Математическая физика, анализ, геометрия 1995, т. 2, № 3/4, с. 296–305

УДК 517.958

The multistability in the stationary scattering problem for a nonlinear mean-field model

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Received November 15, 1994

We consider the stationary scattering problem for the nonlinear mean-field model of wave and particle propagation and the quasi-stationary solutions of the scattering problem for the wave equation with the same nonlinearity. The multistability phenomena are discovered and studied. For the quasi-stationary solutions the asymptotic decomposition is obtained.

In [1] Jona-Lasinio with coauthors has investigated numerically the one-dimentional nonlinear mean-field model of the electron propagation. They obtained results demonstrating selfgenerating oscillations of the transition coefficient in the non-stationary regimes of the wave propagation.

We shall consider the stationary problem for the model and the quasi-stationary solutions of the scattering problem for the wave equation with the same kind of nonlinearity. As for the physical background of the problem we refer to [1] and start with the Schroedinger equation (in dimensionless variables)

$$- i\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} - \alpha V\psi - \beta(\delta(x+1) + \delta(x-1))\psi \tag{1}$$

or the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} - \alpha V\psi - \beta(\delta(x+1) + \delta(x-1))\psi, \tag{1'}$$

where α and β are non-negative costants and the potential $V = V(x)$ depends upon ψ in the following manner:

$$V(x) = \begin{cases} 0, & |x| > 1; \\ \int_{-1}^{1} |\psi|^2 dx, & |x| < 1. \end{cases} \tag{2}$$

The stationary scattering problem consists in finding the solution $\psi(x, t) = e^{-iEt} u(x)$ of (1) with given E, satisfying the conditions

$$\psi(x, t) = \begin{cases} e^{-iEt + ik(x+1)} + Re^{-iEt - ik(x+1)}, & x < -1; \\ Te^{-iEt + ik(x-1)}, & x > 1 \end{cases} \tag{3}$$

with $E = k^2$ $(k \geq 0)$.

For (1′) the solution is of the form

$$\psi(x, t) = e^{-ikt} u(x)$$

with given k and satisfies similar conditions with k instead of E.

The constants T and R are called the transition and the reflection coefficients, respectively.

The paper is organized as follows: we study the stationary solution of (1) in Section 1. The only difference for (1′) is that kt should be substituted instead of Et in all the exponents. The result and even the pictures are the same since our denotations include $k = \sqrt{E}$.

Section 2 devoted to the quasi-stationary solution of the scattering problem for the wave equation. We get the asymptotic expansion of the solution of the scattering problem with slowly changing amplitude of the incident wave.

The last section is devoted to a discussion of the multistability phenomena that manifest themselves in the obtained solution.

It is the pleasure of the second and the third authors to express their gratitude to the University Paris VII for its hospitality.

1. The solution of the stationary scattering problem

Let us define z by the equality

$$z^2 = \alpha V - k^2. \tag{4}$$

Then, since k, $V \geq 0$, we may set $z = \zeta$ or $z = i\zeta$ with $\zeta \geq 0$.

Substituting (3) into (1) we see that (for $z \neq 0$)

$$u(x) = \begin{cases} e^{ik(x+1)} + Re^{-ik(x+1)}, & x < -1; \\ Ae^{zx} + Be^{-zx}, & |x| < 1; \\ Te^{ik(x-1)}, & x > 1, \end{cases} \tag{5}$$

or (for $z = 0$)

$$u(x) = \begin{cases} e^{ik(x+1)} + Re^{-ik(x+1)}, & x < -1; \\ Ax + B, & |x| < 1; \\ Te^{ik(x-1)}, & x > 1, \end{cases} \tag{5'}$$

with some unknown constants A, B, T, R and z.

Using the gluing conditions at the points $x_i = \pm 1$ we find A, B, T, R ($z \neq 0$ being considered as a parameter):

$$\begin{cases} A = 2ike^{-z}(\beta - z - ik)\Delta^{-1}, \\ B = -2ike^{z}(\beta + z - ik)\Delta^{-1}. \end{cases} \tag{6}$$

$$\begin{cases} T = -4ikz\,\Delta^{-1}, \\ R = 2\{(z^2 + k^2 + \beta^2)\sinh 2z + 2\beta z\cosh 2z\}\Delta^{-1}, \end{cases} \tag{7}$$

where Δ equals

$$\Delta = 4\{z\cosh z + (\beta - ik)\sinh z\}\{z\sinh z + (\beta - ik)\cosh z\}. \tag{8}$$

As one can easily check, $|T|^2 + |R|^2 \equiv 1$.

