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On toroidal submanifolds of constant negative curvature

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Earlier M.L. Rabelo and K. Tenenblat have introduced the notion of toroidal submanifolds generated by some curve α and they have constructed immersions of domains of the n-dimensional Lobachevsky space L^n in E^{2n-1} as toroidal submanifolds. Here these submanifolds are reconstructed by a simply way, and in the case $n = 3$ the influence of the torsion κ of the curve α on the geometry of the submanifolds $M^3 \subset E^5$ is investigated. Here the torsion appears in the coefficient of torsion of the special normal basis of M^3. The Grassmann image of its has been constructed.

Introduction

In [1] Rabelo and Tenenblat have introduced the notion of toroidal submanifolds and they have constructed immersions of domains of the n-dimensional Lobachevsky space L^n in the $(2n-1)$-Euclidean space E^{2n-1}, as toroidal submanifolds. We recall here that notion. Let

$$\alpha(x_1) = (f_1(x_1), f_2(x_1), \dots, f_n(x_1)), \quad x_1 \in I \subset \mathbf{R},$$

be a parametrization of a regular curve in E^n such that $f_i(x_1) \neq 0$, $i \geq 2$, $\forall x_1 \in I$. The submanifold parametrized by

$$r(x_1, \dots, x_n) = (f_1(x_1), f_2(x_1)\cos x_2, f_2(x_1)\sin x_2, \dots, f_n(x_1)\cos x_n, f_n(x_1)\sin x_n)$$

is called a *toroidal submanifold* M^n *of* E^{2n-1} *generated by the curve* α.

In [1] such toroidal submanifolds with sectional curvature equal to -1 have been constructed in Theorem 2.1. The results are distinct for $n = 3$ and $n \geq 4$. For $n \geq 4$ the submanifold is generated by a tractrix in E^n.

In this paper we reconstruct these submanifolds for $n = 3$ in a simple way and investigate their properties. The question about the influence of the torsion κ of the curve α on the geometry of the submanifolds $M^3 \subset E^5$ is especially interesting. We show that κ appears in the coefficient of torsion of the normal basis generated by a principal vector of the normal curvature of the submanifold M^3. Moreover, when $\kappa = 0$, this basis parallel translated in the normal bundle. Then we construct the Grassmann image of the submanifold $M^3 \subset E^5$.

For $n \geq 4$ we show that it coincides with the toroidal submanifold with sectional curvature equal to -1, which is the one discovered by F. Shur.

For all these immersions a family of line of curvatures consists of geodesic curves of L^n. In [2] the immersions L^3 in E^5 with this condition were investigated with the view of solving the principal system of immersions L^3 in E^5 (see Theorem 4, p. 382). However, the condition on the toroidal structure of submanifolds gives us new opportunities for a deeper investigation of their behaviour.

1. Case $N = 3$

For simplicity we use the notation $x_1 = t$ and, for $n = 3$, we have the position vector of M^3

$$r(t, x_2, x_3) = (f_1, f_2 \cos x_2, f_2 \sin x_2, f_3 \cos x_3, f_3 \sin x_3),$$

where $f_i = f_i(t)$, $i = 1, 2, 3$. We use the prime to denote derivatives with respect to t.

We have the following expression of the derivatives of r:

$$r_t = (f_1', f_2' \cos x_2, f_2' \sin x_2, f_3' \cos x_3, f_3' \sin x_3),$$

$$r_{x_2} = (0, -f_2 \sin x_2, f_2 \cos x_2, 0, 0),$$

$$r_{x_3} = (0, 0, 0, -f_3 \sin x_3, f_3 \cos x_3).$$

We can take the length of arc of the curve α as the parameter t. Then $\sum_{i=1}^{3} f_i'^2 = 1$ and hence for the metric of M^3 we obtain

$$ds^2 = dt^2 + f_2^2 dx_2^2 + f_3^2 dx_3^2.$$

Since the ds^2 metric has a constant curvature equal to -1, using formulas (37.4) from [3], we obtain the three equations:

