Математическая физика, анализ, геометрия
1995, т. 2, N 1, с. 15-24

УДК 5.517

Polarized modules and Fredholm modules

J. Brodzki
University of Durham

A. Connes, D. Ellwood
Institute des Hautes Etudes Scientifiques, Bures-sur-Yvette, 91440 France

Received December 27, 1994

The basic notion of a polarized module over a C^* algebra A is introduced. A. Such modules are more basic than Fredholm modules: every Fredholm module naturally reduces to a polarized module. It is shown that the polarized module associated to a given Fredholm module is invariant under the action of the group $\mu_{ev}(M)$ up to a canonical isomorphism.

1. Introduction

It is by now a well established fact that the notion of a Fredholm module plays a paramount role in both the formulation of the cycle concept and its geometric analysis in the framework of non-commutative geometry [1]. However, it is easy to see in the context of an ordinary differentiable manifold that the construction of a Fredholm module necessarily involves the choice of an additional structure above and beyond that with which a manifold is naturally endowed. Indeed, one might think of the theory of bounded Fredholm modules as non-commutative conformal geometry as illustrated by the recent constructions of Connes, Sullivan and Teleman [1,2]. On the other hand, the theory of unbounded Fredholm modules may be regarded as the starting point of non-commutative Riemannian geometry [1,3]. In both the bounded and unbounded theory there is a naturally associated quantum calculus which displays several advantages over the classical calculus in the special case of an ordinary manifold. The greater power and flexibility of the quantum calculus over the latter has opened up exciting new "semi-classical" applications in both physics [3,4] and mathematics [5], when the space of interest exhibits a fine structure beyond the reach of ordinary differential geometry.

Our aim here is to free up the notion of a cycle by characterizing the orbits of a bounded Fredholm module under its canonical group of perturbations. More precisely, the various choices involved in the construction of an even Fredholm module over a given C^* algebra A are related by a natural generalization of Beltrami differentials. Given an even Fredholm module (H,F,γ) over A, the action of the generalized Beltrami differentials yield a natural group "$\mu_{ev}(M)$" of perturbations of F by the commutant M of A [1].

Our purpose here is to introduce the more basic notion of a *polarized module* over a C^* algebra A. Such modules are more basic than Fredholm modules in that every Fredholm module naturally reduces to a polarized module. However, the definition of the latter has been carefully designed so as to only capture the content of a Fredholm module up to the action of the group $\mu_{ev}(M)$. In fact we will prove that the polarized module associated to a given Fredholm module is invariant under the action of the group $\mu_{ev}(M)$ up to a canonical isomorphism. Furthermore, the ambiguity in lifting a polarized module to a Fredholm module is again precisely the group $\mu_{ev}(M)$.

The main difference between Fredholm modules and polarized modules is that in the latter the Hilbert space norm is irrelevant. Instead one retains only an indefinite inner product (or Krein space structure) on the underlying Hilbertian space. More precisely, if (H,F,γ) is an even Fredholm module over A, the pair (H,σ) is a Krein space where σ: $\mathrm{H}\times\mathrm{H} \to \mathrm{C}$ denotes the sesquilinear form determined by the Hilbert space inner product $\langle\cdot,\cdot\rangle$ and the Z_2-grading operator γ, i.e., $\sigma(\xi,\eta) := \langle \xi,\gamma\eta \rangle\ \forall \xi, \eta \in \mathrm{H}$. The representation of the C^* algebra is left unchanged, but the presence of the operator F in the triple (H,F,γ) is replaced by the specification of a subspace E of H which is totally isotropic with respect to the Krein space inner product σ. The natural invariance group of a polarized module over A is then by construction the Krein unitary group of the commutant M of A in H. It is a crucial result of the theory that this group be exactly identifiable with the group $\mu_{ev}(M)$ of perturbations of an even Fredholm module by the generalized Beltrami differentials.

Thus the notion of a polarized module provides a nice characterization of the orbits of the group μ_{ev} in the space of even Fredholm modules and hence is likely to be of central importance to the proper understanding of the concept of cycle in non-commutative differential geometry.

