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The solvability of the first main mixed problem of the theory of elasticity in a complete scale of Sobolev spaces

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The unique solvability of the first main problem of the dynamic elasticity theory is proved in a complete scale of Sobolev spaces. The method is based on the properties of the retarded and the advanced elastic potentials.

1. Introduction

The solvability of the Cauchy problem and mixed problems for strong hyperbolic equations and systems of equations in the complete scales of Sobolev spaces was researched by Ya. A. Roitberg in [1-3]. These investigations are continued in this paper. Its objective is to prove the unique solvability of the first main mixed problem of the theory of elasticity in the complete scale of Sobolev spaces which includes, in particular, the spaces of generalized functions with the negative Sobolev norm [4]. We point out that the system of equations of the elasticity theory is not strong hyperbolic one (moreover, V. D. Kupradze called it the "degenerate" hyperbolic [5]). We will use the elastic dynamic potential theory methods as the main tool. The mathematical base of the theory was constructed in [6, 7]. It is shown below that the combination of results obtained in [6, 7] concerning the solvability of mixed problems in the spaces with positive Sobolev norms together with the idea of the "transposition" makes it possible to study the mixed problems in the spaces with negative Sobolev norm.

2. The notations and the formulation of the problem

Let S be a closed hypersurface of class C^∞ which divides $\mathbf{R}^d$, $d \geq 2$ into domains Ω^+ (internal) and Ω^- (external). A displacement of a point $x = (x_1 \ldots, x_d)$ of an elastic medium filling either Ω^+ or Ω^- at the moment t is denoted by $u(X) = (u_1(X), \ldots, u_d(X))$, $X = (x,t)$. In the presence of volume forces with density $q(X) = (q_1(X), \ldots, q_d(X))$ the vector function $u(X)$ satisfies the motion equation

$$\partial_t^2 u(X) + (Au)(X) = q(X). \tag{1}$$

either in $G_T^+ = \Omega^+ \times (0,T)$ or in $G_T^- = \Omega^- \times (0,T)$, $T > 0$. Here $\partial_t = \partial/\partial t$, A is the matrix differential operator of the anisotropic elasticity theory:

$$(Au)_i(X) = -\partial_j \sigma_{ij}(u), \tag{2}$$

$\partial_j = \partial/\partial x_j$. Let us explain the notations used in (2). Here and later on the summation rule with respect to repeating subscripts from 1 to d is used. The equality containing free indices is supposed to be valid for all values of the indices from 1 to d. In (2) $\sigma(u) = \{\sigma_{ij}(u)\}_{i,j=1}^{d}$ is the stress tensor related to the strain tensor $\varepsilon(u) = \{\varepsilon_{ij}(u)\}_{i,j=1}^{d}$, $2\varepsilon_{ij}(u) = \partial_i u_j + \partial_j u_i$ by the Hook law $\sigma_{ij}(u) = a_{ijkh}\varepsilon_{kh}(u)$ where $\{a_{ijkh}\}_{i,j,k,h=1}^{d}$ is the tensor of elastic constants of the medium which satisfies the symmetry and ellipticity conditions [8]. The initial conditions are supposed to be homogeneous

$$u(x, +0) = (\partial_t u)(x, +0) = 0, \qquad x \in \Omega^{\pm}. \tag{3}$$

To formulate the boundary condition let us denote $\Sigma_T = S\times(0,T)$. In the first main problem $I_r^{\pm}$ (internal or external) the displacements of the boundary points are known $u^{\pm}(X) = f(X)$, $x \in \Sigma_T$. Here and later on the superscripts "±" denote the limit values of the corresponding vector functions when their argument tends to Σ_T from within $G_T^{\pm}$. To present the correct formulation of these problems it is necessary to introduce some functional spaces of Sobolev type.

3. The functional spaces

We denote by $H_{m,k}(\mathbf{R}^{d+1})$, $m, k \in \mathbf{R}$, the Sobolev space consisting of d-component real-valued vector functions $u(X) = u(x,t)$ with the finite norm

$$\| u \|_{m,k}^2 = \underset{\mathbf{R}^{d+1}}{S} (1 + |\xi| + |\tau|)^{2m}(1 + |\tau|)^{2k} |\tilde{u}(\xi,\tau)|^2 \, d\xi d\tau,$$

