{N-1}(g_p) & \dots & g_p \\ \delta_1^N(f_1) - 0 & \delta_1^{N-1}(f_1) & \dots & f_1 \\ . & . & \dots & . \\ \delta_N^N(f_N) - 0 & \delta_N^{N-1}(f_N) & \dots & f_N \end{pmatrix} = 0.$$

From the latter equality it follows that

$$\mathrm{Det}\begin{pmatrix} \delta_p^N(g_p) & \delta_p^{N-1}(g_p) & \dots & g_p \\ \delta_1^N(f_1) & \delta_1^{N-1}(f_1) & \dots & f_1 \\ . & . & \dots & . \\ \delta_N^N(f_N) & \delta_N^{N-1}(f_N) & \dots & f_N \end{pmatrix} - \mathrm{Det}\begin{pmatrix} \phi_p & \delta_p^{N-1}(g_p) & \dots & g_p \\ 0 & \delta_1^{N-1}(f_1) & \dots & f_1 \\ . & . & \dots & . \\ 0 & \delta_N^{N-1}(f_N) & \dots & f_N \end{pmatrix} = 0$$

and

$$\phi_p = \mathrm{Det}(W(\delta \mid F))^{-1}\mathrm{Det}(W(\tilde{\delta} \mid \tilde{F})), \tag{2.11}$$

where

$$\tilde{\delta} = \begin{pmatrix} \delta_p & & & 0 \\ & \delta_1 & & \\ & & \ldots & \\ 0 & & & \delta_N \end{pmatrix}, \qquad \tilde{F} = \begin{pmatrix} g_p & 0 & \ldots & 0 \\ f_1 & 0 & \ldots & 0 \\ . & . & \ldots & . \\ f_N & 0 & \ldots & 0 \end{pmatrix}.$$

Lemma 4. *If the elements $f_i \in \mathrm{U}$ satisfy the equations*

$$\sum_{k=0}^{N} \delta_i^k(f_i)a(k) = L_i(f_i), \quad 1 \le i \le N,$$

where $a(k) \in \mathrm{U}$, $L_i \in \mathrm{L}(\mathrm{U})$, $\delta_i \in \Delta(\alpha, \beta)$, and $\delta_i L_i = L_i \delta_i$, then the Wronsky matrix $W(\delta \mid F)$ satisfies the equation

$$\sum_{k=0}^{N} \delta^k(W)b(k) = L(W), \tag{2.12}$$

where

$$\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ldots & \\ 0 & & \delta_N \end{pmatrix}, \qquad L = \begin{pmatrix} L_1 & & 0 \\ & \ldots & \\ 0 & & L_N \end{pmatrix},$$

$b(k) = (b_{ij}(k)) \in \mathrm{Mat}_N(\mathrm{U})$, *and*

$$b_{ij}(k) = \begin{cases} [\alpha^{N-i}, \beta^{i-j}](a(k)) & i \ge j \\ 0 & i < j. \end{cases} \tag{2.13}$$

P r o o f. The matrix equality (2.12) is equivalent to the system

$$\sum_{k=0}^{N} \sum_{p=0}^{N} \delta_i^k(W_{ip})b_{pj}(k) = L_i(W_{ij}), \tag{2.14}$$

where

$$W_{ij} = \delta_i^{N-j}(f_i)$$

are the elements of the Wronsky matrix $W(\delta \mid F)$. Since $\delta_i L_i = L_i \delta_i$, we have

$$\begin{aligned} L_i(W_{ij}) &= L_i(\delta_i^{N-j}(f_i)) = \delta_i^{N-j} L_i(f_i) = \\ &= \delta_i^{N-j}\left(\sum_{k=0}^{n} \delta_i^k(f_i)a(k)\right) = \sum_{k=0}^{n} \delta_i^{N-j}(\delta_i^k(f_i)a(k)). \end{aligned} \tag{2.15}$$

It follows from formula (1.1) that

$$\delta_i^{N-j}(\delta_i^k(f_i)a(k)) = \sum_{m=0}^{N-j} \delta_i^{N-j-m}(\delta_i^k(f_i))[\alpha^{N-j-m}, \beta^m](a(k)) =$$

$$= \sum_{m=0}^{N-j} \delta_i^k \delta_i^{N-j-m}(f_i)[\alpha^{N-j-m}, \beta^m](a(k)) =$$

$$= \sum_{m=0}^{N-j} \delta_i^k(W_{i,j+m})[\alpha^{N-j-m}, \beta^m](a(k)) = \sum_{p=j}^{N} \delta_i^k(W_{ip})[\alpha^{N-p}, \beta^{p-j}](a(k))$$

and, according to (2.13),

$$\delta_i^{N-j}(\delta_i^k(f_i)a(k)) = \sum_{p=0}^{N} \delta_i^k(W_{ip})b_{pj}(k).$$

Substituting this expression into the r.h.s. of equality (2.15), we get the desired equality (2.14).

Example. We will denote by $f(z) \in C^\infty(B(H))$ an arbitrary solution of the differential equation

$$\sum_{k=0}^{n} \partial_1^k(f(z))a(k) = zf(z) + \lambda\partial_2(f(z)),$$

where $\lambda \in \mathbf{C}$, $a(k) \in C^\infty(B(H))$, $z \in B(H)$.

