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A simple proof of Dubinin's Theorem

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Let Ω be a domain formed by removing n radial segments connecting the circles $\{ z : | z | = r_0 \}$ and $\{ z : | z | = 1 \}$ from the unit disk D. Let Ω_0 be a domain of the same type which is invariant with respect to rotation by the angle $2\pi / n$. If $\omega(z)$ and $\omega_0(z)$ are the harmonic measures of the unit circle with respect to these domains, then the inequality

$$\omega(0) \geq \omega_0(0)$$

holds, and the equality is possible only if the domain Ω coincides with Ω_0 up to rotation. This proposition is known as the Gonchar problem which has been proved by Dubinin. The aim of this paper is to give a more simple proof of this theorem.

In 1984 Dubinin [2] proved the following theorem:

Theorem 1. *Let* D *be the unit disk* $\{ z : | z | < 1 \}$ *and let*

$$\Omega = \mathrm{D} \setminus \bigcup_{k=1}^{n} I e^{i\lambda_k}, \quad \lambda_k \in [0, 2\pi), \quad I = [r_0, 1],$$

where $I\xi \overset{\text{def}}{=} \{ \xi z : z \in I \}$ *and let* Ω_0 *be a domain of the same type with* $\lambda_k = 2\pi k / n$. *Let* $\omega(z)$ *and* $\omega_0(z)$ *be the harmonic measures of the unit circle with respect to these domains respectively. Then*

$$\omega(0) \geq \omega_0(0),$$

and the equality is possible only if the domain Ω *coincides with* Ω_0 *up to rotation.*

Dubinin's proof attracted attention of many analysts and a series of papers related to this proof appeared, see, for example, papers [2-5]. This proof was based on Desymmetrization Principle for Dirichlet integrals and on a generalization of a Polya theorem on decreasing of Dirichlet integrals after a circular symmetrization. The generalization of the Polya theorem and its application is sufficiently complicated and here we give the new proof of Dubinin's theorem which does not use any circular symmetrization and uses Desymmetrization Principle only.

Lemma 1 in our paper is a convenient form of Desymmetrization Principle invented by Dubinin [2], Lemmas 3 and 4 are its application to special functions. Lemma 2 establishes the existence of special mappings which is necessary for proving Lemmas 3 and 4. Lemma 5 is trivial, but very essential for proving the main Theorem.

Let us begin with some definitions. Let $u(z)$ be a continuous function defined in a domain B of the complex plain C and smooth in this domain except some finite set L of smooth Jordan arcs. Denote by $D(u, B)$ the Dirichlet integral of this function in this domain:

$$D(u, B) = \iint_B |\operatorname{Grad} u(z)|^2 \, dm_2(z),$$

where m_2 is the plane Lebesgue measure and the integral is taken over all the domain B except the finite set of those Jordan arcs where Grad $u(z)$ is not defined. If the domain B coincides with the whole complex plane C, we will denote the corresponding Dirichlet integral by $D(u)$. So called Minimum Principle (see, for example, [6]) for the Dirichlet integrals is well known: let B_1 be a subdomain of B, if a continuous function $v(z)$ coincides with $u(z)$ on $B \setminus B_1$ and is harmonic inside B_1, then

$$D(v, B) \le D(u, B)$$

and the equality is possible only if $u(z)$ is harmonic inside B_1 (in other words, if $v(z) = u(z)$ on B_1).

Let K_0 and K_1 be two closed sets in the closure of B. Consider all the continuous functions smooth on the B except maybe some finite set of Jordan arcs and take the infimum of Dirichlet integrals over all such functions which are equal to 1 on K_1 and to 0 on K_0. Denote this infimum by $C(K_0, K_1, B)$ and call it the capacity of the capacitor (K_0, K_1) with respect to the domain B. If B coincides with the whole complex plane, we denote the capacity of the corresponding capacitor by $C(K_0, K_1)$. If the domain B and the sets K_0 and K_1 are sufficiently "good" (for example, when the boundary of the domain $B \setminus (K_0 \cup K_1)$ consists of smooth Jordan arcs), then according to Minimum Principle the capacity $C(K_0, K_1)$ may be evaluated as the Dirichlet integral of a potential function $v(z)$ of this capacitor:

$$C(K_0, K_1, B) = D(v, B),$$

where $v(z)$ is a solution of the Laplace equation of the mixed Dirichlet-Neumann boundary problem:

$$v(\xi) = 1, \quad \xi \in K_1,$$
$$v(\xi) = 0, \quad \xi \in K_0,$$
$$\frac{\partial v(\xi)}{\partial n} = 0, \quad \xi \in B \setminus (K_0 \cup K_1),$$

(see, for example, [6]).