Calculating explicitly V in terms of A, B and z and using (4), we obtain equality (9) connecting z with E, α, and β:

$$\begin{aligned} &(E + z^2)\{|z\cosh(z) + \beta\sinh(z)|^2 + E|\sinh(z)|^2\}\times \\ &\times\{|z\sinh(z) + \beta\cosh(z)|^2 + E|\cosh(z)|^2\} = \\ &= \alpha E\{(|z|^2 + E + \beta^2)\frac{\sinh 2(z + \bar z)}{2(z + \bar z)} + \\ &+ (|z|^2 - E - \beta^2)\frac{\sinh 2(z - \bar z)}{2(z - \bar z)} + \beta\sinh(2z)\sinh(2\bar z)\} \end{aligned} \tag{9}$$

(we have also replaced k^2 by E).

This equality is in fact the equation with respect to z, with given E, α, β. But the investigation of the solution $z(E, \alpha, \beta)$ is very complicated. However, it may also be considered as a cubic equation with respect to E, with α, β, z being its parameters. Finding the solution $E(z, \alpha, \beta)$, we get the inverse function to $z(E, \alpha, \beta)$ that can be studied.

Three cases arise from (4): z may be

a) imaginary ($z = i\zeta, \zeta > 0$),

b) real ($z = \zeta > 0$),

c) zero ($z = 0$).

They should be considered separately.

a) Let $z = i\zeta$ with $\zeta > 0$ (this means that $E = k^2 > \alpha V$ and $k > \zeta$). Then (for $\zeta \neq n\pi/2$) equation (9) has a unique positive root. We denote by $E(\zeta)$ this root considered as a function of ζ. If $\zeta \neq n\pi/2$, it is larger than ζ^2. Note that though $E(\zeta)$ is unique for a fixed ζ there may exist many values of ζ with the same $E(\zeta)$.

For $\zeta = n\pi/2$ (9) degenerates to a quadratic equation. If $\alpha < \zeta^2$, there still exists a root $E(\zeta) > \zeta^2$. For $\alpha \geq \zeta^2$ there are no positive roots, and as $\zeta \to n\pi/2$ the root $E(\zeta)$ tends to $+\infty$.

One can easily get main asymptotics of E: they are

$$E \approx \zeta^2 + 2\alpha + 0(1/\zeta) \text{ when } \zeta \to +\infty,$$

$$E \approx (\alpha - \zeta^2)(\zeta - n\pi/2)^{-2} \text{ when } \zeta \to n\pi/2 < \sqrt{\alpha}.$$

When $\zeta \to 0$, the result depends upon the relation between α and β : if

$$\alpha < \alpha_0 = \frac{3\beta^2(1+\beta)^2}{2(4\beta^2 + 6\beta + 3)}, \tag{10}$$

we get

$$E \to +0, \text{ as } \zeta \to 0;$$

if on the contrary $\alpha > \alpha_0$, we get

$$E \to E_0 + 0, \text{ as } \zeta \to 0,$$

where E_0 is the positive root of the limit equation

$$E(E+\beta^2)(E+(1+\beta)^2) = \frac{8\alpha E}{3}\left(E+\beta^2+\frac{3\beta}{2}+\frac{3}{4}\right). \tag{11}$$

This means that for a sufficiently large α and $E < E_0$ a) cannot be the case. In other words, we shall have $E = k^2 < \alpha V$, which is the case b).

b) Now for fixed ζ and small α there are no positive roots of (9); as α grows it reaches the critical value $\alpha_{cr}(\zeta, \beta)$ where a single root of (9) appears, while for $\alpha > \alpha_{cr}(\zeta, \beta)$ there are two roots. As $\alpha \to +\infty$, one of them tends to 0 while the other tends to $+\infty$. Note that this time for $E < E_0$ there exists the unique $\zeta = \zeta(E, \alpha, \beta)$.