Let $W(z^{(1)}, z^{(2)}, \ldots, z^{(N)})$ be the Wronsky matrix obtained by applying the differentiation of ∂_1 to the solutions $f(z^{(1)})$, $f(z^{(2)})$, ... , $f(z^{(N)})$. If $z^{(1)} = z$ and $f(z^{(i)}) \equiv 0$, $2 \le i \le N$, then the corresponding Wronsky matrix will be denoted by $W(z)$.

These matrices belong to the algebra $\mathrm{Mat}_N(C^\infty(B(H)))$ and, since the operators $\partial_1 \in \Delta(I, \partial_1)$, $z + \lambda\partial_2 \in \Delta(I, \lambda\partial_2)$ commute, they satisfy by Lemma 4 the equation

$$\sum_{k=0}^{n} \partial_1^k(W)b(k) = L(W),$$

where $b(k)$ is the triangle-shaped matrix with the elements

$$b_{ij}(k) = \begin{cases} \dfrac{(N-j)!}{(N-i)!(i-j)!}\,\partial_1^{i-j}(a(k)), & i \ge j, \\ 0, & i < j, \end{cases}$$

and L is the diagonal matrix with

$$L_{ii} = \begin{cases} z^{(i)} + \lambda\partial_2 & \text{if } W = W(z^{(1)}, z^{(2)}, \ldots, z^{(N)}), \\ z + \lambda\partial_2 & \text{if } W = W(z). \end{cases}$$

Let the matrix $\Gamma = W(z^{(1)}, z^{(2)},\dots, z^{(N)})$ be invertible in the algebra $\mathrm{Mat}_N(C^\infty B(H))$. Then Theorem 1 yields that the Darboux transform

$$\phi(z) = \partial_1(W(z)) - W(z)\Gamma^{-1}\partial_1(\Gamma)$$

of the matrix $W(z)$ satisfies the equation

$$\sum_{k=0}^{n} \{ Q_k(\phi)b(k) + Q_{k-1}(\phi)D_1(b(k)) \} = (z + \lambda\partial_2)_N(\phi),$$

where

$$D_1(b(k)) = \partial_1(b(k)) - b(k)\gamma\,; \qquad \gamma = \Gamma^{-1}\partial_1(\Gamma);$$

$$Q_k(\phi) = \sum_{k=0}^{n} \partial_1^j(\phi)q_k(j); \qquad q_k(j) = \sum_{p=0}^{k-j} \frac{(p+j)!}{p!j!}\,\partial_1^p(\gamma_l + \partial_1)^{k-j-p}(1).$$

After some elementary transformations this equation can be reduced to the following one:

$$\sum_{k=j}^{n} \partial_1^j(\phi)C(j) = (z + \lambda\partial_2)_N(\phi),$$

where

$$C(j) = \sum_{j=0}^{n} q_k(j)\{ b(k) + D_1(b(k+1)) \}$$

and $b(m) = 0$ for $m > n$.

Note now that only the first row of the matrix $W(z)$ is nonzero. This means that the Darboux transform $\phi(z)$ has only the first row not equal to zero. Hence, $\phi(z) = P\phi(z)$ and, since the corollary of Lemma 3 implies $\phi(z) = \phi(z)P$, we have

$$\phi(z) = P\phi(z)P = \tilde{\phi}(z)P,$$

with $\tilde{\phi}(z) \in C^\infty(B(H))$. Using Lemma 2, we can see now that $\tilde{\phi}(z)$ is the solution of the equation

$$\sum_{k=0}^{n} \partial_1^j(\tilde{\phi})\tilde{C}(j) = z\tilde{\phi} + \lambda\partial_2(\tilde{\phi})$$

in the initial algebra $C^\infty(B(H))$, where the coefficients $\tilde{C}(j) \in C^\infty(B(H))$ are defined by the equation

$$PC(j)P = \tilde{C}(j)P \qquad \tilde{C}(j) \in C^\infty(B(H))$$

(i.e., $\tilde{C}(j)$ is the element of the matrix $C(j)$ standing in the first column of the first row).

If, in particular, the Hilbert space H is of the dimension 1, then z, $z^{(1)}, z^{(2)},\dots, z^{(N)} \in \mathbb{C}$,while $\tilde{\phi}(z)$,$\tilde{C}(j)$ are scalar functions. According to (2.11), we have in this case

$$\tilde{\phi}(z) = \frac{\mathrm{Det}\ W(z, z^{(1)}, z^{(2)},\dots, z^{(N)})}{\mathrm{Det}\ W(z^{(1)}, z^{(2)},\dots, z^{(N)})}\,.$$

One can show now that the transformation $f(z) \to \tilde{\phi}(z)$ is equivalent to sequential appliation of the usual Darboux transform N times.

3. The Darboux transform depending upon functional parameters

All the differential or finite-difference equations we deal with are considered in the algebra $C^\infty(B(H))$. We are going now to construct the Darboux-type transforms for them but this time these transforms will depend not on finite number of number-valued parameters but on finite number of functions.