2. Polarized modules

Let H be a Hilbertian space. This means that H is a locally convex topological vector space isomorphic to a Hilbert space. On other words, there exists and inner product norm on H which induces the same topology.

2.1. Definition. *Let A be a C^*-algebra. A polarized module over A is given by a triple* (H,E,σ) *where:*

1. H *is a Hilbertian space on which A acts by a unitarisable representation.*
2. *E is a closed subspace of* H.
3. *σ is a continuous hermitian invertible sesquilinear form on* H *which is such that:*

(a) E is a totally isotropic subspace with respect to σ, i.e., $\sigma(\xi, \eta) = 0$ for all ξ, η in E.

(b) For all $a \in A$ and ξ, η in H

$$\sigma(a\xi, \eta) = \sigma(\xi, a^*\eta).$$

(c) The operator associated to σ from E to the annihilator $E^\perp$ of E in the topological dual H^* *of* H *is Fredholm.*

(d) For any a in A, the form $(\xi, \eta) \mapsto \sigma(a\xi, \eta)$, where $\xi, \eta \in E$ is compact.

It is necessary to make a few remarks to clarify this definition.

2.2. Remark

1. First note that the requirement that the representation be unitarisable means that there exists a certain inner product norm on H for which the given representation lifts to a $*$-homomorphism from A into $B(\mathrm{H})$. Thanks to work on the similarity problem for non-selfadjoint representations of C^*-algebras, this lifting can be achieved in great

generality. Indeed, using the deep results of [6] on injective von Neumann algebras and the characterization of nuclear C^*-algebras given in [7], one gets the result that every bounded representation of a nuclear C^*-algebra is similar to a $*$-representation (c.f. [8] theorem 3.5 and [9] theorem 4.1). Thus any continuous representation of a nuclear C^*-algebra is unitary in the above sence. Furthermore, thanks to [10] the above still holds for any cyclic representation of an arbitrary C^*-algebra. The requirement of unitarisability which underlies these results is essential in the lifting of a polarized module to a Fredholm module (see section 3).

2. The operator from E to $E^{\perp}$ referred to in 3(*b*) is given by the explicit formula $[T_{\sigma}(\xi)](\eta) := \sigma(\eta, \xi)\ \forall \xi \in E, \eta \in \mathrm{H}$. Note that $T_{\sigma}(\xi) \in E^{\perp}$ by construction as σ is totally isotropic on E.

3. Finally, we say the sesquilinear form defined in 3(d) is compact if for any weakly convergent sequence $\xi_n \to 0$, the associated sequence $\{\sigma_a(\xi_n, \cdot)\} \subset E^*$ converges uniformly to zero bounded subsets of E.

It is also important to remark that polarized modules are often far easier to construct than Fredholm modules. In fact the above definition is motivated by the following classical example.

2.3. E x a m p l e. Let V be a smooth even dimensional compact oriented manifold, $\dim V = 2m$, and let H be the space $L^2(V, \Lambda^m T^*_{\mathbb{C}})$ of square integrable middle dimensional forms. By construction H is Hilbertian* , however using only the differentiable structure and the orientation we have a hermitian sesquilinear form

$$\sigma(\omega_1, \omega_2) = \int_V \omega_1 \wedge \bar{\omega}_2$$

that is totally isotropic on the subspace E of exact forms in $L^2(V, \Lambda^m T^*_{\mathbb{C}})$. It is easy to check that the triple (H, E, σ) thus defined satisfies all the conditions given in definition 2.1.

Before we proceed further, it is interesting to remark that the pair (H, σ) is a special type of indefinite inner product space called a Krein space. In fact, the transition from Fredholm modules to Polarized modules can be understood as one from Z_2-graded Hilbert space to Krein space. Basically a Krein space is just an indefinite inner product space isomorphic to Hilbert space. More precisely, let H be a C-linear space and σ: $\mathrm{H} \times \mathrm{H} \to \mathrm{C}$ a hermitian sesquilinear form on H. We recall;

2.4. D e f i n i t i o n. *The pair* (H, σ) *is called a Krein space if there exists a pair of Hilbert spaces* H_1 , H_2 *such that:*