$$f_2'' = f_2, \quad f_3'' = f_3, \quad f_2' f_3' = f_2 f_3.$$

From the two equations we have

$$f_2 = A e^t + B e^{-t}, \quad f_3 = C e^t + D e^{-t},$$

where A, B, C, D are some constants and the third equation implies that

$$AD + BC = 0. \tag{3}$$

From $f_1'^2 = 1 - f_2'^2 - f_3'^2$ we have

$$f_1(t) = \pm \int_0^t \sqrt{a_1 - a_2 e^{2t} - a_3 e^{-2t}}\, dt + c,$$

where $a_1 = 1 - 2(AB + CD)$, $a_2 = A^2 + C^2$, $a_3 = B^2 + D^2$ and c is an arbitrary constant. Then we have the necessary condition

$$2(AB + CD) < 1.$$

If $A^2 + C^2 \neq 0$ and $B^2 + D^2 \neq 0$, then the function f_1 is defined in an interval of the t-axis. Let us unvestigate this case. First we find the points where $f_1' = 0$, which is equivalent to

$$a_2 e^{4t} - a_1 e^{2t} + a_3 = 0. \tag{4}$$

Hence

$$e^{2t} = \frac{a_1 \pm \sqrt{a_1^2 - 4a_2a_3}}{2a_2}.$$

Using condition (3) we obtain that $\Delta = a_1^2 - 4a_2a_3 = (1 - 4AB)(1 - 4CD)$.

The case $a_1^2 - 4a_2a_3 = 0$ does not occur because the expression in the left hand side of (4) would be positive for all t except one point. Therefore, we can assume that $a_1^2 - 4a_2a_3 \neq 0$. By setting

$$2t_1 = \ln\left(\frac{a_1 - \sqrt{\Delta}}{2a_2}\right), \quad 2t_2 = \ln\left(\frac{a_1 + \sqrt{\Delta}}{2a_2}\right),$$

the curve α is defined in the interval (t_1, t_2) and its arc length l is equal to

$$l = t_2 - t_1 = \frac{1}{2}\ln\left(\frac{a_1 + \sqrt{\Delta}}{a_1 - \sqrt{\Delta}}\right). \tag{5}$$

Since

$$a_1^2 - \Delta = 4(AB - CD)^2,$$

if $B \to 0$, $D \to 0$ and $|A|, |C| <$ const, then $l \to \infty$.

Now, we calculate the curvature k and the torsion κ of the curve α. We remember that t is a natural parameter of α. Therefore

$$k^2 = |\alpha''|^2 = (f_1'')^2 + f_2^2 + f_3^2 = 1 - 2a_1 + \frac{\Delta}{a_1 - a_2 e^{2t} - a_3 e^{-2t}}. \tag{6}$$

Consequently, when $t \to t_i$, the curvature $k \to \infty$. From this equality we have

$$a_1 - a_2 e^{2t} - a_3 e^{-2t} = \frac{\Delta}{k^2 + 2a_1 - 1}. \tag{7}$$

Then, using the formula for the torsion of a space curve in the natural parametrization,

$$\kappa = \frac{(r' r'' r''')}{k^2},$$

the equations $f_i'' = f_i$, $i = 2, 3$ imply that

$$(r'r''r''') = \begin{vmatrix} f_1' & f_1'' & f_1''' \\ f_2' & f_2 & f_2' \\ f_3' & f_3 & f_3' \end{vmatrix} = \begin{vmatrix} f_1' & f_1'' & f_1''' - f_1' \\ f_2' & f_2 & 0 \\ f_3' & f_3 & 0 \end{vmatrix} = (f_1''' - f_1')(f_2'f_3 - f_3'f_2). \tag{8}$$

Hence

$$\kappa = \frac{(f_1'' - f_1')(f_2'f_3 - f_3'f_2)}{k^2}. \tag{9}$$

It is easy to obtain that $f_2'f_3 - f_3'f_2$ is a constant number given by

$$f_2'f_3 - f_3'f_2 = 2(AD - BC).$$

We set

$$T = 2(AD - BC),$$

and obtain that

$$f_1''' - f_1' = -\frac{f_2^2 + f_3^2 + 1 - f_2'^2 - f_3'^2 - (f_2'f_3 - f_3'f_2)^2}{(1 - f_2'^2 - f_3'^2)^{3/2}} =$$

$$= \frac{4a_2a_3 - a_1^2}{(a_1 - a_2e^{2t} - a_3e^{-2t})^{3/2}}.$$

Hence the torsion of the curve α is given by the expression

$$\kappa = \frac{2(4a_2a_3 - a_1^2)(AD - BC)}{(a_1 - a_2e^{2t} - a_3e^{-2t})^{3/2}k^2}. \tag{10}$$