We introduce the following notations. Let μ be a finite Borel measure with a compact support $\Omega \subset \mathbb{C}$ and $\int d\mu(\xi) = \mu$. Let H be a Hilbert space of the dimension r ($= 1,2,\dots,\infty$) and a scalar product $(\cdot,\cdot)_H$. We denote by H_1 the Hilbert space of the H-valued functions $f(\xi)$, $\xi \in \mathbb{C}$, with the scalar product

$$(f,g)_{H_1} = \int \big(f(\xi),g(\xi)\big)_H \, d\mu(\xi).$$

All the constant functions ($f(\xi) \equiv f \in H$) belong obviously to the space H_1 and form a subspace of H_1 that is naturally isomorphic to H. The operator Π defined by the formula

$$(\Pi f)(\xi) = \mu^{-1} \int f(\xi) d\mu(\xi)$$

is the orthogonal projector on this subspace that we will identify with H. Then the algebra $B(H)_1$ is obviously the extension of $B(H)$ while the algebra $C^\infty(B(H_1))$ — that of $C^\infty(B(H))$. We will identify $B(H)$ with $\Pi B(H_1)\Pi$ and $\Pi C^\infty(B(H_1))\Pi$ with $C^\infty(B(H))$.

One can extend the algebra $C^\infty(B(H_1))$ to $\mathrm{Mat}_N(C^\infty(B(H_1)))$ and identify the initial one with the subalgebra $P\mathrm{Mat}_N(C^\infty(B(H_1)))P$ of $\mathrm{Mat}_N(C^\infty(B(H_1)))$.

In what follows we will use the notations of Section 2. In particular, A_N is the diagonal matrix of $\mathrm{Mat}_N(C^\infty(B(H_1)))$ with the same element $A \in C^\infty(B(H_1))$ on the diagonal.

It is easy to check that the operators P and Π_N commute, while the element $\Pi \in B(H_1)$ is identified with the idempotent $P\Pi_N = \Pi_N P$ of the algebra $C^\infty(B(H_1))$.

Let now be $\mathrm{U} = C^\infty(B(H))$, $\mathrm{U}_1 = C^\infty(B(H_1))$, $\mathrm{U}_2 = \mathrm{Mat}_N(C^\infty(B(H_1)))$. Taking into account the identifications made above, we see that $\mathrm{U} \subset \mathrm{U}_1 \subset \mathrm{U}_2$ and

$$\mathrm{U} = \Pi \mathrm{U}_1 \Pi = \Pi_N P\, \mathrm{U}_2 \Pi_N P = P\Pi_N \mathrm{U}_2 P\Pi_N ; \qquad \mathrm{U}_1 = P\mathrm{U}_2 P.$$

To simplify the calculation, in what follows, we will consider the simplest case when $\dim H = r = 1$ so that the algebra $C^\infty(B(H))$ coincides with that of usual infinitely differentiable scalar functions. In the case of any finite r the algebra $C^\infty(B(H))$ coincides with that of infinitely differentiable matrix-valued functions but all the considerations follow the same way without considerable complications.

Since we consider now the one-dimensional space H, its extension H_1 consists of the scalar functions $f(\xi) \in L_2(d\mu)$.

The algebra $B(H_1)$ contains the commutative subalgebra $T(H_1)$ that consists of all the operators of multiplication by the bounded functions $T(\xi)$:

$$T(\xi)(f) = T(\xi)f(\xi), \qquad \sup_{\xi \in \Omega} | T(\xi) | < \infty.$$

Consequently the algebra $C^\infty(B(H_1))$ contains the commutative subalgebra $C^\infty(T(H_1))$. All the operators of the algebra $B(H_1)$ commute with differentiations ∂_i and homomorphisms δ_i.

Now we will deal with one important special case.
Consider the equations

$$\partial_1^n(y) = \xi^n y, \qquad \partial_1^n(y) = z^n y,$$

where ξ^n, $z^n \in T(H_1)$ are the operators of multiplication by the function ξ^n (the support Ω of $d\mu$ is compact!) and by a constant function $T(\xi) \equiv z^n$, that is by the complex number z^n.

The first equation has the solution $\Gamma \in C^\infty(B(H_1))$ which is an operator-valued function of the form

$$\Gamma = \sum_{k=0}^{n-1} C_k(\xi) \exp(\varepsilon^k \xi x_1) L_k, \tag{3.1}$$

where $\varepsilon = \exp(2\pi i/n)$ and $C_k(\xi)$ are arbitrary functions bounded on the compact Ω, and L_k are some arbitrary operators of the algebra $B(H_1)$.

The second equation is solved by the functions

$$\exp(\varepsilon^k z x_1) \in C^\infty(T(H_1)) \subset C^\infty(B(H_1)),$$

with $k = 0,1,\ldots,n-1$.

Theorem 1 implies that if Γ (see (3.1)) is invertible in the algebra $C^\infty(B(H_1))$, then the Darboux transform

$$\phi(\varepsilon^k z, x_1) = \partial_1 \left(e^{\varepsilon^k z x_1}\right) - e^{\varepsilon^k z x_1}\gamma = e^{\varepsilon^k z x_1}(\varepsilon^k z - \gamma),$$

where

$$\gamma = \Gamma^{-1}\partial_1(\Gamma) \in C^\infty(B(H_1))$$

is the solution of the equation

$$Q_n(\phi) - Q_{n-1}(\phi)\gamma = z^n \phi,$$

since in this case $a(n) = 1$ and $a(k) = 0$ for $k < n$.