1. $\mathrm{H} \equiv H_1 \times H_2$,

---

* Note that a choice of a conformal structure is sufficient to define the action of the restriction of the Hodge $*$-operator to the space of square integrable middle dimensional forms. Then H becomes a Hilbert space under the usual inner product

$$\langle \omega_1, \omega_2 \rangle = \int_V \omega_1 \wedge * \bar{\omega}_2.$$

2. $\sigma(\xi,\eta) = \langle \xi_1, \eta_1 \rangle_1 - \langle \xi_2, \eta_2 \rangle_2 \quad (\forall \xi, \eta \in \mathrm{H})$
where (ξ_1, ξ_2) *and* (η_1, η_2) *are the elements of* $H_1 \times H_2$ *corresponding to* $\xi, \eta \in \mathrm{H}$.

The basic geometry of such spaces is nicely explained in [11]. For a more comprehensive treatment see [12-14] and the references contained there in. The original motivation behind the study of Krein spaces arose from various problems in quantum physics and its structure continues to underly certain constructions in quantum field theory [15].

For our purposes it will be more convenient to understand the relation of Krein space to Hilbert space through the existence of certain types of inner product norms. We recall

2.5. Definition. *Let* H *be a* C*-linear space and* $\sigma\colon \mathrm{H}\times\mathrm{H} \to \mathrm{C}$ *a hermitian sesquilinear form on* H. *Then the pair* (H,σ) *is called an indefinite innerproduct space if there exists a certain inner product norm* $\|\cdot\|$ *on* H *which satisfies*

$$\|\xi\| = \sup_{|\eta| \le 1} |\sigma(\xi,\eta)| \qquad (\xi, \eta \in \mathrm{H}).$$

Such a norm will be referred to a finite unitary norm on H.

Now let (H,σ) be a Krein space. Then by definition H is Hilbertian since it can be identified with the direct sum of two Hilbert spaces. Moreover, under this identification the sesquilinear form σ and the direct sum inner product $\langle\cdot,\cdot\rangle$ are related by the self adjoint involution $\gamma \in B(\mathrm{H})$ associated with the splitting, i.e.,

$$\sigma(\xi,\eta) = \langle \xi, \gamma\eta \rangle$$

for all $\xi, \eta \in \mathrm{H}$. These remarks lead to the following useful lemma.

2.6. Lemma. *The pair* (H,σ) *is a Krein space if and only if there exists a finite unitary norm* $\|\cdot\|$ *under which* H *is complete. Then* σ *is bounded with respect to* $\|\cdot\|$ *and the associated inner product* $\langle,\rangle$ *can be expressed uniquely in terms of* σ *according to the following formula*

$$\langle \xi, \eta \rangle = \sigma(\xi, \gamma\eta)$$

where $\gamma = \gamma^* = \gamma^{-1}$ *is a bounded linear operator on* H.

The above result is elementary in the theory of Krein spaces and can be found for instance in [11] where the involution γ is referred to as defining operator of the finite unitary norm $\|\cdot\|$. We will later give a detailed construction of the finite unitary norms involved in lifting a Krein unitary representation on Hilbertian space to a self adjoint representation on Hilbert space. The existence of such norms provides the crucial ingredient necessary for promotion of a polarized module to a Fredholm module (see theorem 3.1). Indeed the general significance of polarized modules arises from their relation with Fredholm modules. Before explaining this relation we first recall the definition of an even Fredholm module.

2.7. Definition. *Let* A *be a* C^**-algebra. An even Fredholm module over* A *is given by an involutive representation of* A *on a Hilbert space* H, *together with a bounded self adjoint operator* F *and* Z_2*-grading operator* γ *such that:*

1. $F^2 = 1$ *and* $[F,a]$ *is a compact operator for every* $a \in A$,

2. *$\gamma F + F\gamma$ is a finite rank operator,*
3. $[\gamma, a] = 0 \ \forall a \in A$.

Note that the above definition is a slight generalization of that contained in [1] in that the operator F and the grading γ are only required to anticommute modulo finite rank operators. We are now ready to state the central result of this section.