As $\zeta \to 0$, we come to equation (11) as in the case a), but this time the largest positive root approaches E_0 from below. In this case (9) has no roots larger than E_0.

c) In the case $\zeta = 0$ representation (5) of the solution in the domain $-1 < x < 1$ is no longer valid. In this case we have (5′). Going through all the calculations (that are rather easy), we come to the limit equation (11) that was considered above.

The computer simulation gives the picture fig. 1: the larger is α the higher is the curve shown in $k(\zeta) = \sqrt{E(\zeta)}$.

2. The quasi-stationary solution of the scattering problem for the wave equation

The physical considerations show that for the wave equation the mean-field approximation may be valid only in the case when the time it takes the wave to cross the nonlinear area is small when compared with that of the considerable change in the wave's characteristics. In our model this means that the incident wave should be modulated by some slowly changing signal. Since the speed of the wave in dimensionless variables equals 1, we should introduce the small parameter ε in the modulating function and obtain the incident wave that looks like the following;

$$\psi_{in}(x, t) = e^{ik(x+1-t)}\varphi(\varepsilon(x+1-t)) \tag{12}$$

for $t < 0$ and $x < -1$. For technical reasons we shall assume that $\log(\varphi(y))$ is a smooth function.

The scattering problem in this case consists of finding the solution of the wave equation (1′) satisfying the following conditions at infinity:

$$\psi(x,t)=\begin{cases}\psi_{in}(x-t)+\psi_R(x+t), & x>1;\\ \psi_T(x-t), & x<-1\end{cases}$$

with some unknown ψ_{in} and ψ_R.

Since the wave changes to a considerable extent, we must assume that the characteristic time considered is at least of the order $O(\varepsilon^{-1})$. We shall seek for the solution of the problem in the form (cf. (5))

$$\psi(x,t)=\begin{cases}e^{ik(x+1-t)}\varphi(\varepsilon(x+1-t))+\\ +e^{-ik(x+1+t)}\varphi(-\varepsilon(x+1+t))R(\varepsilon(x+1+t)), & x<-1;\\ e^{-ikt}\varphi(-\varepsilon t)\,u(x,\varepsilon t), & |x|<1;\\ e^{ik(x-1-t)}\varphi(\varepsilon(x-1-t))\,T(\varepsilon(x-1-t)), & x>1,\end{cases}\tag{13}$$

where the functions u, T and R have the asymptotic expansions

$$u(x,\varepsilon t)=u_0(x,\varepsilon t)+\varepsilon\,u_1(x,\varepsilon t)+\ldots+\varepsilon^n u_n(x,\varepsilon t);$$

$$R(x)=R_0(x)+\varepsilon\,R_1(x)+\ldots+\varepsilon^n R_n(x);$$

$$T(x)=T_0(x)+\varepsilon T_1(x)+\ldots+\varepsilon^n\,T_n(x).$$

Inserting these formulas to equation (1′) and glueing the solutions at $x=\pm 1$, we get the set of problems for all the terms of the expansion. Due to the introduction of "the slow time" $\varepsilon t=\tau$, they are analogous to those for the stationary problem. To shorten the notations we write everywhere $u_j(x)$ for $u_j(x,\varepsilon t)$ and φ for $\varphi(-\varepsilon t)$; the dot like in $\dot u$ stands for the derivative in respect to the "slow time $\varepsilon t=\tau$" (and not just t):

We get for the zero order

$$\begin{cases}u_0''(x)+k^2u_0(x)-\alpha\varphi^2\left[u_0(x)\displaystyle\int_{-1}^{1}|u_0(x)|^2dx\right]=0,\\ (\beta-ik)u_0(-1)-u_0'(-1)=-2ik,\\ (\beta-ik)u_0(1)+u_0'(1)=0;\end{cases}\tag{14}$$

for the first order we get the linear problem

$$\begin{cases}u_1''(x)+k^2u_1(x)-\alpha\varphi^2\left[u_1(x)\displaystyle\int_{-1}^{1}|u_0(x)|^2dx+2u_0(x)\displaystyle\int_{-1}^{1}\mathrm{Re}(u_0(x)\overline{u_1}(x))dx\right]=\\ \qquad=-2ik[\dot u_0(x)\varphi-u_0(x)\varphi'],\\ \varphi\,[(\beta-ik)u_1(-1)-u_1'(-1)]=-2\varphi'+u_0\varphi'(-1)-\dot u_0(-1)\varphi\\ \varphi\,[(\beta-ik)u_1(1)+u_1'(1)=u_0\varphi'(-1)-\dot u_0(-1)\varphi\end{cases}\tag{15}$$

and for the order j $(1 < j < n)$ the same kind with the different r.h.s. that is a sum of the terms $u_0, u_1, \ldots, u_{j-i}$, with the coefficients that are equal to integrals of bilinear combinations of the same functions.

