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The solvability of the second main mixed problem of the theory of elasticity in a complete scale of Sobolev spaces

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The unique solvability of the second main problem of the dynamic elasticity theory is proved in a complete scale of Sobolev spaces which include the spaces with negative norm.

In this article we continue the investigations made in [1] concerning the mixed problems of the elastisity theory. All the notations and definitions from [1] are valid here.

1. The formulation of the problem

We look for a vector function $u(X)$, $X \in G_T^{\pm}$ satisfying the motion equation

$$\partial_t^2 u(X) + (Au)(X) = q(X), \qquad X \in G_T^{\pm} \tag{1}$$

and the homogeneous initial conditions

$$u(x, +0) = (\partial_t u)(x, +0) = 0, \qquad x \in \Omega^{\pm} \tag{2}$$

where A is the matrix differential operator of the anisotropic elasticity theory [1]. The boundary conditions in the problems $\mathrm{II}_r^{\pm}$ (internal or external) have the form $(T_\nu u)^{\pm}(X) = g(X)$, $X \in \Sigma_T$, where T_ν is the normal boundary stress operation acting by

$$(T_\nu u)_i(X) = \sigma_{ij}(u)\nu_j(x),$$

$\sigma_{ij}(u)$ are the components of the stress tensor of the medium and $\nu(x) = (\nu_1(x),\dots,\nu_d(x))$ is an outward unit normal to S.

A vector function $u \in H_{r;1,0}(G_T^{\pm})$ will be called *the generalized solution of the problems* $\mathrm{II}_r^{\pm}$ if it satisfies the variational equation

$$\left(\sigma_{ij}(u), \varepsilon_{ij}(\eta)\right)_{0,T} - \left(\partial_t u, \partial_t \eta\right)_{0,T} = (q, \eta)_{0,T} \pm \langle g, \eta \rangle_{0,T}$$

for any element $\eta \in H_{a;1,0}(G_T^{\pm})$. The "advanced" problems $\mathrm{II}_a^{\pm}$ are formulated quite similarly. The only difference from the problems $\mathrm{II}_r^{\pm}$ is that the initial conditions

$$u(x, T-0) = (\partial_t u)(x, T-0) = 0, \qquad x \in \Omega^{\pm}$$

are given instead of (2). Let us consider the case $q = 0$. It was shown in [2] that the solving operators $\overset{\circ}{R}{}^{\pm}_{2,r}$, $\overset{\circ}{R}{}^{\pm}_{2,a}$ of the problems $\mathrm{II}^{\pm}_r$, $\mathrm{II}^{\pm}_a$ which set the correspondence between the boundary value g and the solutions of these problems in this case perform the maps

$$\overset{\circ}{R}{}^{\pm}_{2,r} : H_{r;\, m,\, k}(\Sigma_T) \to H_{r;\, m+3/2,\, k-1}(G^{\pm}_T),$$
$$\overset{\circ}{R}{}^{\pm}_{2,a} : H_{a;\, m,\, k}(\Sigma_T) \to H_{a;\, m+3/2,\, k-1}(G^{\pm}_T), \tag{3}$$

which were continuous for all $m \geq -1/2$, $k \in \mathbf{R}$.

2. Normal boundary stress operators

Considering case $q = 0$ let us introduce the operators $N^{\pm}_r$ and $N^{\pm}_a$ setting correspondence between displacements $u(X)$ of boundary points of the medium and the normal boundary stresses. Now we give the correct definition for these operators. Let $f \in H_{r;\,1/2,\,1/2}(\Sigma_T)$ and $u = \overset{\circ}{R}{}^{\pm}_{1,r} f \in H_{r;\,1,\,0}(G^{\pm}_T)$ be the solution of the problems $\mathrm{I}^{\pm}_r$ with the boundary value f [1]. Further, let v be an arbitrary element of the space $H_{a;\,1,\,0}(G^{\pm}_T)$, $\gamma^{\pm}_a v = z \in H_{a;\,1/2,\,0}(\Sigma_T)$. We define operators $N^{\pm}_r$ on the element f by the equalities

$$\pm \langle N^{\pm}_r f, z \rangle_{0,T} = \left(\sigma_{ij}(u), \varepsilon_{ij}(v)\right)_{0,T} - \left(\partial_t u, \partial_t v\right)_{0,T}. \tag{4}$$

The operators $N^{\pm}_a$ are defined in the same way. It was shown in [2, 3] that the normal boundary stress operators for all $m \geq -1/2$, $k \in \mathbf{R}$ perform the continuous maps

$$N^{\pm}_r : H_{r;\, m+1,\, k}(\Sigma_T) \to H_{r;\, m,k}(\Sigma_T),$$
$$N^{\pm}_a : H_{a;\, m+1,\, k}(\Sigma_T) \to H_{a;\, m,\, k}(\Sigma_T). \tag{5}$$

Lemma 1. *For all* $f_r \in C^{\infty}_r(\overline{\Sigma}_T)$, $f_a \in C^{\infty}_a(\overline{\Sigma}_T)$ *the equalities*

$$\langle N^{\pm}_r f_r, f_a \rangle_{0,T} = \langle f_r, N^{\pm}_a f_a \rangle_{0,T} \tag{6}$$

are valid.