According to (1.11), (1.12), this equation, is

$$\sum_{j=0}^{n} \partial_1^j(\phi)(q_n(j) - q_{n-1}(j)\gamma) = z^n \phi, \tag{3.2}$$

where

$$q_p(j) = \begin{cases} \sum_{m=0}^{p-j} \frac{(m+j)!}{m!j!} \partial_1^m(\gamma_l + \partial_1)^{p-j-m}(1), & j \le p, \\ 0, & j > p. \end{cases} \tag{3.2'}$$

Note that the coefficients of the equation obtained above and its solution belong to the algebra $C^\infty(B(H_1))$. But if we manage to find the coefficients in (3.1) so that the conditions of Lemma 2 (with $P = \Pi$) are satisfied, then we will receive, after making the projection, the equation and its solution in the initial algebra $C^\infty(B(H))$ of the usual scalar functions.

Let $C_0(x) \equiv 1$ and $L_0 = I$ in formula (3.1). Then,

$$\Gamma = e^{\xi x_1} + \sum_{k=1}^{n-1} C_k(\xi) e^{\varepsilon^k \xi x_1} L_k,$$

$$\partial_1(\Gamma) = \Gamma\xi + \sum_{k=1}^{n-1} C_k(\xi) e^{\varepsilon^k \xi x_1}(\varepsilon^k \xi L_k - L_k \xi),$$

and if the operators L_k satisfy the equations

$$\varepsilon^k \xi L_k - L_k \xi = r_k \Pi \tag{3.3}$$

(here r_k are arbitrary operators of the algebra $B(H_1)$), then we get

$$\gamma = \xi + \gamma_1 \Pi, \tag{3.3'}$$

with

$$\gamma_1 = \Gamma^{-1} \sum_{k=1}^{n-1} C_k(\xi) e^{\varepsilon^k \xi x_1} r_k = \gamma_1(\xi, x_1).$$

Since the operator Π transforms the vectors $f(\xi)$ of the space H_1 into constants, then $r_k \Pi$ is an integral operator with the kernel $r_k(\xi)\Pi(\xi')$:

$$(r_k \Pi f)(\xi) = \int r_k(\xi)\Pi(\xi') f(\xi') d\mu(\xi'),$$

where $r_k(x) = r_k(1)$ and $\Pi(\xi') = \mu^{-1}$. Hence the operator γ_1 is that of multiplication by the function $\gamma_1(\xi, x_1)$, that can be found from the equation

$$(\Gamma\gamma_1)(\xi) = \sum_{k=1}^{n-1} C_k(\xi) e^{\varepsilon^k \xi x_1} r_k(\xi). \tag{3.4}$$

Now it follows from (3.3′) that if (3.3) holds, then the Darboux transforms are

$$\phi(\varepsilon^k z) = e^{\varepsilon^k z x_1}\left(\varepsilon^k z - \xi - \gamma_1(\xi, x_1)\Pi\right)$$

It is obvious that if the function $\phi(\varepsilon^k z)$ solves equation (3.2), so does

$$\psi(\varepsilon^k z) = (\varepsilon^k z - \xi)^{-1}\phi(\varepsilon^k z) = e^{\varepsilon^k z x_1}\left(1 - (\varepsilon^k z - \xi)^{-1}\gamma_1(\xi, x_1)\Pi\right)$$

and

$$\Pi\psi(\varepsilon^k z) = e^{\varepsilon^k z x_1}(\Pi - \Pi(\varepsilon^k z - \xi)^{-1}\gamma_1(\xi, x_1)\Pi) = \Pi\psi(\varepsilon^k z)\Pi = \tilde{\psi}(\varepsilon^k z)\Pi.$$

Here the scalar function $\tilde{\psi}(\varepsilon^k z) \in C^\infty(B(H))$ equals

$$\tilde{\psi}(\varepsilon^k z) = e^{\varepsilon^k z x_1}\left(1 - \mu^{-1}\int (\varepsilon^k z - \xi)^{-1}\gamma_1(\xi, x_1)d\mu(\xi)\right). \tag{3.5}$$

(We consider only the values of z for which $\varepsilon^k z \notin \Omega$.) According to Lemma 2, the function $\tilde{\psi}(\varepsilon^k z) = \tilde{\psi}(\varepsilon^k z, x_1)$ satisfies the equation

$$\sum_{j=0}^{n} \partial_1^j(\tilde{\psi})\tilde{s}(j) = z^n\tilde{\psi}, \tag{3.6}$$

with the coefficients

$$\tilde{s}(j) = \Pi(q_n(j) - q_{n-1}(j)\gamma)\Pi \in C^\infty(B(H)), \tag{3.6'}$$

which also are scalar functions.

Using the identities

$$z^n - \xi^n \equiv (\varepsilon^k z - \xi)((\varepsilon^k z)^{n-1} + (\varepsilon^k z)^{n-2}\xi + \ldots + \xi^{n-1}),$$

one can transform formula (3.5) to the form

$$\tilde{\psi} = e^{\varepsilon^k z x_1}\left(1 - \mu^{-1}\int \frac{(\varepsilon^k z)^{n-1} + (\varepsilon^k z)^{n-2}\xi + \ldots + \xi^{n-1}}{z^n - \xi^n}\gamma_1(\xi, x_1)d\mu(\xi)\right) =$$

$$= e^{\varepsilon^k z x_1} + \sum_{j=0}^{n-1} \partial_1^j(e^{\varepsilon^k z x_1})\mu^{-1}\int \frac{\xi^{n-1-j}}{\xi^n - z^n}\gamma_1(\xi, x_1)d\mu(\xi).$$

So the solutions $\tilde{\psi} = \tilde{\psi}(\varepsilon^k z, x_1)$ of equation (3.6) may be obtained from the functions $e^{\varepsilon^k z x_1}$ by applying the same differential operator of the order $n - 1$:

$$D_{n-1} = \sum_{j=0}^{n-1} d_j(z^n, x_1)\partial_1^j + 1, \tag{3.7}$$

where

$$d_j(z^n, x_1) = \mu^{-1}\int \frac{\xi^{n-1-j}}{\xi^n - z^n}\gamma_1(\xi, x_1)d\mu(\xi). \tag{3.7'}$$

Since the functions $e^{\varepsilon^k z x_1}$ $(0 \le k \le n - 1)$ form the basis of the solutions of the equation $\partial_1^n(y) = z^n y$, the operator D_{n-1} transforms any solution of this equation into the solution of equation (3.6).