2.8. Theorem. *Let* (H,F,γ) *be an even Fredholm module and* V *the positive eigenspace of* F. *Then there is canonically associated polarized module* (H,F,σ) *where*

$$\sigma(\xi, \eta) = \langle \xi, \gamma\eta \rangle$$

and

$$E = \{ \xi \in V;\ \sigma(\xi,\eta) = 0 \quad \forall \eta \in V \}.$$

Moreover, the action of the group $\mu_{ev}(M)$ *on* (H,F,γ) *leaves the associated polarized module invariant up to a canonical isomorphism.*

P r o o f. We leave H and the representation of A unchanged but forget about the Hilbert space norm. It follows from the definition of σ that it is a bounded continuous sesquilinear form on H. It is hermitian because γ is self-adjoint and invertible because γ is invertible. By construction the subspace E is closed and totally isotropic with respect to σ. Moreover, E is of finite codimension in V because F and γ anticommute modulo finite rank operators. Let us define an operator $T_\sigma\colon E \to E^\perp \subset \mathrm{H}^*$ by $[T_\sigma](\eta) = \sigma(\xi, \eta)\ \forall \xi \in \mathrm{H}$ and $\eta \in E$. It is clear that this is a well defined map from the subspace E to its annihilator $E^\perp$ in H^*. But the conjugate linear isomorphism defined by the inner product on H, that identifies H^* with $\overline{\mathrm{H}}$ the complex conjugate of H, transforms T_σ to the restriction of γ to $E \subset V$. Thus the operator T_σ is Fredholm as required.

The next step is to check that the form σ satisfies the required compactness condition. We have that for any $a \in A$

$$\sigma(a\xi, \eta) = \langle a\xi, \gamma\eta \rangle.$$

Thus for $a \in A$ the form $(\xi, \eta) \mapsto \sigma(a\xi, \eta)$ is compact if and only if the operator $P\gamma aP$ is compact where $P := (1 + F)/2$.

Working modulo the ideal of finite operators in H, F and γ anticommute and thus

$$P\gamma aP = \gamma(1 - P)aP = \gamma(1 - F)a(1 + F)/4$$

is compact since $[F,a]$ is compact. This finishes the proof of the first part of Theorem 2.8.

To prove the second part, we first need to recall some basic facts about the group $\mu_{ev}(M)$ of perturbations of (H,F,γ) by the commutant M of A [1]. By definition, $\mu(M)$ is the space of matrices

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_2(M)$$

which satisfy the identity

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} a^* & -b^* \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} a^* & -b^* \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows easily that

$$aa^* - bb^* = 1,\ ab^* = ba^*,\ a^*a - b^*b = 1,\ a^*b = b^*a \tag{2.9}$$

and that the matrices of this type form a group. Indeed, this group turns out to be the natural group of perturbations of an odd Fredholm module (H,F) over A and is isomorphic to $GL_1(M)$ as explained in [1]. However, in the case an even Fredholm module, we must also account of the fact that the algebra A acts in a Z_2-graded Hilbert space. Then the relevant group of symmetries is restricted to the subgroup $\mu_{ev}(M)$ consisting of matrices as above but where a is even and b is odd as operators in H.

Now let (H,σ,E) be the polarized module associated with a given even Fredholm module (H,F,γ) over A, and $U(M,\sigma)$ be the group of Krein unitary operators in the commutant M of A, i.e.,

$$U(M,\sigma) = \{\, x \in GL_1(M);\ x^\dagger = x^{-1} \,\} \tag{2.10}$$

where $x^\dagger$ denotes the Krein adjoint of x with respect to σ. Be construction $U(M,\sigma)$ is the natural invariance group of the polarized module (H,σ,E). The following proposition identifies the group $U(M,\sigma)$ with the group $\mu_{ev}(M)$ of perturbations of (H,F,γ) by the commutant M of A.