From the assumption that $\Delta \neq 0$ we deduce that the curve α is a plane curve if and only if $AD - BC = 0$. From (3) this condition is satisfied if and only if $A = C = 0$ or $B = D = 0$.

Formulas (7) and (10) give us that k and κ are connected by the algebraic relation

$$\kappa = -\frac{T(k^2 + 2a_1 - 1)^{3/2}}{\Delta^{1/2}k^2}.$$

We observe that the two numbers A and B (or C and D) cannot vanish simultaneously in consequence of the regularity of the submanifold M^3.

In the case $\kappa \neq 0$ the curve α is a space curve whose projection on the plane (f_2, f_3) is an arc of a hyperbola. In fact, we have

$$Cf_2 - Af_3 = (CB - DA)e^{-t},$$

$$Df_2 - Bf_3 = (AD - CB)e^{t}.$$

Hence

$$CDf_2^2 + ABf_3^2 = -(DA - CB)^2. \tag{11}$$

Condition (3) means that there exists a number λ such that

$$C = \lambda A, \quad D = -\lambda B.$$

Therefore, equation (11) can be rewritten in the form

$$\lambda^2 f_2^2 - f_3^2 = 4\lambda^2 AB.$$

If the curve α is a plane curve, then this hyperbola is degenerated into a pair of straight lines with an intersection. Now, we investigate the geometric properties of the submanifold $M^3 \subset E^5$. The immersions L^n in E^{2n-1} always have plane normal connections. Therefore, there exists a basis of normal vector fields with coefficients of torsion identically equal to zero. But now, we shall construct special normal vector fields whose coefficients of torsion will be related to the torsion κ of the curve α.

We take the family of coordinate curves on M^3 with the parameter t. Since these curves are geodesic curves their curvature vectors lie in the normal plane. We take the field of unit vectors along them

$$n_1 = \frac{1}{k}(f_1'', f_2 \cos x_2, f_2 \sin x_2, f_3 \cos x_3, f_3 \sin x_3). \tag{12}$$

The second field of normal vectors n_2 we calculate using the cross-product

$$n_2 = \frac{1}{\nu}\left[r_t r_{x_2} r_{x_3} n_1\right],$$

where ν is the norm of this cross-product. After some calculations we obtain

$$n_2 = \frac{1}{\nu}(f_3' f_2 - f_2' f_3, \xi \cos x_2, \xi \sin x_2, \eta \cos x_3, \eta \sin x_3), \tag{13}$$

where

$$\xi = f_1' f_3 - f_1'' f_3', \quad \eta = f_1'' f_2' - f_1' f_2,$$

$$\nu = \sqrt{(f_3' f_2 - f_2' f_3)^2 + \xi^2 + \eta^2}.$$

As the fields of the normal basis n_1, n_2 do not depend on x_2 and x_3, the coefficients of torsion $\mu_{12/2}$ and $\mu_{12/3}$ are equal to zero. Finally, we have

$$\mu_{12/1} = (n_1 n_{2x_1}) = -(n_{1x_1} n_2).$$

The derivative n_{1t} has the form

$$n_{1t} = \frac{1}{k}(f_1''', f_2' \cos x_2, f_2' \sin x_2, f_3' \cos x_3, f_3' \sin x_3) - \frac{k'}{k} n_1 .$$

Consequently, the coefficient of torsion

$$\mu_{12/1} = \frac{-1}{k\nu}(f_1''' (f_3' f_2 - f_2' f_3) + \xi f_2' + \eta f_3') = \frac{(f_1''' - f_1')(f_2' f_3 - f_3' f_2)}{k\nu}.$$