It still remains to find the operators L_k solving equations (3.3) with the integral operators (3.3) with the kernels $r_k(\xi)\Pi(\xi')$ (here $r_k(\xi) = r_k(1)$ and $\Pi(\xi') = \mu^{-1}$) as their r.h.s. We will look for special solutions of these equations in the form of integral operators with the kernels $L_k(\xi, \xi')$. According to (3.3), they may be found from the equations

$$\varepsilon^k\xi L_k(\xi, \xi') - L_k(\xi, \xi')\xi' = r_k(\xi)\Pi(\xi').$$

So the operators with the kernels

$$L_k(\xi, \xi') = \frac{r_k(\xi)\Pi(\xi')}{\varepsilon^k \xi - \xi'}$$

are some special solutions of (3.3). One can add to these operators the solutions of the homogeneous equations

$$\varepsilon^k \xi m_k(\xi, \xi') - m_k(\xi, \xi')\xi' = 0.$$

It is easy to check that these equations are solved by the operators m_k defined by the relations

$$(m_k f)(\xi) = m_k(\xi)\chi_k(\xi) f(\varepsilon^k \xi), \tag{3.8}$$

where $\chi_k(\xi)$ are the characteristic functions of the sets

$$\Omega_k = \{\xi \mid \xi \in \Omega, \varepsilon^k \xi \in \Omega\},$$

while $m_k(\xi)$ is an arbitrary function bounded on this set.

Summing up, we can state that equations (3.3) are solved by the operators L_k of the form

$$(L_k f)(\xi) = m_k(\xi)\chi_k(\xi) f(\varepsilon^k \xi) + r_k(\xi)\mu^{-1} \int \frac{f(\xi')}{\varepsilon^k \xi - \xi'} d\mu(\xi'),$$

on the assumptions that these equalities define correctly some bounded operators $L_k \in B(H_1)$.

The general sufficient conditions of this correctness are obtained in [5]. Supposing that these conditions are fulfilled, we get the following expression for Γ:

$$(\Gamma f)(\xi) = e^{\xi x_1} f(\xi) + \sum_{k=1}^{n-1} e^{\varepsilon^k \xi x_1} \left(\nu_k(\xi)\chi_k(\xi) f(\varepsilon^k \xi) + \rho_k(\xi) \int \frac{f(\xi')}{\varepsilon^k \xi - \xi'} d\mu(\xi') \right),$$

where

$$\nu_k(\xi) = C_k(\xi) m_k(\xi), \qquad \rho_k(\xi) = C_k(\xi) r_k(\xi)\mu^{-1}.$$

Now equation (3.4) that is used to find the function $\gamma_1(\xi) = \gamma_1(\xi, x_1)$ takes the form

$$e^{\xi x_1}\gamma_1(\xi) + \sum_{k=1}^{n-1} e^{\varepsilon^k \xi x_1} \left(\nu_k(\xi)\chi_k(\xi)\gamma_1(\varepsilon^k \xi) + \rho_k(\xi) \int \frac{\gamma_1(\xi')}{\varepsilon^k \xi - \xi'} d\mu(\xi') \right) = \sum_{k=1}^{n-1} e^{\varepsilon^k \xi x_1} \rho_k(\xi). \tag{3.9}$$

To sum up the result we formulate the following

Theorem 2. *If Equation* (3.9) *has the unique solution* $\gamma_1(\xi) = \gamma_1(\xi, x_1)$, *then the differential operator* D_{n-1} *defined by equalities* (3.7) *and* (3.7') *transforms the solutions of the equation* $\partial_1^n(y) = z^n y$ *into the solutions of Equation* (3.6) *with coefficients that can be found from formulae* (3.2'), (3.6') *with* $\xi + \gamma_1(\xi, x_1)$ *substituted for* γ.

Remark. Formulae (3.2′) and (3.6′) are very cumbersome. It is much easier to find the coefficients $\tilde{s}(j)$ by substituting the r.h.s. of relation (3.5) with $k = 0$ into Equation (3.6), multiplying the resulting expression by e^{-zx_1} and noting that the coefficients of all non-negative powers of z must be 0.

For example, if $n = 3$, then the coefficients $\tilde{s}(j)$ of Equation (3.6) are

$$\tilde{s}(3) = 1,\ \tilde{s}(2) = 0,\ \tilde{s}(1) = 3\mu^{-1}\int \gamma_1(\xi, x_1)d\mu(\xi),$$

$$\tilde{s}(0) = 3\mu^{-1}\int (\partial_1^2\gamma_1(\xi, x_1) + \xi\partial_1\gamma_1(\xi, x_1))d\mu(\xi) +$$

$$+ 3\mu^{-2}\int \gamma_1(\xi, x)d\mu(\xi)\int \partial_1\gamma_1(\xi, x_1)d\mu(\xi).$$

The equation $\partial_1^n(y) = z^n y$ whose solutions are transformed by the operator D_{n-1} into the solutions of equation (3.6) (see Theorem 2) is the simplest one among the n-th order equations.