2.11. Proposition. *The group $\mu(M)$ is isomorphic to the group $GL_1(M)$ of invertible elements in M, while the subgroup $\mu_{ev}(M)$ is isomorphic to the Krein unitary group $U(M,\sigma) \subset GL_1(M)$.*

Proof. Let $m = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ be an element of $\mu(M)$; in the graded case we shall assume that a is even and b is odd. Let $x = a + b$. It follows from the identities (2.9) that

$$(a + b)(a - b)^* = (a - b)^*(a + b) = 1$$

so that x is an invertible element of M with inverse $x^{-1} = (a - b)^*$. The map $m \mapsto x$ gives a homomorphism from $\mu(M)$ to $GL_1(M)$. In the graded case, we have that

$$\gamma x^* \gamma = \gamma(a + b)^* \gamma = (a - b)^* = x^{-1}.$$

Moreover, by construction the Krein adjoint and the Hilbert space adjoint are related by the grading operator γ according to $\gamma x^* \gamma = x^\dagger$. Thus it follows that

$$\gamma x^* \gamma = x^\dagger = x^{-1}$$

and so x is in $U(M,\sigma)$ as required.

Conversely, if $x \in GL_1(M)$, then let

$$a = (a + \gamma x \gamma)/2,$$
$$b = (x - \gamma x \gamma)/2.$$

This determines the inverse map and so establishes the required isomorphism in the ungraded case. In the graded case we check that wnen x is Krein unitary then a is even and b is odd as required.

Now let (H,F,γ) be an even Fredholm module and $m \in \mu_{ev}(M)$. Then (H,F',γ), where $F' = m(F) = (aF + b)(bF + a)^{-1}$, is an even Fredholm module satisfying the same summability condition. This defines the action of the group $\mu_{ev}(M)$ in the space of even Fredholm modules. To complete the proof of theorem 2.8 we need to show that the polarized modules associated to (H,F,γ) and (H,F',γ) are canonically isomorphic.

Let x be the element of $U(M,\sigma)$ corresponding to m under the above isomorphism, and let V and V' be the $+1$ eigenspaces of F and F' respectively. Then $F'\xi = \xi$ for some $\xi \in V'$ means that

$$(aF + b)(bF + a)^{-1}\xi = \xi \tag{2.12}$$

which may be rewritten as $(aF + b)\eta = (bF + a)\eta$, where $\eta = (bF + a)^{-1}\xi$. This implies that $F\eta = \eta$, and so $\xi \in V'$ if and only if $\eta \in V$. But then (2.12) gives

$$\xi = (aF + b)\eta = (a + b)\eta = x\eta$$

and thus $V' = xV$.

We now show that the map x: H → H induces an isomorphism of polarized modules. Since x is an element of the commutant of A, this map is a map of A modules and so does not change the representation of A. Furthermore, since $x \in U(M,\sigma)$ we have $\sigma(x\xi, x\eta) = \sigma(\xi, \eta)$, for any ξ and η in H. But by definition

$$E' = \{\, \xi \in V'; \sigma(\xi, \eta) = 0 \ \forall\ \eta \in V' \,\}$$

and so $E' = xE$ also. To summarise, the isomorphism $x \in U(M,\sigma)$ maps the subspace E to $E' = xE$ and preserves both the representation of A and the form σ. This ends the proof of Theorem 2.8.

3. From a polarized module to a Fredholm module

In this last section we go backwards and show that a polarized module can be promoted to a Fredholm module. Indeed, we now prove the exact converse to theorem 2.8.

3.1. Theorem. *To any polarized module one can canonically associate an orbit of Fredholm modules under the Krein unitary group of the commutant.*

The proof of this theorem will be an explicit construction of the required orbit of Fredholm modules and is split into several steps listed below. First of all, since H is hilbertian and representation assumed unitarisable (e.g. A is nuclear or the representation cyclic), there exists an innerproduct $\langle,\rangle_1$ on H that lifts the given representation to a $*$-representation in $B(\mathrm{H})$. From now on we shall assume that H is equipped with this Hilbert space structure. Then (the Krein innerproduct) σ becomes a bounded invertible hermitian sesquilinear form on Hilbert space, and thus corresponds to a unique invertible bounded selfadjoint operator T_1 through the formula

$$\sigma(\xi, \eta) = \langle\, \xi, T_1\eta \,\rangle_1$$

for all $\xi, \eta \in \mathrm{H}$.