In what follows, the main role will be played by $2n$-fold mappings of the complex plane that are continuous and analytic except maybe some finite set of smooth Jordan arcs. Since the Dirichlet integral is conformally invariant, then it is expected that such mappings will preserve Dirichlet integrals in some sense.

Let us consider in details some of these mappings.

Let $s(t)$ be a continuous 2π-periodic function with the following properties:

a) the range of $s(t)$ is the interval $[0, \pi/n]$;

b) the equation $s(t) = a$ has exactly $2n$ solutions on any period as $a \in (0, \pi/n)$;

c) $|s'(t)| = 1$ everywhere, except some finite set of points on any period of the function $s(t)$.

With every such function we associate a $2n$-fold mapping $S(z)$ of the complex plane $\mathbf{C}$ onto the sector $A_n = \{z : \arg z \in [0, \pi/n]\}$ defined as

$$S(z) = |z| e^{is(\arg z)}. \tag{1}$$

This mapping is $2n$-fold in the following sense: if we identify the rays $\{z : \arg z = 0\}$ and $\{z : \arg z = \pi/n\}$, then the preimage of any point $z \in A_n$ contains exactly $2n$ points. The mappings of the type described above preserve Dirichlet integrals in the following sense:

Lemma 1. *Let $u(z)$ be a continuous function defined on the angle A_n and smooth everywhere on this angle except maybe some finite set of smooth Jordan arcs. Let $D(u, A_n)$ be its Dirichlet integral. Then the equality*

$$D(u, A_n) = \frac{1}{2n} D(u \circ S) \tag{2}$$

holds, where $D(u \circ S)$ is the Dirichlet integral of the function

$$u \circ S(z) = u(S(z)), \quad z \in \mathbf{C},$$

over the whole complex plane.

P r o o f. Divide the angle A_n into a finite number of the angles $A_{n,j}$ whose interiors do not contain the critical rays of the mapping $S(z)$ (those rays whose arguments are the critical points of the function $s(t)$. The preimage of any $A_{n,j}$ is a union of $2n$ angles of the same opening and with nonintersecting interiors. The restriction of the mapping $S(z)$ on any angle $A_{n,j}$ is a superposition of a rotation and a mirror reflection which do not change the value of the Dirichlet integral. So relation (2) follows from additivity of the Dirichlet integral. The lemma is proved. □

The following our step is to construct the mappings of S-type possessing some special properties. Since the mapping S is completely defined by the function $s(t)$ with properties a) – c), we restrict ourselves to constructing the functions $s(t)$. The following lemma is very useful for our further applications.

Lemma 2. *Let L be a 2π-periodic set containing exactly n closed intervals on a period with nonintersecting interiors (some of these intervals may be one-point sets and some of them may have common endpoints). Let $|L|$ be the total length of these intervals evaluated on a period. Then there exists a function $s(t)$ possessing properties a) – c) and such that*

$$s^{-1}[0, |L|/(2n)] = L. \tag{3}$$

P r o o f. Let $f(t)$ be a continuous 2π-periodic function such that $|f'(t)| = 1$ except some finite set of points on a period. Let the range of this function contain the segment

$[0, |L|/(2n)]$. Let I be the range of all the critical points of the function $f(t)$. The points of I will be called critical levels. The preimage of any critical level is a set which consists of a finite number of minimum and maximum points of the function $f(t)$ on a period. If the number of the minimum points coincides with that of the maximum points for some critical level, then this level is called complete. Otherwise, the level is called incomplete. Let l be a critical level of a function $f(t)$ possessing properties a) and c). Introduce the deficiency of the level l as

$$d(f, l) = \#_{max}(l) - \#_{min}(l),$$

where $\#_{max}(l)$ and $\#_{min}(l)$ are the numbers of the maxima and the minima of the function $f(t)$ on a period which correspond to the level l. It is not very difficult to note that for checking whether a function $f(t)$ satisfying conditions a) and c) satisfies condition b) as well, we have to verify that any its critical level of the interval $(0, \pi/n)$ is complete or, in other words, $d(f, l) = 0$ for any $l \in (0, \pi/n)$. Indeed, if so, then the number $\#(r, a)$ of the solutions of the equation

$$f(t) = a, \quad a \in (0, \pi/n),$$

on a period does not depend on a. This number is even since the function $f(t)$ is 2π-periodic and continuous. Since $|f'(t)| = 1$

$$\text{mes}\,(f^{-1}(E) \cap [0, 2\pi]) = \#(f, a)\,\text{mes}\,(E) = \#(f)\,\text{mes}\,(E)$$

for any open set $E \in (0, \pi/n)$. This implies immediately that $\#(f) = \#(f, a) = 2n$.