Using equation (9), we obtain

$$\mu_{12/1} = \kappa \frac{k}{\nu}. \tag{14}$$

Let us consider the quotient κ/ν. We have

$$\xi = \frac{f_3 - f_2' T}{\sqrt{1 - f_2'^2 - f_3'^2}}, \qquad \nu = -\frac{f_2 + f_3' T}{\sqrt{1 - f_2'^2 - f_3'^2}}.$$

Therefore

$$\nu^2 = -1 + \frac{1 - 4(AD - BC)^2}{a_1 - a_2 e^{2t} - a_3 e^{-2t}}.$$

If we compare this expression with k^2 in (6), we see that they are very similar. Using (7), we obtain

$$\nu^2 = C_1 k^2 + C_2,$$

where

$$C_1 = \frac{1 - 4(AD - BC)^2}{\Delta}, \quad C_2 = C_1(2a_1 - 1) - 1.$$

Consequently,

$$\mu_{12/1} = \kappa \frac{k}{\sqrt{C_1 k^2 + C_2}},$$

where C_1 and C_2 are some constants. Since the second fundamental forms

$$II^1 = k\, dt^2 - f_2^2\, dx_2^2 - f_3^2\, dx_3^2,$$

$$II^2 = -f_2 \xi\, dx_2^2 - f_3 \eta\, dx_3^2$$

are diagonal, the coordinate curves are lines of curvature. In particular, t-curves are the lines of curvature and r'' is the principal vector of curvature. We call the basis n_1, n_2 the *special invariant normal basis.* Thus we can formulate the result:

Theorem. *On the toroidal submanifold $M^3 \subset E^5$ with curvature equal to -1 the coefficient of torsion $\mu_{12/1}$ of the special invariant basis is expressed in terms of the curvature k and the torsion κ of the curve α by the relation*

$$\mu_{12/1} = \kappa \frac{k}{\sqrt{C_1 k^2 + C_2}},$$

where C_1 and C_2 are constants.

It is obvious for $\kappa = 0$ that the coefficient $\mu_{12/1}$ also vanishes. Since we have explicit expressions for n_1 and n_2, it is easy to obtain the Grassmann image of M^3. We get for its Plücker coordinates

$$p^{12} = \sigma_1 \cos x_2, \qquad p^{13} = \sigma_1 \sin x_2,$$

$$p^{14} = \sigma_2 \cos x_3, \qquad p^{15} = \sigma_2 \sin x_3,$$

$$p^{23} = 0, \; p^{45} = 0,$$

$$p^{24} = \sigma_3 \cos x_2 \cos x_3 , \quad p^{25} = \sigma_3 \cos x_2 \sin x_3 ,$$

$$p^{34} = \sigma_3 \sin x_2 \cos x_3 , \quad p^{35} = \sigma_3 \sin x_2 \sin x_3 ,$$

where σ_i , $i = 1, 2, 3$ are functions of t:

$$\sigma_1 = \frac{f_1''\xi - f_2 T}{k\nu}, \quad \sigma_2 = \frac{f_1''\eta - f_3 T}{k\nu}, \quad \sigma_3 = \frac{f_2\eta - f_3\xi}{k\nu}.$$

Consequently, the Grassmann image Γ^3 lies in some 8-dimensional Euclidean space E^8. We introduce three Euclidean spaces E_1 , E_2 and E_3 mutually orthogonal with coordinates and position vectors given by

$$E_1 : r_1 = \{ p^{12}, p^{13} \}, \quad E_2 : r_2 = \{ p^{14}, p^{15} \},$$

$$E_3 : r_3 = \{ p^{24}, p^{25}, p^{34}, p^{35} \},$$

We also choose three unit vectors $a_i \in E_i$ defined by

$$a_1 = \{ \cos x_2 \sin x_2 \}, \quad a_2 = \{ \cos x_3 \sin x_3 \},$$
$$a_3 = \{ \cos x_2 \cos x_3, \cos x_2 \sin x_3, \sin x_2 \cos x_3, \sin x_2 \sin x_3 \}.$$

The position vector p of the Grassmann image Γ^3 is a linear combination of the vectors a_i ,

$$p = \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3 .$$

If x_2 and x_3 are fixed, then a_i will be constant vectors. Hence the image of a t-curve is not in general a plane curve in a three-dimensional Euclidean space. As $\sum_{i=1}^{3} \sigma_i^2 = 1$, this curve lies on a 2-dimensional unit sphere. When t is fixed and x_2 and x_3 are changing, then r_1 and r_2 describe two circles and r_3 describes the Clifford torus.