Let us denote by T_E the restriction of T_1 to the subspace E. Then for any ξ, η in E

$$\langle\, \xi, T_E\eta \,\rangle_1 = \sigma(\xi, \eta) = 0$$

which shows that T_E maps E to its orthogonal complement $E^\perp$. Moreover, we see immediately that the choice of Hilbert space structure on H, which provides a (conjugate-linear) isomorphism of $E^\perp$ with the annihilator of E in the space H^* of continuous linear functionals on H, identifies T_E with the operator $E \to E^\perp$ determined by the form σ. Thus T_E is a Fredholm operator, and so is its adjoint T_E^*: $E^\perp \to E$. Therefore, relative to the orthogonal decomposition $\mathrm{H} = E \oplus E^\perp$ we can represent the operator T_1 by the following matrix

$$T_1 = \begin{pmatrix} 0 & T_E^* \\ T_E & S \end{pmatrix}$$

where S: $E^\perp \to E^\perp$ is a self-adjoint operator. It is not difficult to see that the operator T_1 is an element of the commutant M of the algebra A. Indeed, we have that for any a in A and any two vectors ξ and η in H

$$\sigma(\xi, a\eta) = \langle \xi, T_1 a\eta \rangle_1 ,$$

while, on the other hand,

$$\sigma(a^*\xi, \eta) = \langle a^*\xi, T_1\eta \rangle_1 = \langle \xi, aT_1\eta \rangle_1 .$$

The left hand sides of the above equalities are the same by the definition of (H,E,σ), and thus $T_1 a = aT_1$ for any a in A.

Now by hypothesis T_1 invertible and selfadjoint. Thus its polar decomposition $T_1 = \gamma \mid T_1 \mid$ yields an isometric involution $\gamma = \gamma^* = \gamma^{-1}$.

Finally, since any a in A commutes with T_1 and $\mid T_1 \mid$, we have that the representation of A in H is unitary with respect to the modified inner product $\langle \xi, \eta \rangle := \langle \xi, \mid T_1 \mid \eta \rangle_1$ $\forall \xi, \eta \in \mathrm{H}$.

3.2. Lemma. *The space $E \oplus \gamma E$ is of finite codimension in* H.

P r o o f. The operator γ_E: $E \to E^\perp$ is Fredholm. The codimension of the space $E \oplus \gamma E$ is — Index γ_E.

Let P be the orthogonal projection onto the subspace E' with respect to the scalar product $\langle , \rangle$. We denote by $F = 2P - 1$ the corresponding involution. Then it follows from the above proposition that we do not have $\gamma F = F\gamma$ unless the index of γ is zero. However, we do have $\gamma F = -F\gamma$ modulo finite rank as required.

The next step in the proof of Theorem 3.1 is to check that the commutator $[F,a]$ is a compact operator for all a in A. We use the assumption that $\sigma(a\xi, \eta)$ is compact on E, which with our choice of the Hilbert space structure is equivalent to $P\gamma aP$ being a compact operator on H. Now $\gamma Pa\gamma P$ is also compact and

$$\gamma P\gamma aP = \gamma(P\gamma PaP + P\gamma(1 - P)aP) = (1 - P)aP$$

where we use that $P\gamma(1 - P) = \gamma(1 - P)$ and $P\gamma P = 0$.

It follows by taking the adjoint that $Pa(1 - P)$ is compact, too, so that the commutator $[P,a]$ is compact. It is now clear that $[F,a]$ is a compact operator for all $a \in A$.

Our considerations thus far depended on a particular choice of the Hilbert space structure, and we now want to study this dependence in more detail. Assume then that we have chosen a different scalar product $\langle , \rangle'$ on H. This scalar product is related to our original one by means of a positive invertible bounded operator x which satisfies

$$\langle \xi, \eta \rangle' = \langle x\xi, x\eta \rangle.$$

The Hilbert space H with the new scalar product will be a $*$-representation of the algebra A if and only if x is an element of the commutant M of A.