where the variables $\xi \in \mathbf{R}^d$, $\tau \in \mathbf{R}$, are dual to x, t, respectively, $\tilde{u}(\xi,\tau)$ is the Fourier transform of $u(x,t)$. The space $H_{r;m,k}(\mathbf{R}_+^{d+1})$, $(H_{a;m,k}(\mathbf{R}_{T,-}^{d+1}))$ is the subspace of $H_{m,k}(\mathbf{R}^{d+1})$ which consists of elements vanishing when $t<0$ $(t>T)$. The spaces $H_{r;m,k}(G_T^{\pm})$, $H_{a;m,k}(G_T^{\pm})$ are formed by restrictions onto $G_T^{\pm}$ of elements of the corresponding spaces $H_{r;m,k}(\mathbf{R}_{T,+}^{d+1})$, $H_{a;m,k}(\mathbf{R}_{T,-}^{d+1})$, their norms are induced by the norms of these spaces. For example, the norm of the space $H_{r;m,k}(G_T^{\pm})$ is defined as follows:

$$\| u \|_{m,k,G_T^{\pm}} = \inf_{\hat{u} \in H_{r;m,k}(\mathbf{R}_+^{d+1})} \| \hat{u} \|_{m,k},$$

where infimum is taken all over the set of extensions $\hat{u} \in H_{r;m,k}(\mathbf{R}_+^{d+1})$ of the element $u \in H_{r;m,k}(G_T^{\pm})$. The spaces $\overset{0}{H}_{r;m,k}(\mathbf{R}_T^{d+1})$ and $\overset{0}{H}_{a;m,k}(\mathbf{R}_T^{d+1})$ where $\mathbf{R}_T^{d+1} = \mathbf{R}^d\times(0,T)$ are defined in the same way. Finally, the space $H_{r;m,k}(G_T^{\pm})$, $(H_{a;m,k}(G_T^{\pm}))$, is the subspace in $H_{r;m,k}(\mathbf{R}_T^{d+1})$, $(H_{a;m,k}(\mathbf{R}_T^{d+1}))$ which consists of elements u such that $\operatorname{supp} u \subset \overline{G_T^{\pm}}$.

Let us turn to spaces of vector functions defined on the boundary hypersurface Σ_T. When $S = \mathbf{R}^{d-1}$ the spaces $H_{m,k}(S\times\mathbf{R})$, m, $k \in \mathbf{R}$, consist of d-component vector functions $u(x', t)$ with the finite norm

$$\| u \|^2_{m,k;S\times\mathbf{R}} = \int_{\mathbf{R}^d} (1 + | \xi' | + | \tau |)^{2m}(1 + | \tau |)^{2k} | \tilde{u}(\xi',\tau) |^2 \, d\xi' d\tau,$$

where the variable $\xi' \in \mathbf{R}^{d-1}$ is dual to x'. In the general case these spaces are defined using the resolution of identity and the corresponding local coordinates [4]. Let us denote by $H_{r;m,k}(S\times\mathbf{R}_+)$, $(H_{a;m,k}(S\times\mathbf{R}_{T,-})$, $\mathbf{R}_{T,-} = \{ t \in \mathbf{R}: t < T \})$ the subspace in $H_{m,k}(S\times\mathbf{R})$ which consists of elements vanishing when $t < 0$ $(t > T)$. The spaces $H_{r;m,k}(\Sigma_T)$ and $H_{a;m,k}(\Sigma_T)$ are formed by restrictions onto Σ_T of elements of the corresponding spaces $H_{r;m,k}(S\times\mathbf{R}_+)$, $H_{a;m,k}(S\times\mathbf{R}_{T,-})$, their norms are induced by the norms of these spaces.

For $m > 1/2$, $k \in \mathbf{R}$, let us define the trace operators $\gamma_r^{\pm}$, $\gamma_a^{\pm}$ mapping continuously $H_{r;m,k}(G_T^{\pm})$, $H_{a;m,k}(G_T^{\pm})$ onto $H_{r;m-1/2,k}(\Sigma_T)$, $H_{a;m-1/2,k}(\Sigma_T)$, respectively. The vector function $u \in H_{r;1,0}(G_T^{\pm})$ is called the generalized solution of the problems $I_r^{\pm}$ if $\gamma_r^{\pm} u = f$ and u satisfies the variational equation

$$\left(\sigma_{ij}(u), \varepsilon_{ij}(\eta)\right)_{0,T} - \left(\partial_t u, \partial_t \eta\right)_{0,T} = (q, \eta)_{0,T} \tag{4}$$

for any element $\eta \in \mathring{H}_{a;1,0}(G_T^{\pm})$. In (4) the same notation $(\cdot,\cdot)_{0,T}$ is used for the scalar product in the space $L^2(G_t^{\pm})$ and the space $\left[L^2(G_t^{\pm})\right]^d$. This, however, will not cause any confusion.