For the last term in the expansion we have a more complicated nonlinear problem:

$$\varphi\left\{u_n'' + k^2 u_n(x) - \alpha\varphi^2\Big[\sum_{i+j+k\geq n} \varepsilon^{i+j+k-n} u_i(x)(2-\delta_{j,k})\int_{-1}^{1}\mathrm{Re}(u_j(x)\overline{u_k}(x))dx\Big]\right\}=$$

$$= 2ik[\dot u_{n-1}(x)\varphi - u_{n-1}(x)\varphi'] + \varphi'' u_{n-2}(x) - 2\varphi'\dot u_{n-2}(x) + \varphi\,\ddot u_{n-2}(x) +$$
$$+\varepsilon\,[\varphi'' u_{n-1}(x) - 2\varphi'\dot u_{n-1}(x) + \varphi\,\ddot u_{n-1}(x) - 2ik\varepsilon\,(\dot u_n(x)\varphi - u_n(x)\varphi')] +$$
$$+\varepsilon^2\,[\varphi'' u_n(x) - 2\varphi'\dot u_n(x) + \varphi\,\ddot u_n(x)] \tag{16}$$

where

$$\delta_{i,j} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

The conditions at $x = \pm 1$ take the form

$$\begin{cases}\varphi\,[(\beta - ik)u_n(-1) - u_n'(-1) + \varepsilon\dot u_n(-1)] - u_n(-1)\,\varphi' = u_{n-1}(-1)\,\varphi' - \dot u_{n-1}(-1)\varphi, \\ \varphi\,[(\beta - ik)u_n(1) + u_n'(1) + \varepsilon\dot u_n(1)] - \varepsilon u_n(1)\,\varphi' = u_{n-1}(1)\,\varphi' - \dot u_{n-1}(1)\varphi.\end{cases} \tag{17}$$

In order to prove that the expansion obtained is the asymptotic one we must study one by one all the problems from (13) to (16)-(17).

Problem (13) coincides with that considered for the stationary case and has a corresponding solution depending on $\tau = \varepsilon t$ as a parameter. The solution and its derivatives with respect to x and to $\tau = \varepsilon t$ remain smooth while there is no singularity. It appears when the derivative $\partial u_0/\partial\tau$ vanishes. Since the problem depends upon τ only via the term $\alpha\varphi^2$, and φ is smooth, the singularity appears at the point where $\partial u_0/\partial\alpha = 0$. As we assume that $k = k(\alpha, \zeta) = \text{const}$, this is the case when $\partial k/\partial\zeta = 0$ (at the points of local extrema of the curve $k(\zeta)$), see fig. 1.

Problem (14) depends upon τ via the known parameter φ, $u_i(x, \tau)$, and functions in the r.h.s. composed of $u_i(x, \tau)$ with $i < j$. All of them are smooth and bounded (recall that we suppose $\log\varphi$ to be a smooth function). Problem (14) is linear (over the real numbers) and has a smooth solution while the respective homogeneous problem has only zero solution. The solution of the homogeneous problem can be expressed explicitly via k, ζ, $\alpha\varphi^2$, and β (cf. Section 1). It is non zero only for a single value of φ. So at least for φ less than a certain value of order 1 (not depending on ε) it is smooth.

To obtain the estimate for the last term let us note that the sum

$$U_n(x, t) = \varphi(x, \varepsilon t)e^{-ikt}\sum_{k=0}^{n-1}\varepsilon^k u_k(x, \varepsilon t)$$