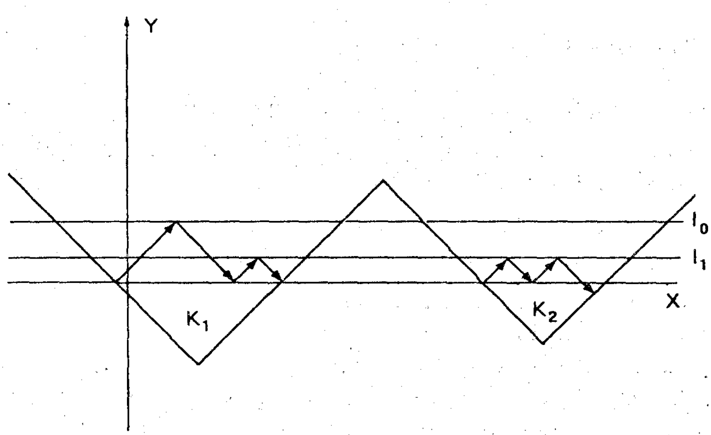

Fig. 1

Let us denote by L_c the 2π-periodic set consisting of n closed intervals on a period that are complementary to the intervals of the set L (this set has the same structure as the set L, it may contain one-point sets and some of its intervals may have common endpoints). Consider the function

$$f(t) = \text{dist}\,(t, L) - \text{dist}\,(t, L_c) + \frac{|\,L\,|}{2n}\,.$$

Let I be the set of the critical levels of this function. It is not difficult to note that the deficiencies of the levels greater than $|\,L\,|/(2n)$ are nonnegative and the deficiencies of the levels less than $|\,L\,|/(2n)$ are nonpositive. The deficiency of the level $|\,L\,|/(2n)$ may have an arbitrary sign, this depends on the number of common endpoints of the intervals forming the set L and on the number of intervals consisting of a single point.

Starting with the function $f(t)$ satisfying conditions a) and c), we will modify it to satisfy condition b).

We make all the modification on the set $\{f \le |\,L\,|/(2n)\}$ (on the set $\{f \ge |\,L\,|/(2n)\}$ they may be done in the same way). The scheme will be the following: first we describe the single step of the modification that allows us either to get rid of one of the intervals where the function $f(t)$ is negative (greater than π/n in the case $\{f \ge |\,L\,|/(2n)\}$) or to get rid of all the critical levels with the negative deficiencies (with the positive ones in the case $\{f \ge |\,L\,|/(2n)\}$.

Consider the set of the critical levels $l_0 \ge l_1 \ge \ldots \ge l_k$, $l_j \in (0, |\,L\,|\,(2n))$, $l_0 \le |\,L\,|\,(2n)$, and take an interval where $f(t) < 0$. To modify the function $f(t)$ replace its graph with the line which is the path of the ray ejected from the left endpoint of the interval in the direction of the vector $(1, 1)$ and making mirror reflection between the lines $\{y = l_0\}$ and $\{y = 0\}$ consequently. The deficiency of the level $\{y = l_0\}$ is enlarged by 1 after any such reflection in the line $\{y = l_0\}$ (the initial deficiency of this level is negative) We proceed with such reflections between the lines $\{y = l_0\}$ and $\{y = 0\}$ while either the path of the ray meets the graph of the function $f(t)$ or the level l_0 gets complete. In the latter case we proceed with the reflections between the line $\{y = 0\}$ and the next critical level $\{y = l_1\}$, etc. This process have to be stopped either if the path meets the graph of the function $f(t)$ or if we get rid of the critical levels $\{l_0, l_1, \ldots, l_k\}$ at all (in the latter case we prolongate the path of the ray without any reflection up to meeting the graph of the function f).