The solution $u \in H_{a;1,0}(G_T^{\pm})$ of the first main "advanced" mixed problem $I_a^{\pm}$ is defined in a similar way. The only difference between the problems $I_r^{\pm}$ and $I_a^{\pm}$ is that instead of the initial conditions (3) the conditions $u(x,T-0) = (\partial_t u)(x,T-0) = 0$, $x \in \Omega^{\pm}$ are given in the problem $I_a^{\pm}$.

In case $q = 0$ we denote the solving operators of the problems $I_r^{\pm}$, $I_u^{\pm}$ by $\mathring{R}^{\pm}_{1,r}$, $\mathring{R}^{\pm}_{1,a}$, respectively. It was shown in [7] that for all $m \geq 1/2$, $k \in \mathbf{R}$, the operators $\mathring{R}^{\pm}_{1,r}$, $\mathring{R}^{\pm}_{1,a}$ setting a correspondence between the boundary value f and the (generalized) solutions of the problems $I_r^{\pm}$, $I_a^{\pm}$ perform the continuous maps:

$$\mathring{R}^{\pm}_{1,r}: H_{r;m,k}(\Sigma_T) \to H_{r;m+1/2,k-1/2}(G_T^{\pm}),$$
$$\mathring{R}^{\pm}_{1,a}: H_{a;m,k}(\Sigma_T) \to H_{a;m+1/2,k-1/2}(G_T^{\pm}). \tag{5}$$

In what follows, the solutions of the formulated problems will be represented in a form of elastic retarded (or advanced) potentials. The properties of them are presented in the next two sections.

4. Elastic single-layer porentials

Let $\Phi^r(X)$ and $\Phi^a(X)$ be the "retarded" and the "advanced" fundamental solutions of the equation (1), respectively, which satisfy the "causality" condition: $\Phi^r(x,t) = 0$ when $t < 0$, $\Phi^a(x,t) = 0$, when $t > 0$. They are $d \times d$ — matrixes whose columns are denoted by $\Phi_j^r(X)$, $\Phi_j^a(X)$, $j = 1,\ldots,d$. It is easy to check that the following equality is valid:

$$\Phi^r(X)^T = \Phi^a(-X), \tag{6}$$

where the superscript "T" denotes the transposition of a matrix.

Elastic retarded and advanced single-layer potentials with d-component densities $\alpha_r(X)$, $\alpha_a(X)$ are defined as

$$(V_r\alpha_r)(X) = \underset{\Sigma_T}{S} (\Phi_j^r(X - Y), \alpha_r(Y))\, e_j ds_Y,$$

$$(V_a\alpha_a)(X) = \underset{\Sigma_T}{S} (\Phi_j^a(X - Y), \alpha_a(Y))\, e_j ds_Y,$$

respectively, where e_j is the j-th unit vector of the coordinate system, $(\cdot,\cdot)$ is the scalar product in $\mathbf{R}^d$. It is evident that for smooth densities α_r, α_a such that $\operatorname{supp} \alpha_r \subset S \times (0,T]$, $\operatorname{supp} \alpha_a \subset S \times [0,T)$ both the potentials satisfy the homogeneous equation (1) and the homogeneous initial conditions for $t = 0$ or $t = T$, respectively. The introduced potentials bear the boundary operators V_r, V_a setting correspondence between the densities α_r, α_a and the values of the corresponding potentials on Σ_T. The properties of the operators V_r, V_a in incomplete scales of Sobolev spaces were researched in [6,7]. It is proved below (Theorem 1) that these properties are valid in the complete scales of these spaces.

We denote by $\langle \cdot,\cdot \rangle_{0,T}$ the real scalar product in $\left[L^2(\Sigma_T)\right]^d$ and remark that the spaces $H_{r;m,k}(\Sigma_T)$ and $H_{a;-m,-k}(\Sigma_T)$ are dual with respect to the duality defined by the form $\langle \cdot,\cdot \rangle_{0,T}$. Let us introduce the spaces $C_r^\infty(\overline{\Sigma}_T)$ and $C_a^\infty(\overline{\Sigma}_T)$ consisting of vector functions of class $C^\infty(\overline{\Sigma}_T)$ with the supports lying in $S \times (0,T]$ and in $S \times [0,T)$, respectively.

Lemma 1. *For all $\alpha_r \in C_r^\infty(\overline{\Sigma}_T)$, $\alpha_a \in C_a^\infty(\overline{\Sigma}_T)$ the equalities*

$$\langle V_r\alpha_r, \alpha_a \rangle_{0,T} = \langle \alpha_r, V_a\alpha_a \rangle_{0,T} \tag{7}$$

are valid.