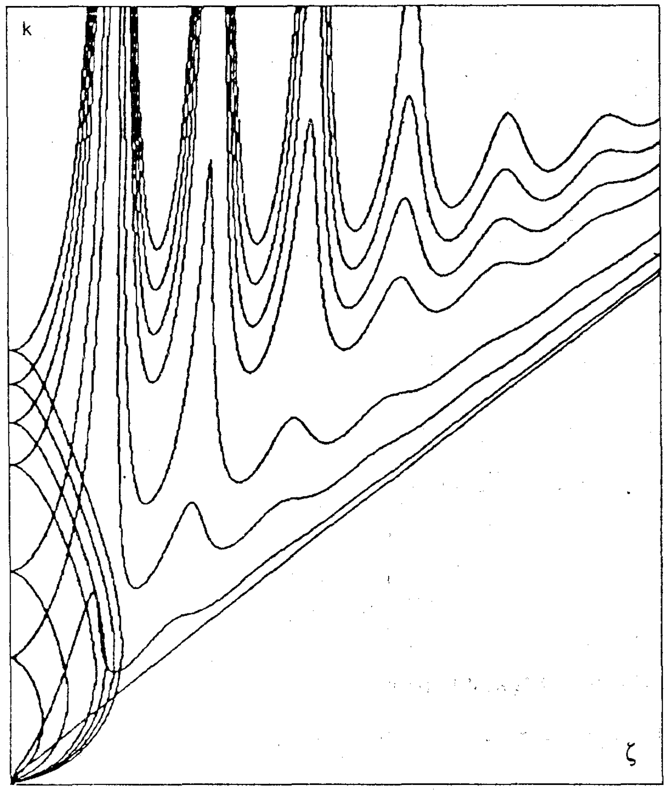

Fig. 1. The relation between k and ζ. The picture shows graphs of functions $k = k(\zeta) = \sqrt{E(\zeta)}$ for $\beta = 2$ and $\alpha = 0$ (the straight line), 1, 4, 8, 16, 20, 24 and 28. On the horizontal axis ζ varies from 0 to $3.2\pi \approx 10.052$, while on the vertical one k varies from 0 to 15. The singularities at the points $\zeta = n\pi/2$ are clearly visible as peaks. The larger is α the higher is the curve showing $k(\zeta) = \sqrt{E(\zeta)}$. The ovals in the left lower corner correspond to the case b) with real $z = \zeta = 4, 8, 16, 20, 24, 28$.

satisfies the equation

$$\ddot{U}_n(x,t) = U_n''(x,t) - \alpha U_n(x,t) \int_{-1}^{1} |U_n(x,t)|^2 dx + \varepsilon^n F(x,t) \tag{18}$$

where F is a known smooth function.

On the other hand, simple considerations following the scheme of the proof of the existence of the solution of a hyperbolic system (see [3]) show that the initial scattering problem (1)-(3) has a solution that we shall now denote by U^*.

* After this paper was completed G. Jona-Lasinio has kindly shown the authors his preprint [4] where the same kind of statement was proved for a more general case.

Subtracting (18) from the equation for the exact solution U, we get the equation for the difference $\delta = U - U_n$:

$$\ddot{\delta}(x, t) = \delta''(x, t) - \alpha/2[\,\delta(x, t)\int_{-1}^{1}\left(\left|\,U_n(x, t)\,\right|^2 + \left|\,U(x, t)\,\right|^2\right)dx +$$

$$+ (U_n(x, t) + U(x, t))\int_{-1}^{1}\mathrm{Re}\,((U_n(x, t) + U(x, t))\,\bar{\delta}(x, t))dx] \; - \; \varepsilon^n F(x, t).$$

Using the conventional energy estimates and taking into account that δ vanishes for $t = 0$ and the boundary conditions, one can prove that δ has the order n in ε.

We see that if the incident wave starts from the stationary one and then changes its amplitude slowly enough, the solution follows closely the sequence of corresponding stationary solutions while the amplitude does not reach the first minimum of the curve $k = k(\zeta)$. Meanwhile the transition coefficient may oscillate and so oscillates the amplitude of the transferred wave.