Denote now the modified function by $f(t)$. If this function has at least one critical level of negative deficiency and an interval of negativity simultaneously, then proceed the modification in the same way. (The possibilities which we get after two such steps are shown in Fig. 1. The initial function $f(t)$ on Figure 1 has two critical levels with the deficiencies $d(f, l_0) = -1$ and $d(f, l_1) = -3$. Position K_1 in Figure 1 corresponds to the case when we meet the graph of the function $f(t)$ at some point where $f(t) \ge 0$, position K_2 presents the case when we get rid of all the critical levels $\{l_0, l_1, \ldots, l_k\}$.

So after a finite number of steps of modification of the function $f(t)$ we get a new one which either has no intervals of negativity or has no critical levels on the interval $(0, |\,L\,|/(2n)]$ with the negative deficiencies (i.e. all the critical levels of the interval $(0, |\,L\,|/(2n))$ are complete and only the level $|\,L\,|/(2n)$ may be incomplete but with a positive deficiency).

Let us show that in the first case the function $f(t)$ has no critical levels on the interval $(0, |\,L\,|/(2n))$. Indeed, let l_* be a critical level of the function $f(t)$. By construction, all the deficiencies of the function $f(t)$ on this interval have nonnegative deficiencies, so the equation

$$f(t) = l, \quad l \in (0, \mid L \mid /(2n)),$$

has not more than $2n$ solutions (the number of the solutions does not increase when l decreases). Since for $l < l_*$ the number of the solutions is less than $2n$, it follows from the relation $\mid f'(t) \mid = 1$ that

$$\text{mes}\,\{ f^{-1}\{ (0, \mid L \mid /(2n)) \} \cap [0, 2\pi] \} < \mid L \mid$$

The latter inequality implies that the function $f(t)$ in this case must be negative on some interval, but this contradicts to the hypothesis $f(t) \ge 0$. Hence, the function $f(t)$ has no incomplete critical levels on the interval $(0, \mid L \mid /(2n))$.

Making the same modification of the function $f(t)$ on the set $\{ f(t) \ge \mid L \mid /(2n) \}$ (ejecting the rays in the direction of the vector $(1, -1)$ from the left endpoints of the intervals where $f(t) > \pi / n$ and using the lines $\{ y = \pi / n \}$ and $\{ y = l_j \}$ with the critical levels $l_0 \ge l_1 \ge \ldots \ge l_k$, $l \in [\mid L \mid /(2n), \pi / n)$, $l_0 \ge \mid L \mid /(2n)$, for the reflections) we get the new function with all its critical levels of the interval $(0, \pi / n)$ being complete. According to the remark made at the very beginning of the proof, this function satisfies all the conditions of the lemma. The proof is completed. □

The following two lemmas form the frame of the proof of the main theorem.

Lemma 3. *Let $u(z)$ and $u_0(z)$ be two continuous functions on the closed unit disc, equal to $- \log \mid z \mid$ on the boundaries and harmonic inside the domains Ω and Ω_0 respectively. Then the inequality*

$$D(u) \le D(u_0)$$

holds and the equality is possible only if these domains coincide up to rotation.

P r o o f. Without loss of generality we can suppose that one of the slits of the domain Ω_0 lies on the positive ray. By Lemma 2 there exists a mapping $S(z)$ that transfers the radial slits of the domain Ω onto the segment $[r_0, r_1]$. So the function $u_0 \circ S$ has the same boundary values on Ω as the function u. Since the function $u_0(z)$ is invariant with respect to rotation by the angle $2\pi / n$ and the mirror reflection in the real axis, then by Lemma 1 and Minimum Principle we have

$$D(u_0) = 2nD(u_0, A_n) = D(u_0 \circ S) \ge D(u). \tag{4}$$

The equality in (4) is possible only in the case if the function $u \circ S$ is harmonic inside Ω. Since $\partial\, u_0(re^{i\theta}) / \partial\, \theta > 0$ as $r \in (0, 1)$, $\theta \in (0, \pi / n)$, (which is not difficult to verify), then to provide the smoothness of the function $u_0 \circ S$ inside Ω the mapping $S(z)$ must have no critical ray the image of which lies inside A_n. The latter is possible only if the function $s(t)$ by means of which we have constructed the mapping $S(z)$, has the form

$$s(t) = \text{dist}\left(t, \{ 2\pi / n + h \}_{k = -\infty}^{\infty}\right)$$

for some real h. Hence, the domain Ω must coincide with Ω_0 up to rotation.The lemma is proved. □