P r o o f. Let $\alpha_r \in C_r^\infty(\overline{\Sigma}_T)$, $\alpha_a \in C_a^\infty(\overline{\Sigma}_T)$ and $\alpha_{r,k}$, $\alpha_{a,k}$ are their k-th components

$$\langle V_r\alpha_r, \alpha_a \rangle_{0,T} = \underset{\Sigma_T}{S}\underset{\Sigma_T}{S} \alpha_{a,k}(X) \Phi_{ki}^r(X - Y)\, \alpha_{r,i}(Y)\, ds_Y ds_X =$$

$$\underset{\Sigma_T}{S} \alpha_{r,i}(X) \Big(\underset{\Sigma_T}{S} \Phi_{ki}^r(Y - X)\, \alpha_{a,k}(Y)\, ds_Y \Big) ds_X .$$

It follows from (6) that $\Phi_{ki}^r(Y - X) = \Phi_{ik}^a(X - Y)$, as needed for (7).

Theorem 1. *For all $m, k \in \mathbf{R}$, the operators V_r, V_a perform the continuous injective maps*:

$$V_r: H_{r;m,k}(\Sigma_T) \to H_{r;m+1,k-1}(\Sigma_T),$$
$$V_a: H_{a;m,k}(\Sigma_T) \to H_{a;m+1,k-1}(\Sigma_T),$$

their ranges are dense in the corresponding spaces.

The inverse operators extended from the dense sets onto these spaces for all m, $k \in \mathbf{R}$, perform the continuous maps:

$$V_r^{-1}: H_{r;m+1,k-1}(\Sigma_T) \to H_{r;m,k-1}(\Sigma_T),$$
$$V_a^{-1}: H_{a;m+1,k-1}(\Sigma_T) \to H_{a;m,k-1}(\Sigma_T).$$

P r o o f. In case $m \geq -1/2$ case the statements of the theorem were proved in [6,7]. For $m < -1/2$ the operators V_r, V_a are defined on the spaces $H_{r;m,k}(\Sigma_T)$, $H_{a;m,k}(\Sigma_T)$, respectively, by the equalities $V_r = V_a^*$, $V_a = V_r^*$. Obviously the reasons for this definition are contained in the statements of Lemma 1. The continuity of the maps V_r, V_a follows immediately from the continuity of the operators V_a, V_r for $m > -1/2$. Another statements of the theorem are evident.

Let us introduce the operators $V_{r,\pm}$, $V_{a,\pm}$ setting correspondence between densities α_r, α_a and the values of the corresponding potentials $(V_r\alpha_r)(X)$, $(V_a\alpha_a)(X)$ for $X \in G_T^{\pm}$. It was shown in [7] that these operators perform the continuous maps:

$$V_{r,\pm}: H_{r;m,k}(\Sigma_T) \to H_{r;m+3/2,k-1}(G_T^{\pm}),$$
$$V_{a,\pm}: H_{a;m,k}(\Sigma_T) \to H_{a;m+3/2,k-1}(G_T^{\pm})$$

for $m \geq -1/2$, $k \in \mathbf{R}$. In the next section it is proved that these statements are valid for all $m, k \in \mathbf{R}$.

In conclusion of this section we take into consideration the equalities $\mathring{R}_{1,r}^{\pm} = V_{r,\pm} V_r^{-1}$, $\mathring{R}_{1,a}^{\pm} = V_{a,\pm} V_a^{-1}$ where $\mathring{R}_{1,r}^{\pm}$, $\mathring{R}_{1,a}^{\pm}$ are the solving operators introduced in the third section.

5. Elastic volume potentials

Elastic retarded and advanced volume potentials with the d-component densities q_r, q_a are defined as

$$(U_r q_r)(X) = \underset{\mathbf{R}_T^{d+1}}{S} \left(\Phi_j^r(X - Y), q_r(Y)\right) e_j \, dY,$$
$$(U_a q_a)(X) = \underset{\mathbf{R}_T^{d+1}}{S} \left(\Phi_j^a(X - Y), q_a(Y)\right) e_j \, dY, \tag{9}$$

respectively. The properties of the potentials are given in the following statements.