Relating the new product to the form σ we see as before that there exists an involution γ' such that

$$\sigma(\xi, \eta) = \langle \xi, \gamma\eta \rangle = \langle \xi, \gamma'\eta \rangle'.$$

But then

$$\langle \xi, \gamma\eta \rangle = \langle x\xi, x\gamma'\eta \rangle$$

so that $\gamma' = x^{-2}\gamma$. Since γ' is an involution we have that

$$x^{-1} = \gamma x \gamma.$$

Let us denote $a = (x + \gamma x \gamma)/2$ and $b = (x - \gamma x \gamma)/2$. Then a and b are self-adjoint elements of the commutant M. It is clear that $\gamma a \gamma = a$ and $\gamma b \gamma = -b$. We note that a and b commute since

$$x(\gamma x \gamma) = xx^{-1} = x^{-1}x = (\gamma x \gamma)x$$

and so

$$a^2 - b^2 = (a + b)(a - b) = u\gamma u\gamma = 1.$$

This demonstrates that $x = a + b$ is in the subgroup $U(M,\sigma)$ of $GL_1(M)$ isomorphic to the even group of perturbations $\mu_{ev}(M)$. This complete the proof Theorem 3.1.

References

1. *A. Connes*, Non-Commutative Geometry. Academic Press, New York (1994).
2. *A. Connes, D. Sullivan, and N. Teleman*, Quasi-conformal mappings, operators on Hilbert space and local formula for characteristic classes.— Preprint IHES/M/93/38 (1993).
3. *A. Connes*, Non-commutative geometry and physics. — In: Gravitation and Quantization. Proceedings of the Les Houches school. Ed. by B. Julia. North-Holland (July 1992).
4. *A. Connes and J. Lott*, Particle models and noncommutative geometry.— Nucl. Phys. (Proc. Suppl.) (1990), v. B18, p. 29–47.
5. *A. Connes and D. Sullivan*, Quasi-circles ect. — Preprint IHES (1993).
6. *A. Connes*, Classification of injective factors. — Ann. Math. (1976), v. 104, p. 73–116.
7. *M. D. Choi and E. D. Effros*, Nuclear C^*-algebra and injectivity. The general case. — Ind. Univ. Math.J. (1977), v. 26, p. 443–446.
8. *J. W. Bruce*, The similarity problem for cyclic representation of C^*-algebras.— Proc. Amer. Math. Soc. (1981), v. 81, p. 817–833.
9. *E. Christensen*, On non-self adjoint representations of operator algebras.— Amer. J. Math. (1981), v. 103, p. 817–833.
10. *U. Haagerup*, Solution of the similarity problem for cyclic representations of C^*-algebras.— Ann. Math. (1983), v. 118, p. 215–240.
11. M. Tomita, Operators and operator algebras in Krein spaces I; spectral analysis in Pontrjagin space.— Preprint Kyushu University (1980).
12. *T. Ya. Azizov and I. S. Iokhvidov*, Linear operators in spaces with an indefinite metric. Wiley-Interscience, New York (1989).

13. *J. Bognor*, Indefinite inner product spaces. Springer, Berlin (1974).
14. *I. S. Iokhvidov, M. G. Krein, and H. Langer*, Introduction to the spectral theory of operators in spaces with an indefinite metric. Academie-Verlag, Berlin (1992).
15. *K. L. Nagy*, State vector spaces with an indefinite metric in quantum field theory. Noordhoff, Groningen (1966).

Поляризованные модули и модули Фредгольма

Дж. Бродски, А. Конн, Д. Эллвуд

Введено основное понятие поляризованного модуля над C^*-алгеброй A. Такие модули являются более общими, чем модули Фредгольма: любой модуль Фредгольма естественным образом редуцирует поляризованный модуль. Показано, что поляризованный модуль, ассоциированный с данным модулем Фредгольма, инвариантен относительно действия группы $\mu_{ev}(M)$ с точностью до канонического изоморфизма.

Поляризовані модулі та модулі Фредгольма

Дж. Бродскі, А. Конн, Д. Эллвуд

Введено основне поняття поляризованого модуля над C^*-алгеброю A. Такі модулі є більш загальними, ніж модулі Фредгольма: будь-який модуль Фредгольма природним способом редукує поляризований модуль. Показано, що поляризований модуль, ассоційований с даним модулем Фредгольма, є інваріантним відносно дії групи $\mu_{ev}(M)$ з точністю до канонічного изоморфізму.