Now we will consider an equation of the general form (in the algebra $C^\infty(B(H))$ of the scalar functions)

$$\partial_1^n(y) - \sum_{j=0}^{n-1} \partial_1^j(y)a_j(x_1) = \xi^n y \tag{3.10}$$

and denote by $\mathrm{E}(\xi) = \mathrm{E}(\xi, x_1)$ its solution with the initial data

$$\partial_1^k(\mathrm{E}(\xi, x_1))\big|_{x_1=0} = \xi^k, \quad 0 \le k \le n-1. \tag{3.11}$$

The functions $\mathrm{E}(\varepsilon^k\xi, x_1)$, with $\varepsilon = \exp 2\pi i/n$, $0 \le k \le n-1$, and $\xi \ne 0$, form a basis of the solutions of this equation. Let $W = W(\xi, x_1)$ be its Wronsky matrix made of these functions:

$$W = \begin{pmatrix} \partial_1^{n-1}(\mathrm{E}(\xi, x_1)) & \dots & \mathrm{E}(\xi, x_1) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \partial_1^{n-1}(\mathrm{E}(\varepsilon^{n-1}\xi, x_1)) & \dots & \mathrm{E}(\varepsilon^{n-1}\xi, x_1) \end{pmatrix}. \tag{3.12}$$

According to Lemma 4, the matrix W satisfies the equation

$$\partial_1^n(W) - \sum_{j=0}^{n-1} \partial_1^j(W)b_j(x_1) = \xi^n W, \tag{3.13}$$

with the coefficients $b_j(x_1) \in \mathrm{Mat}_n(C^\infty(B(H)))$ that can be expressed in terms of the function $a_j(x_1)$ using formula (2.13).

We can pass to the algebras $\mathrm{Mat}_n(C^\infty(B(H_1)))$ and $C^\infty(B(H_1)))$ and regard ξ^n and $E(\varepsilon^k\xi, x_1)$ as operators of multiplication by these functions. Then the matrix W appears to be a solution of Equation (3.13) in the algebra $\mathrm{Mat}_n(C^\infty(B(H_1)))$. Any matrix

$c(\xi) \in \mathrm{Mat}_N(B(H_1))$ with the elements that are the operators of multiplication by the functions $c_{ij}(\xi)$ commutes with the operators ∂_1 and ξ^n, while any diagonal matrix L_n with the same operator $L \in B(H_1)$ on its diagonal commutes with the matrices $b_j(x_1)$ and the operator ∂_1. Therefore the operator-valued matrices

$$\Gamma = W(\xi, x_1) + C(\xi)W(\xi, x_1)L_n \tag{3.14}$$

satisfy equation (3.13) as well. According to (2.6), we have

$$\partial_1(W) = \partial_1^n(W)SP + WS(1_n - P),$$

where $\partial_1^n(W)SP$ is the matrix that has all the columns but the first one is equal to zero, while the first one is composed of the elements $\partial_1^n(\mathrm{E}(\varepsilon^k\xi, x_1))$. It follows from Equation (3.10) that

$$\partial_1^n(\mathrm{E}(\varepsilon^k\xi, x_1)) = \sum_{j=0}^{n-1} \partial_1^j(\mathrm{E}(\varepsilon^k\xi, x_1))a_j(x_1) + \xi^n\mathrm{E}(\varepsilon^k\xi, x_1),$$

so that

$$\partial_1^n(W)SP = W\left(A(x_1) + SP\xi^n\right)$$

and

$$\partial_1(W) = W\left(A(x_1) + SP\xi^n + S(1_n - P)\right), \tag{3.15}$$

where the matrix

$$A(x_1) = \begin{pmatrix} a_{n-1}(x_1) & 0 & \dots & 0 \\ a_{n-2}(x_1) & 0 & \dots & 0 \\ \dots & . & \dots & . \\ \dots & . & \dots & . \\ a_0(x_1) & 0 & \dots & 0 \end{pmatrix}$$

commutes with L_n. Hence,

$$\begin{aligned} \partial_1(\Gamma) &= W\left(A(x_1) + SP\xi^n + S(1_n - P)\right) + C(\xi)W\left(A(x_1) + SP\xi^n + S(1_n - P)\right)L_n \\ &= \Gamma\left(A(x_1) + SP\xi^n + S(1_n - P)\right) + C(\xi)WSP(\xi^n L_n - L_n\xi^n) \end{aligned}$$

and if Γ is invertible in the algebra $\mathrm{Mat}_n(C^\infty(B(H_1)))$, then

$$\gamma = \Gamma^{-1}\partial_1(\Gamma) = A(x_1) + SP\xi^n + S(1_n - P) + \gamma_1, \tag{3.16}$$

where $\gamma_1 = W^{-1}\gamma_0$ and γ_0 is the solution of the equation

$$\gamma_0 + C(\xi)WL_n W^{-1}\gamma_0 = C(\xi)WSP(\xi^n L_n - L_n\xi^n). \tag{3.17}$$

Let

$$C(\xi) = PC(\xi)\begin{pmatrix} C_1(\xi) & C_2(\xi) & \dots & C_n(\xi) \\ 0 & 0 & \dots & 0 \\ \dots & . & \dots & . \\ \dots & . & \dots & . \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$L(f)(\xi) = \sum_{k=0}^{n-1} m_k(f)\xi + \Lambda(f)(\xi),$$

$$\Lambda(f)(\xi) = r(\xi)\mu^{-1}\int \frac{f(\eta)}{\xi^n - \eta^n}\, d\mu(\eta),$$

and the operators m_k be defined by (3.8).