Theorem 2. *For all $m, k \in \mathbf{R}$, the operators U_r, U_a perform the continuous maps:*

$$U_r\colon \overset{\circ}{H}_{r;m,k}(G_T^{\pm}) \to H_{r;m+2,k-1}(G_T^{\pm}),$$
$$U_a\colon \overset{\circ}{H}_{a;m,k}(G_T^{\pm}) \to H_{a;m+2,k-1}(G_T^{\pm}). \tag{10}$$

Proof. We present the proof of the theorem for the case of operator U_r. Let us introduce for all $\kappa \geq 0$ spaces $H_{r;m,k,\kappa}(\mathbf{R}_+^{d+1})$ consisting of (generalized) d-component vector functions $u(X) = u(x,t)$, vanishing when $t < 0$, whose Laplace transforms $\widehat{u}(x,p)$ have the following properties:

1. $\widehat{u}(x,p)$ performs a holomorphic map from the right-hand half-plane $\mathbf{C}_\kappa = \{ p = \sigma + i\tau\colon \sigma > \kappa \}$ into the standard Sobolev space $[W_1^2(\mathbf{R}^d)]^d$.

2. $\sup\limits_{\sigma>\kappa} \int\limits_{\mathbf{R}^{d+1}} (1 + |\xi| + |\sigma + i\tau|)^{2m}(1 + |\sigma + i\tau|)^{2k} |\widetilde{u}(\xi,\sigma + i\tau)|^2 \, d\xi d\tau < \infty$, where

$\widetilde{u}(\xi, p)$ is the Fourier-Laplace transform of $u(x,t)$. The norm in this space is defined as follows:

$$\| u \|^2_{m,k;\kappa} = \int\limits_{\mathbf{R}^{d+1}} (1 + |\xi| + |\kappa + i\tau|)^{2m}(1 + |\kappa + i\tau|)^{2k} |\widetilde{u}(\xi,\kappa + i\tau)|^2 \, d\xi d\tau. \tag{11}$$

We denote by $H_{r;m,k,\kappa}(\mathbf{R}_T^{d+1})$ the space of restrictions onto $\mathbf{R}_T^{d+1}$ of elements of the space $H_{r;m,k,\kappa}(\mathbf{R}_+^{d+1})$ with the corresponding induced norm. Note that in case $T < \infty$ the spaces $H_{r;m,k}(\mathbf{R}_T^{d+1})$ and $H_{r;m,k,\kappa}(\mathbf{R}_T^{d+1})$ coincide as sets and their norms are equivalent.

Let us define the operator U_r on elements of the space $H_{r;m,k,\kappa}(\mathbf{R}_+^{d+1})$ by (9) where the integration is, however, performed over $\mathbf{R}_+^{d+1}$ instead of $\mathbf{R}_T^{d+1}$. On choosing and fixing $\kappa > 0$ let us study the properties of the operator U_r in the spaces $H_{r;m,k,\kappa}(\mathbf{R}_+^{d+1})$.

Let $q_r = (q_{r,1},\ldots,q_{r,d}) \in \left[C_0^\infty\left(\mathbf{R}_+^{d+1}\right)\right]^d$. The Fourier–Laplace transform of the vector function $\left(U_r q_r\right)(x,t)$ has the form

$$\left(\widetilde{U_r q_r}\right)(\xi,p) = \widetilde{\Phi}^{\,r}_{ij}(\xi,p)\widetilde{q}_{r,i}(\xi,p)e_j\,, \tag{12}$$

where $\widetilde{\Phi}^{\,r}_{ij}(\xi,p)$ and $\widetilde{q}_{r,i}(\xi,p)$ are the Fourier-Laplace transforms of $\Phi^r_{ij}(x,t)$ and $q_{r,i}(x,t)$, respectively. From the equalities

$$p^2\widetilde{\Phi}^r_{ij}(\xi,p) + a_{ilkh}\xi_l\xi_k\widetilde{\Phi}^r_{hj}(\xi,p) = \delta_{ij}$$

after multiplying them by $\overline{\widetilde{\Phi}^r_{ij}}(\xi,p)$ and summing up in i we get the system of equations

$$p^2 |\widetilde{\Phi}^{\,r}_{ij}(\xi,p)|^2 + a_{ilkh}\xi_l\xi_k\widetilde{\Phi}^{\,r}_{hj}(\xi,p)\overline{\widetilde{\Phi}^r_{ij}}(\xi,p) = \overline{\widetilde{\Phi}^r_{jj}}(\xi,p) \tag{13}$$

(not sum with respect to j!). Separating the real and the imaginary parts in (13) and taking into account the symmetry properties of the elastic constants: $a_{iklh} = a_{khil} = a_{hkil}$ [8] after elementary transformations we obtain the estimates

$$|\widetilde{\Phi}^r_{j}(\xi,p)| \leq c\kappa^{-1} |p| (|p|^2 + |\xi|^2)^{-1}, \qquad j = 1,\ldots,d, \tag{14}$$

with a positive constant c not depending on $p \in \overline{C}_\kappa$. From (12) and (14) we have

$$\left| \left(\widetilde{U_r q_r}\right)(\xi,p) \right|^2 \leq c_\kappa (1 + |p|)^2 (1 + |p| + |\xi|)^{-4} |\tilde{q}_r(\xi,p)|^2.$$

This estimate together with (11) proves the continuity of the maps

$$U_r: H_{r;m,k,\kappa}(\mathbf{R}_+^{d+1}) \to H_{r;m+2,k-1,\kappa}(\mathbf{R}_+^{d+1}) \tag{15}$$

for all $m, k \in \mathbf{R}$.