Lemma 4. *Let L be a set consisting of at most n arcs of the unit circle and let a set $\tilde{L}$ consist exactly of n arcs on the unit circle and be invariant with respect to rotation by the angle $2\pi/n$. Let us suppose that the sets L and $\tilde{L}$ have the same total lengths and let $R = \{ t < |z| < t^{-1} \} \setminus L$ and $R_0 = \{ t < |z| < t^{-1} \} \setminus \tilde{L}$ for some $t \in (0, 1)$.*

If functions $v(z)$ and $v_0(z)$ are continuous in the closure R, harmonic inside R and R_0 respectively, both vanishing on the circles $\{ |z| = t^{\pm 1} \}$ and equal to 1 on the sets L and $\tilde{L}$ respectively, then

$$D(v) \le D(v_0).$$

P r o o f. The proof is similar to that of the previous lemma. Without loss of generality we will suppose that the set $\tilde{L}$ contains the arc $\{ e^{i\theta} : \theta \in (- |L|/(2n), |L|/(2n)) \}$, where $|L| = |\tilde{L}|$ is the total length of the intervals forming the set L. We will assume that the set L is a union of n arcs with nonintersecting interiors (some of which may have common endpoints). By Lemma 2 there exists a mapping $S(z)$ such that

$$S^{-1}(\tilde{L} \cap A_n) = L.$$

Hence,

$$D(v_0) = 2nD(v_0, A_n) = D(v_0 \circ S) \ge D(v). \qquad (5)$$

The lemma is proved. □

Since the functions $v_0(z)$ and $v(z)$ in Lemma 4 are potential functions of the capacitors $(\tilde{L}, \{ |z| < t \})$ and $(L, \{ |z| < t \})$ with respect to the unit disc $\mathbf{D} = \{ z : |z| < 1 \}$, then the latter Lemma yields the relation for their capacities:

C o r o l l a r y 1.

$$C(L, t\mathbf{D}, \mathbf{D}) \le C(\tilde{L}, t\mathbf{D}, \mathbf{D}).$$

The following lemma is simple but very useful.

Lemma 5. *Let R be an annulus $\{ z : |z| \in (t, 1) \}$ and let R_* be the set formed by removing some set of radial segments from this annulus. Consider the capacitor formed by the pair of circumferences $K_0 = \{ z : |z| = 1 \}$ and $K_1 = \{ z : |z| = t \}$. The capacity of this capacitor does not depend on whether we evaluate it with respect to R or do this with respect to R_*. In other words,*

$$C(K_0, K_1, B_*) = C(K_0, K_1, B).$$

P r o o f. In the both cases the potential functions of these capacitors are the same and coincide with $\log |z| / \log t$. The lemma is proved. □

P r o o f o f t h e m a i n t h e o r e m. Let $f(z)$ and $f_0(0)$ be conformal mappings of the domains Ω and Ω_0 onto the unit disk $D = \{ |z| < 1 \}$ such that $f(0) = f_0(0) = 0$. Denote the images of the unit circle under these mappings by L and L_0 respectively. Let the domains Ω and Ω_0 do not coincide up to rotation. If $|L|$ and $|L_0|$ are the total lengths of the sets L and L_0, then the assertion of the theorem is equivalent to the following:

$$|L| > |L_0|. \tag{6}$$

Let us prove this.

Note that

$$|f'(0)| < |f_0'(0)|. \tag{7}$$

Indeed, let $u(z)$ and $u_0(z)$ be functions continuous on the closed unit disc and harmonic inside Ω and Ω_0 with the boundary values $-\log|z|$ on $\partial\Omega$ and on $\partial\Omega_0$ respectively. The functions $u(z)$ and $u_0(z)$ may be considered as superharmonic ones on the unit disc $\mathbf{D}$. Denote their Riesz measures by μ and μ_0 respectively. Since these measures are supported by the boundaries of Ω and Ω_0 respectively, then by the Green formula we have

$$\log|f'(0)| = \log\left|\frac{f(z)}{z}\right|_{z=0} = u(0) = \int_{z\in\mathbf{D}} (-\log|z|)\, d\mu(z) =$$

$$= \int_{z\in\mathbf{D}} G(z,0)\, d\mu(z) = \int_{z\in\mathbf{D}} u(z)\, d\mu(z) = \int\int_{\xi, z\in\mathbf{D}} G(z,\xi)\, d\mu(\xi)\, d\mu(z) = D(u),$$

where $G(\cdot\,,\cdot)$ is the Green function of the unit disc. This relation together with the similar one for the function $u_0(z)$ and Lemma 3 imply (7).