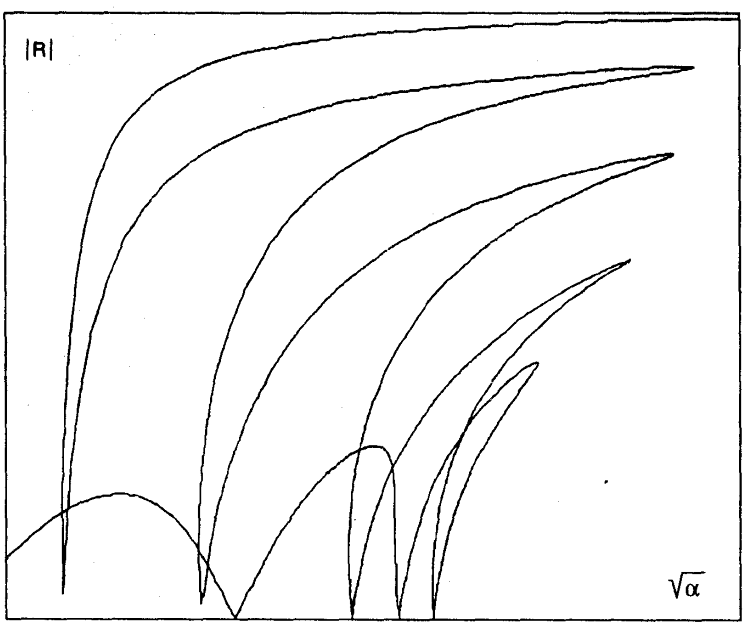

Fig. 2. The relation between the reflection coefficient $|\,R\,|$ and the amplitude of the incident wave for $k = 100$, $\beta = 2$.The values of $|\,R\,|$ from 0 to 1 are depicted along the vertical axis while that of $\sqrt{\alpha}$ (= amplitude) along the horizontal one (growing from 0 at the left end to $\sqrt{28} \approx 5.29$ at the right one). The resonant transparancies are clearly visible as the graph touches the bottom. The multistability proclaims itself through the existence of several values of $|\,R\,|$ corresponding to the same amplitude.

3. Discussion

The multistability (bistability, see [5]) is the phenomenon appearing when the wave is transmitted through nonlinear media. It consists of the non-uniqueness of the transmission and reflection coefficients for given parameters (the momentum and the amplitude) of the incident wave. It is studied in several papers ([7], also see the list in [6]).

It is of no wonder that a nonlinear equation in general may have multiple solutions. What's interesting in this simple case is the physical meaning of the behaviour exhibited and the possibility to get the rigorous and explicit solutions that allow one to study the genesis and behaviour of multistability for different values of parameters.

The close study of the results exposed above leads to the following conclusions:

For weak nonlinearity (for small α or k) there is no multistability.

There are obviously as many different stationary solutions of the scattering problem with the same incident wave $e^{-i(Et-kx)}$ as the number of intersections of the curve $y=k(\zeta)$ with the horizontal line $y=E$.

For nonlinearity (α, the amplitude of the incident wave, is large enough) there appear "forbidden" values of the momentum ζ in the area $|x|<1$. Namely for $|\zeta|^2<\alpha$ ξ cannot be equal to $k\pi/2$. Nevertheless, a solution exists for all energies $E>0$. As ζ approaches these forbidden values (while a remains constant) the reflection coefficient tends to 0. This is not a wonder since at these points E tends to infinity.

The dependence of the reflection (or transmission) coefficient on the amplitude of the incident wave is also of some interest. A complicated picture (Fig. 2) appears that describes the implicit function $R(\alpha)$ generated by two non-monotonic ones $R(\zeta)$ and $\alpha(\zeta)$.

The results on the quasi-stationary regimes obtained in the second part of the paper show that except at some special points of bifurcations this solution behaves properly, following the sequence of stationary states. But when combined with the function $R(\alpha)$, they may lead to a complicated behaviour of R as a function of time.

The behaviour at the bifurcation points (that is at the local minima of the function $k(\zeta)$, see Fig. 1) is unstable and too complicated to get the explicit solution.

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Мультистабильность в стационарной задаче рассеяния для нелинейной модели типа среднего поля

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Рассмотрены стационарная задача рассеяния для нелинейной модели типа модели среднего поля и квазистационарные решения соответствующего волнового уравнения с нелокальной нелинейностью. Изучено явление мультистабильности в стационарной задаче и получено асимптотическое разложение квазистационарных решений.

Мультистабільність в стаціонарній задачі розсіяння для нелінійної моделі типу середнього поля

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Розглянуто стаціонарна задача розсіяння для нелінійної моделі типу моделі середнього поля та квазістаціонарні рішення відповідного хвильового рівняння з нелокальною нелінійністю. Вивчено явище мультистабільності в стаціонарній задачі і одержано асимптотичний розклад квазістаціонарних рішень.