Let l be an operator of extension of vector functions from $\mathbf{R}_T^{d+1}$ onto $\mathbf{R}_+^{d+1}$ which maps $H_{r;m,k,\kappa}(\mathbf{R}_T^{d+1})$ into $H_{r;m,k,\kappa}(\mathbf{R}_+^{d+1})$ continuously. The operator of restriction of vector functions onto $\mathbf{R}_T^{d+1}$ is denoted by π. We remark that as follows from the causality condition ($\Phi^r(x,t) = 0$ when $t < 0$) for any vector function q_r the equality

$$(U_r q_r)(X) = (\pi U_r l q_r)(X), \qquad X \in \mathbf{R}_T^{d+1},$$

is valid. Using continuity of the map (15) and the equivalence of the norms of the spaces $H_{r;m,k}(\mathbf{R}_T^{d+1})$, $H_{r;m,k,\kappa}(\mathbf{R}_T^{d+1})$ we obtain the continuity of the maps

$$U_r: H_{r;m,k}(\mathbf{R}_T^{d+1}) \to H_{r;m+2,k-1}(\mathbf{R}_T^{d+1}).$$

The continuity of the maps (10) follows from this statement immediately. The continuity of the operators U_a is proved in the same way.

Let us introduce for $m > -3/2$, $k \in \mathbf{R}$, the operators $U_{r,\pm} = \gamma_r^{\pm} U_r$, $U_{a,\pm} = \gamma_a^{\pm} U_a$ performing continuous maps:

$$\begin{aligned} U_{r,\pm}&: \overset{0}{H}_{r;m,k}(G_T^{\pm}) \to H_{r;m+3/2,k-1}(\Sigma_T), \\ U_{a,\pm}&: \overset{0}{H}_{a;m,k}(G_T^{\pm}) \to H_{a;m+3/2,k-1}(\Sigma_T). \end{aligned} \tag{16}$$

Lemma 2. *For all vector functions* $\alpha_r \in C_r^\infty(\overline{\Sigma}_T)$, $\alpha_a \in C_a^\infty(\Sigma_T)$, q_r, $q_a \in C^\infty(\mathbf{R}_T^{d+1})$ *such that* $\operatorname{supp} q_r \subset \Omega^{\pm} \times (0,T]$, $\operatorname{supp} q_a \subset \Omega^{\pm} \times [0,T)$ q_r *and* q_a *are finite in the case of the domain* Ω^-, *the equalities are valid:*

$$\begin{aligned} \langle U_{r,\pm} q_r, \alpha_a \rangle_{0,T} &= \left(q_r, V_{a,\pm}\alpha_a\right)_{0,T}, \\ \langle U_{a,\pm} q_a, \alpha_r \rangle_{0,T} &= \left(q_a, V_{r,\pm}\alpha_r\right)_{0,T}. \end{aligned} \tag{17}$$

P r o o f. Let us check the validity of the first equality in (17)

$$\langle U_{r,\pm} q_r, \alpha_a \rangle_{0,T} = \int_{\Sigma_T} \int_{\mathbf{R}_T^{d+1}} \alpha_{a,k}(X) \Phi_{ki}^r (X - Y) q_{r,i}(Y) dY ds_X =$$

$$= \int_{\mathbf{R}_T^{d+1}} q_{r,i}(X) \left(\int_{\Sigma_T} \Phi_{ki}^r (Y - X) \alpha_{a,k}(Y) ds_Y \right) dX.$$

The relation $\Phi^{r}_{ki}(Y-X) = \Phi^{a}_{ik}(Y-X)$ proves the required equality. The second equality in (17) is proved just like the first one.