Then $\xi^n L_n - L_n \xi^n = r(\xi)\Pi_n$ and $\gamma_0 = \tilde{\gamma}_0(\xi, x_1)P\Pi_n$, provided that Equation (3.17) has a unique solution. The scalar function $\tilde{\gamma}_0(\xi, x)$ satisfies the equation

$$\tilde{\gamma}_0(\xi) + \tilde{T}(\tilde{\gamma}_0)(\xi) = \sum_{k=1}^{n} C_k(\xi) r(\xi) \mathrm{E}(\varepsilon^{k-1}\xi, x_1), \tag{3.18}$$

where the operator $\tilde{T} \in C^\infty(B(H_1))$ is defined by the formula

$$\tilde{T} = PC(\xi)W(\xi, x_1)L_n W^{-1}(\xi, x_1)P.$$

It follows from (3.18) that

$$PC(\xi)W(\xi, x_1)m_k W(\xi, x_1)^{-1}P = PC(\xi)W(\xi, x_1)W(\varepsilon^k\xi, x_1)^{-1}Pm_k.$$

Since multiplication by the matrix S from the left (right) hand side shifts cyclically the rows (columns) of a matrix by one step upwards (to the right), we have

$$W(\varepsilon^k\xi, x_1) = S^k W(\xi, x_1)$$

and

$$PC(\xi)W(\xi, x_1)W(\varepsilon^k\xi, x_1)^{-1}Pm_k = PC(\xi)S^{-k}Pm_k = C_{k+1}(\xi)m_k P.$$

Hence,

$$PC(\xi)W(\xi, x_1)\left(\sum_{k=0}^{n-1} m_k\right)W^{-1}(\xi, x_1) = \left(\sum_{k=0}^{n-1} C_{k+1}(\xi)m_k\right)P. \tag{3.19}$$

Next, we have, due to (3.15) and (3.11), that

$$\partial_1 (W(\xi, x_1)W(\eta, x_1)^{-1}) = (\xi^n - \eta^n)W(\xi, x_1)SPW(\eta, x_1)^{-1}$$

and

$$W(\xi, 0)W(\eta, 0)^{-1} = (\omega_{ij}(\xi, \eta)),$$

where

$$\omega_{ij}(\xi,\eta)) = \frac{\xi^n - \eta^n}{n\eta^{n-1}(\varepsilon^{i-j}\xi - \eta)}.$$

Therefore, the matrix

$$(W(\xi,x_1)W(\eta,x_1)^{-1}) = W(\xi,0)W(\eta,0)^{-1} + (\xi^n - \eta^n)\int_0^{x_1} W(\xi,t)SPW(\eta,t)^{-1}\,dt$$

has the following elements:

$$(\xi^n - \eta^n)\left\{\frac{1}{n\eta^{n-1}(\varepsilon^{i-j}\xi - \eta)} + K_{ij}(\xi,\eta,x_1)\right\},$$

where K_{ij} are the elements of the matrix $\int_0^{x_1} W(\xi,t)SPW(\eta,t)^{-1}\,dt$ which is analytic in ξ, η and $PC(\xi)W\Lambda_n W^{-1}P$ is an integral operator with the kernel

$$\sum_{p=1}^{n} C_p(\xi)r(\xi)\left\{\frac{1}{n\eta^{n-1}(\varepsilon^{i-j}\xi - \eta)} + K_{p1}(\xi,\eta,x_1)\right\} \tag{3.20}$$

It follows from (3.19), (3.8) and (3.20) that

$$\tilde{T}(f)(\xi) = \sum_{p=1}^{n}\Big\{C_p(\xi)m_{p-1}(\xi)\chi_{p-1}(\xi)f(\varepsilon^{p-1}\xi) + C_p(\xi)r(\xi)\int\left\{\frac{1}{n\eta^{n-1}(\varepsilon^{i-j}\xi - \eta)} + K_{p1}(\xi,\eta,x_1)\right\}f(\eta)\,d\mu(\eta)\Big\}$$

and Equation (3.18) has the form

$$\tilde{\gamma}_0(\xi) + \sum_{p=1}^{n}\Big\{\nu_p(\xi)\chi_{p-1}(\xi)\tilde{\gamma}_0(\varepsilon^{p-1}\xi) + \rho_k(\xi)\int\left\{\frac{1}{n\eta^{n-1}(\varepsilon^{i-j}\xi - \eta)} + K_{p1}(\xi,\eta,x_1)\right\}\tilde{\gamma}_0(\eta)\,d\mu(\eta)\Big\} = \sum_{p=1}^{n}\rho_p(\xi)E(\varepsilon^{p-1}\xi,x_1), \tag{3.21}$$

where $\nu_p(\xi) = C_p(\xi)m_{p-1}(\xi)$, $\rho_p(\xi) = C_p(\xi)r(\xi)$.