Relation (7) yields that we can take a $t>0$ to provide

$$f(\{|z|<\varepsilon\}) \subset \{|\xi|<t\}) \subset f_0(\{|z|<\varepsilon\}) \tag{8}$$

for some positive $\varepsilon>0$. Let $\mathbf{T}$ be the unit circle $\mathbf{T}=\{z : |z|=1\}$ and let $\varepsilon\mathbf{T}=\{z : |z|=\varepsilon\}$. By Lemma 5 the capacities $C(\mathbf{T}, \varepsilon\mathbf{T}, \Omega)$ and $C(\mathbf{T}, \varepsilon\mathbf{T}, \Omega_0)$ coincide. Since any capacity is invariant with respect to conformal mappings, then

$$C(L, f(\{\varepsilon\mathbf{T}\}, \mathbf{D}) = C(L_0, f_0(\{\varepsilon\mathbf{T}\}, \mathbf{D}),$$

or, accounting for (8),

$$C(L, t\mathbf{T}, \mathbf{D}) > C(L_0, t\mathbf{T}, \mathbf{D}).$$

Let $\widetilde{L}$ be a set of n arcs on the unit circle and invariant with respect to rotation by the angle $2\pi/n$. If $|\widetilde{L}| = |L|$, then by Corollary of Lemma 4

$$C(\widetilde{L}, t\mathbf{T}, \mathbf{D}) \geq C(L, t\mathbf{T}, \mathbf{D}) > C(L_0, t\mathbf{T}, \mathbf{D}).$$

Since both the sets $\widetilde{L}$ and L_0 consist of n arcs and are invariant with respect to rotation by the angle $2\pi/n$, we may suppose without loss of generality that either $\widetilde{L} \subseteq L$ or $\widetilde{L} \supseteq L$. The latter inequality yields that $\widetilde{L} \supset L$ is true and $\widetilde{L} \neq L$. So we have

$$|\widetilde{L}| = |L| > |L_0|.$$

The theorem is proved. □

References

1. *V.N. Dubinin*, On the change of harmonic measure under symmetrization.— Math. USSR Sb.(1985), v. 52, pp. 267–273.
2. *A. Baernstein II*, Dubinin's symmetrization theorem. Complex analysis I. Proc. Univ. of Maryland 1985-86.— Lect. Notes in Mathematics. (1987), v. 1275, Springer-Verlag.
3. *A. Baernstein II*, On the harmonic measure of slit domains.— Complex Variables (1987),v. 9, pp. 131–142.
4. *A. Baernstein II*, An extremal property of meromorphic functions with N-fold symmetry (to appear).
5. *J.R. Quine*, On the harmonic measure of slit domains.— Complex Variables (1989), v. 6, pp. 223–231.
6. *L V. Ahlfors*, Conformal Invariants (1973).

Простое доказательство теоремы Дубинина

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Пусть Ω — область, которая образована удалением n радиальных сегментов из единичного круга D, соединяющих окружности $\{z : |z| = r_0\}$ и $\{z : |z| = 1\}$. Пусть Ω_0 — область того же типа, инвариантная относительно вращения на угол $2\pi/n$. Если $\omega(z)$ и $\omega_0(z)$ — гармонические меры единичной окружности относительно этих областей, то выполнено неравенство

$$\omega(0) \geq \omega_0(0),$$

и равенство возможно только, если область Ω совпадает с Ω_0 с точностью до вращения. Это предложение известно как задача Гончара, решение которой было найдено Дубининым. Цель настоящей работы — дать более простое доказательство этого утверждения.

Просте доведення теореми Дубініна

О.Є. Фринтов

Нехай Ω — область, яка утворена вилученням n радіальних сегментів з одиничного кругу D, що зв'язують кола $\{z : |z| = r_0\}$ та $\{z : |z| = 1\}$. Нехай Ω_0 — область того ж типу, інваріантна відносно обертання на кут $2\pi/n$.Якщо $\omega(z)$ та $\omega_0(z)$ є гармонічні міри одиничного кола відносно цих областей, тоді виконується нерівність

$$\omega(0) \geq \omega_0(0),$$

і рівність можлива тільки, коли область Ω збігається з Ω_0 з точністю до обертання. Ця пропозиція відома як задача Гончара, рішення якої було знайдено Дубініним. Мета роботи — просте доведення цього твердження.