The results of Lemma 2 make it possible to define the operators $V_{r,\pm}$, $V_{a,\pm}$, $U_{r,\pm}$, $U_{a,\pm}$ on the corresponding Sobolev spaces with the negative norm. For $m \le -3/2$, $k \in \mathbf{R}$, we define the operators $U_{r,\pm}$, $U_{a,\pm}$ in the spaces $\mathring{H}_{r;m,k}(G^{\pm}_T)$, $\mathring{H}_{a;m,k}(G^{\pm}_T)$, respectively, by $U_{r,\pm} = V^{*}_{a,\pm}$, $U_{a,\pm} = V^{*}_{r,\pm}$. From the continuity of the maps (8) and the duality of the spaces $H_{r;m,k}(G^{\pm}_T)$, $\mathring{H}_{a;-m,-k}(G^{\pm}_T)$ and $H_{a;m,k}(G^{\pm}_T)$, $\mathring{H}_{r;-m,-k}(G^{\pm}_T)$, it follows that the maps (16) are continuous for all $m, k \in \mathbf{R}$. The operators $V_{r,\pm}$ $V_{a,\pm}$ are defined on spaces $H_{r;m,k}(\Sigma_T)$, $H_{a;m,k}(\Sigma_T)$ for $m < -1/2$, $k \in \mathbf{R}$, by $V_{r,\pm} = U^{*}_{a,\pm}$, $V_{a,\pm} = U^{*}_{r,\pm}$. The continuity of the maps (8) for all $m, k \in \mathbf{R}$, follows from the continuity of the maps (16). Let us formulate these results as a theorem.

Theorem 3. *The operators $V_{r,\pm}$ $V_{a,\pm}$ perform the maps (8) and the operators $U_{r,\pm}$, $V_{a,\pm}$ perform the maps (16) which are continuous for all $m, k \in \mathbf{R}$.*

The introduced operators U_r, U_a have been defined on the spaces $\mathring{H}_{r;m,k}(G^{\pm}_T)$, $\mathring{H}_{a;m,k}(G^{\pm}_T)$, respectively. In conclusion of this section let us introduce the operators $U^{\pm}_r$, $U^{\pm}_a$ acting in the spaces $H_{r;m,k}(G^{\pm}_T)$, $H_{a;m,k}(G^{\pm}_T)$. Let $l^{\pm}_r$ and $l^{\pm}_a$ be operators of extension of vector functions from $G^{\pm}_T$ onto $\mathbf{R}^{d+1}_T$ which perform the continuous maps:

$$l^{\pm}_r \colon H_{r;m,k}(G^{\pm}_T) \to H_{r;m,k}(\mathbf{R}^{d+1}_T),$$

$$l^{\pm}_a \colon H_{a;m,k}(G^{\pm}_T) \to H_{a;m,k}(\mathbf{R}^{d+1}_T).$$

Let $\pi^{\pm}$ be the operators of restriction of vector functions from $\mathbf{R}^{d+1}_T$ onto $G^{\pm}_T$. We define the operators $U^{\pm}_r$, $U^{\pm}_a$ by $U^{\pm}_r = \pi^{\pm} U_r l^{\pm}_r$, $U^{\pm}_a = \pi^{\pm} U_a l^{\pm}_a$. Clearly, for all $m, k \in \mathbf{R}$, these operators perform the continuous maps:

$$U^{\pm}_r \colon H_{r;m,k}(G^{\pm}_T) \to H_{r;m+2,k-1}(G^{\pm}_T),$$

$$U^{\pm}_a \colon H_{a;m,k}(G^{\pm}_T) \to H_{a;m+2,k-1}(G^{\pm}_T). \tag{18}$$

For $m > -3/2$, $k \in \mathbf{R}$, the operators $U^{\pm}_{r,\pm} = \gamma^{\pm}_r U^{\pm}_r$, $U^{\pm}_{a,\pm} = \gamma^{\pm}_a U^{\pm}_a$ perform the continuous maps:

$$U^{\pm}_{r,\pm} \colon H_{r;m,k}(G^{\pm}_T) \to H_{r;m+3/2,k-1}(\Sigma_T),$$

$$U^{\pm}_{a,\pm} \colon H_{a;m,k}(G^{\pm}_T) \to H_{a;m+3/2,k-1}(\Sigma_T). \tag{19}$$

Note that the operators $U^{\pm}_r$, $U^{\pm}_a$ (and hence the operators $U^{\pm}_{r,\pm}$, $U^{\pm}_{a,\pm}$) depend on the choice of the operators of extension $l^{\pm}_r$, $l^{\pm}_a$.