Finally, let $y = y(z,x_1)$ be an arbitrary solution of Equation (3.10), with $\xi = z$. Then the Wronsky matrix

$$\mathbf{Y} = \mathbf{Y}(z,x_1) = \begin{pmatrix} \partial_1^{n-1}(y) & \partial_1^{n-2}(y) & \dots & \partial(y) \\ 0 & 0 & \dots & 0 \\ \dots & . & \dots & . \\ \dots & . & \dots & . \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

satisfies Eqs. (3.13) and (3.15) with $\xi = z$ and its Darboux transform

$$\phi = \phi(z, x_1) = \partial_1(\mathbf{Y}) - \mathbf{Y}\,\Gamma^{-1}\,\partial_1(\Gamma) =$$

$$= \mathbf{Y}\{\,A(x_1) + SPz^n + S(\mathbf{1}_n - P) - A(x_1) + SP\xi^n - S(\mathbf{1}_n - P) - \gamma_1\,\} =$$

$$= (z^n - \xi^n)\mathbf{Y}\{\,SP - (z^n - \xi^n)^{-1}W^{-1}(\xi, x_1)\widetilde{\gamma}_0(\xi, x_1)P\Pi_n\,\}$$

satisfies the equation

$$\sum_{j=0}^{n} \partial_1^n(\phi)C(j) = z^n\phi, \tag{3.22}$$

where

$$C(j) = \sum_{k=j}^{n} q_k(j)\{\,b_k(x_1) + \partial_1(b_{k+1}(x_1)) - b_{k+1}(x_1)\gamma\,\}, \tag{3.22'}$$

$$q_k(j) = \sum_{p=0}^{k-j} \frac{(p+j)!}{p!j!}\,\partial_1^p(\gamma_l + \partial_1)^{k-j-p}(1), \tag{3.22''}$$

and $b_n(x_1) = 1$, $b_{n+1}(x_1) = 0$. It is obvious that

$$\psi = (z^n - \xi^n)^{-1}\phi = \mathbf{Y}\{\,SP - (z^n - \xi^n)^{-1}W^{-1}(\xi, x_1)\widetilde{\gamma}_0(\xi, x_1)P\Pi_n\,\} \tag{3.23}$$

also satisfies equation (3.22). Since the operator $\Pi_n = \Pi_n^2$ commute with $\mathbf{Y}$ and S, we have

$$P\Pi_n\psi = P\Pi_n\psi P\Pi_n = \widetilde{\psi}P\Pi_n, \tag{3.24}$$

where the scalar function $\widetilde{\psi} = \widetilde{\psi}(z, x_1) \in C^\infty(B(H))$, according to Lemma 3, is a solution of the equation

$$\sum_{j=0}^{n} \partial_1^j(\widetilde{\psi})\,\widetilde{c}(j) = z^n\widetilde{\psi}, \tag{3.25}$$

where the coefficients $\widetilde{c}(j) \in C^\infty(B(H))$ are defined by the formula

$$\widetilde{c}(j)P\Pi_n = P\Pi_n\,C(j)P\Pi_n. \tag{3.26}$$

It follows from (3.23) and (3.24) that

$$\widetilde{\psi}(z, x_1) = y(z, x_1) + \sum_{j=1}^{n} \partial^{n-j}(y(z,x_1)) \cdot d_j(z^n,x),$$

where the scalar functions $d_j(z^n,x) \in C^{\infty}(B(H))$ are the elements of the first column of the matrix

$$\mu^{-1} \int (\xi^n - z^n)^{-1} W^{-1}(\xi, x_1)\widetilde{\gamma}_0(\xi, x_1)d\mu(\xi).$$

Thus, we arrive at the following theorem.

Theorem 3. *If Eq.* (3.21) *has the unique solution* $\widetilde{\gamma}_0(\xi) = \widetilde{\gamma}_0(\xi, x_1)$, *then the differential operator*

$$D_{n-1} = \sum_{j=1}^{n} d_j(z^n, x_1)\partial_1^{n-j} + 1$$

transforms the solutions of Eq. (3.10) *into the solutions of Eq.* (3.25) *with the coefficients that can be found from the formulae* (3.22′), (3.22″), (3.26) *with*

$$\gamma = A(x_1) + SP\xi^n + S(1_n - P) + W^{-1}(\xi, x_1)\widetilde{\gamma}_0(\xi, x_1)P\Pi_n .$$

N o t e. *The functions* $v_p(\xi)$, $\rho_k(\xi)$ *and the measure* $d\mu(\eta)$ *are arbitrary parameters on which the above obtained Darboux transform depends.*

Acknowledgements

This work was partially suported by CNRS (France) and the State Committee of the Ukraine on Science and Engineering.

References

1. *G. Darboux*, Sur une proposition relative a une équation linéaire, C. R. Acad. Sci. Paris **54** (1882).
2. *M. M. Crum*, Associated Sturm-Liouville systems, Quart. — J. Math. (1855), Ser. 2, v. 6, p. 121—127.
3. *V. B. Matveev*, Darboux Transformation and Explicit Solutions of the Kadomtsev-Petviashvily Equation Depending on Functional Parameters.— Lett. Math. Phys. (1979), v. 3, p. 213—216.
4. *A. B. Shabat, A. P. Veselov*, Dressing Chain and Spectral Theory of Schrodinger Operator, Forschungsinstitut fur Mathematik, ETH Zurich, Nov. 1992.
5. *V. A. Marchenko*, Nonlinear Equations and Operator Algebras, D. Reidel, Dordrecht 1987.