2. Case $N \geq 4$

In conclusion we investigate the toroidal submanifold $M^n \subset E^{2n-1}$ with constant curvature equal to -1, for the general case $n \geq 4$. We obtain the system of equations

$$f_i'' = f_i , \quad f_i' f_j' = f_i f_j , \quad i, j = 2,\dots, n.$$

Hence

$$f_i = A_i e^t + B_i e^{-t},$$

where

$$A_i B_j + B_i A_j = 0, \quad i, j = 2,\dots, n.$$

It is not difficult to prove that all $A_i = 0$ or all $B_i = 0$. In fact, we choose three different, indices i, j, k and the corresponding equations

$$A_i B_j + A_j B_i = 0, \tag{15}$$

$$A_i B_k + A_k B_i = 0, \tag{16}$$

$$A_i B_k + B_j A_k = 0. \tag{17}$$

Suppose, for example, that $B_i \neq 0$. We must remark that, in consequence of the regularity of the submanifold M^n, for every index i the corresponding constants A_i and B_i are not identically zero at the same time. From (15) we obtain

$$A_j = -\frac{A_i B_j}{B_i}.$$

Substituting this relation in (17), we get

$$B_j(-A_i B_k + A_k B_i) = 0. \tag{18}$$

From (16) and (18) we have

$$B_j A_i B_k = 0.$$

We have to analyse three possibilities. The first one is $B_j = 0$. This implies that $A_j \neq 0$, but since $B_i \neq 0$, we have a contradiction with (15). The second one is $B_k = 0$ and again from (16) we end up with a contradiction. The third case is $A_i = 0$ for which from (16) we obtain that $B_i A_k = 0$, but since $B_i \neq 0$ we necessarily have that $A_k = 0$.

Thus we have only one solution $A_i = A_k = A_j = 0$ and the expressions for f_i assume the forms

$$f_1 = \int \sqrt{1 - c^2 e^{-2t}}\, dt, \quad f_i = B_i e^{-t}, \quad i = 2, \ldots, n,$$

where $c^2 = \sum_{i=1}^{n} B_i^2$. With the choice of a new parameter $\tau = t - \ln c$ we can obtain the coincidence with the example of F. Shur.

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О тороидальных подмногообразиях постоянной отрицательной кривизны

Ю.А. Аминов, М.Л. Рабело

Ранее М. Рабело и К. Тененблат ввели понятие тороидальных подмногообразий, порождаемых некоторой кривой α, и построили погружения областей n-мерного пространства Лобачевского L^n в E^{2n-1} в виде тороидальных подмногообразий. Здесь эти подмногообразия строятся простым способом, и в случае $n = 3$ изучено влияние кручения κ кривой α на геометрию подмногообразия $M^3 \subset E^5$. Показано, что κ проявляет себя в коэффициенте кручения специального базиса нормалей M^3 так, что когда $\kappa = 0$, то этот базис — *параллельно переносимый*. Строится грассманов образ M^3.

Про тороідальні підмноговиди постійної від'ємної кривини

Ю.А. Амінов, М.Л. Рабело

Раніше М. Рабело та К. Тененблат ввели поняття про тороідальні підмноговиди, породжені кривою α, і побудували занурення областей n-вимірного простору Лобачевського L^n в E^{2n-1} у вигляді тороідальних підмноговидів. У роботі ці занурення побудовані простим засобом, і у випадку $n = 3$ вивчено вплив скруту κ кривої α на геометрію підмноговиду $M^3 \subset E^5$. Доведено , що скрут κ з'являється у коефіцієнті скруту спеціального базису нормалей M^3 таким чином, що коли $\kappa = 0$, то цей базис паралельно переносимий. Будується грассманів образ M^3.