6. The solvability of the first main mixed problem

Let us suppose that the right-hand side of the equation (1) $q \in C^\infty\left(\overline{G_T^\pm}\right)$, $\operatorname{supp} q \subset \Omega^\pm \times (0,T]$ and q is finite in the case of domain G_T^-. Also suppose that the boundary value $f \in C_r^\infty(\Sigma_T)$. It is evident that the solutions of the problems $I_r^\pm$ can be represented in the form

$$u(X) = (U_r q)(X) + (V_{r,\pm} V_r^{-1}(f - U_{r,\pm} q))(X), \qquad X \in G_T^\pm.$$

In case $\operatorname{supp} q \subset \overline{\Omega^\pm} \times (0,T]$ we represent the solutions of these problems in the form

$$u(X) = (U_r^\pm q)(X) + (V_{r,\pm} V_r^{-1}(f - U_{r,\pm}^\pm q))(X), \qquad X \in G_T^\pm. \tag{20}$$

Let us introduce the solving operators $R_{1,r}^\pm$ of the problems $I_r^\pm$ which set the correspondence between the pair $\{q, f\}$ and the solutions $u(X)$, $X \in G_T^\pm$, of these problems. Our objective is to study the possibility of extending operators $R_{1,r}^\pm$ by continuity onto products of Sobolev spaces including spaces with the negative norm.

Theorem 4. *For all $k \in \mathbf{R}$ the operators $R_{1,r}^\pm$ perform the continuous maps:*

$$R_{1,r}^\pm : \begin{cases} H_{r;m,k}(G_T^\pm) \times H_{r;m+3/2,k-1}(\Sigma_T), & m \geq -1 \\ & \qquad \to H_{r;m+2,k-3/2}(G_T^\pm). \\ \mathring{H}_{r;m,k}(G_T^\pm) \times H_{r;m+3/2,k-1/2}(\Sigma_T), & m \leq -1 \end{cases}$$

P r o o f. First we consider the case $m \geq -1$. From (20) and from the continuity of the maps (18), (19) it follows that $U_r^\pm q \in H_{r;m+2,k-1}(G_T^\pm)$, $f - U_{r,\pm}^\pm q \in H_{r;m+3/2,k-1}(\Sigma_T)$ for any $q \in H_{r;m,k}(G_T^\pm)$, $f \in H_{r;m+3/2,k-1}(\Sigma_T)$. Recalling that $V_{r,\pm} V_r^{-1} = \mathring{R}_{1,r}^\pm$ and using continuity of the maps (5) we obtain the required statement for all $m \geq -1$.

Now let us suppose that $m \leq -1$, $q \in \mathring{H}_{r;m,k}(G_T^\pm)$, $f \in H_{r;m+3/2,k-1/2}(\Sigma_T)$ and write the solutions of the problems $I_r^\pm$ in the form

$$u = U_r q + V_{r,\pm}\left(V_r^{-1} f - V_r^{-1} U_{r,\pm} q\right).$$

Noticing that $V_r^{-1} U_{r,\pm} = \left(V_{a,\pm} V_a^{-1}\right)^* = \left(\mathring{R}_{1,a}^\pm\right)^*$ and using continuity of the maps (5) we have the continuity of the operators

$$V_r^{-1} U_{r,\pm} : \mathring{H}_{r;m,k}(G_T^\pm) \to H_{r;m+1/2,k-1/2}(\Sigma_T) \tag{21}$$

for all $m \leq -1$, $k \in \mathbf{R}$. From (21) and the statements of Theorem 1 it follows that

$$V_r^{-1} f - V_r^{-1} U_{r,\pm}\, q \in H_{r;m+1/2,k-1/2}(\Sigma_T).$$

The statement of the theorem for $m \leq -1$ follows from Theorem 3.

R e m a r k. *The solving operators $R^{\pm}_{1,a}$ of the problem $I^{\pm}_{u}$ are introduced just like the operators $R^{\pm}_{1,r}$. It is evident that these operators perform the continuous maps:*

$$R^{\pm}_{1,a}: \begin{cases} H_{a;m,k}(G^{\pm}_{T}) \times H_{a;m+3/2,k-1}(\Sigma_T), & m \geq -1 \\ \mathring{H}_{a;m,k}(G^{\pm}_{T}) \times H_{a;m+3/2,k-1/2}(\Sigma_T), & m \leq -1 \end{cases} \to H_{a;m+2,k-3/2}(G^{\pm}_{T})$$

for all $k \in \mathbf{R}$.

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Разрешимость первой основной смешанной задачи теории упругости в полной шкале соболевских пространств

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Доказана однозначная разрешимость первой основной смешанной задачи динамической теории упругости в полной шкале соболевских пространств. Метод доказательства основан на свойствах запаздывающих и опережающих упругих потенциалов.

Розв'язуваність першої основної змішаної задачі теорії пружності у повній шкалі соболєвських просторів

І. Ю. Чудінович

Доведена однозначна розв'язуваність першої основної змішаної задачі динамічної теорії пружності у повній шкалі соболєвських просторів. Метод доведення грунтується на властивостях запізнілих та випереджаючих пружних потенціалів.