P r o o f. Let us write (4) for $f = f_r$, $z = f_a$, $v = \overset{\circ}{R}{}^{\pm}_{1,a} f_a$. Writing equalities

$$\pm \langle N^{\pm}_a f_a, f_r \rangle_{0,T} = \left(\sigma_{ij}(v), \varepsilon_{ij}(u)\right)_{0,T} - \left(\partial_t v, \partial_t u\right)_{0,T}$$

and taking into account the symmetry properties of the form

$$\left(\sigma_{ij}(u), \varepsilon_{ij}(v)\right)^{*}_{0,T} = a_{ijkh}\left(\partial_k u_h, \partial_i v_j\right)_{0,T}\,\tilde{}$$

[4] we obtain (6).

The formula (6) makes it possible to define the operators $N^{\pm}_r$, $N^{\pm}_a$ on spaces $H_{r;\,m,\,k}(\Sigma_T)$, $H_{a;\,m,\,k}(\Sigma_T)$ for $m < 1/2$, $k \in \mathbf{R}$ by

$$N^{\pm}_r = (N^{\pm}_a)^{*}, \qquad N^{\pm}_a = (N^{\pm}_r)^{*}. \tag{7}$$

The validity of the next statement follows from (5), (7) immediately.

Lemma 2. The operators $N_r^{\pm}$, $N_a^{\pm}$ perform the maps (5) which are continuous for all $m, k \in \mathbf{R}$.

3. The properties of operators generated by elastic dynamic potentials

Elastic retarded and advanced double-layer potentials with d-component densities $\beta_r(X), \beta_a(X)$ defined on Σ_T are introduced by

$$(W_r\beta_r)(X) = \underset{\Sigma_T}{S}\left((T_{\nu(y)}\Phi_j^r)(X-Y), \beta_r(Y)\right)e_j\, ds_Y,$$

$$(W_a\beta_a)(X) = \underset{\Sigma_T}{S}\left((T_{\nu(y)}\Phi_j^a)(X-Y), \beta_a(Y)\right)e_j\, ds_Y$$

where $T_{\nu(y)}$ is the normal boundary stress operation acting with respect to the variable y. It is evident that both the potentials with densities $\beta_r \in C_r^\infty(\overline{\Sigma}_T), \beta_a \in C_a^\infty(\overline{\Sigma}_T)$ satisfy the homogeneous equation (1) and the homogeneous initial conditions for $t = 0$ or $t = T$, respectively. Moreover, it was shown in [2] that both the potentials with the densities from the classes presented above (and much wider ones) could be repesented in the form

$$\left.\begin{aligned}(W_r\beta_r)(X) &= \left(V_{r,\pm} N_r^{\mp}\beta_r\right)(X),\\ (W_a\beta_a)(X) &= \left(V_{a,\pm} N_a^{\mp}\beta_a\right)(X),\end{aligned}\right\} \qquad X \in G_T^{\pm}.$$

Let us introduce operators $W_{r,\pm}$, $W_{a,\pm}$ setting correspondence between densities β_r, β_a and the values of the corresponding potentials $(W_r\beta_r)(X)$, $(W_a\beta_a)(X)$, $X \in G_T^{\pm}$. From the equalities $W_{r,\pm} = V_{r,\pm} N_r^{\mp}$, $W_{a,\pm} = V_{a,\pm} N_a^{\mp}$ and from the statements of Lemma 2 and Theorem 3 [1] the continuity of the maps

$$\begin{aligned}&W_{r,\pm} : H_{r;\, m,\, k}(\Sigma_T) \to H_{r;\, m+1/2,\, k-1}(G_T^{\pm}),\\ &W_{a,\pm} : H_{a;\, m,\, k}(\Sigma_T) \to H_{a;\, m+1/2,\, k-1}(G_T^{\pm})\end{aligned} \tag{8}$$

follows for all $m, k \in \mathbf{R}$.

Now we introduce the operators of limit values of double-layer potentials by the equalities

$$W_r^{\pm} = V_r N_r^{\mp}, \quad W_a^{\pm} = V_a N_a^{\mp}.$$

The well-known formulae for jumps of elastic potentials [5] in our notations take the form

$$\begin{aligned}&(N_r^+ - N_r^-)V_r = I, \qquad (N_a^+ - N_a^-)V_a = I,\\ &W_r^+ - W_r^- = -I, \qquad W_a^+ - W_a^- = -I\end{aligned} \tag{9}$$

where I is the unit operator.

Lastly, let us define double-layer potential normal boundary stress operators by the equalities

$$F_r = N_r^+ W_r^+ = N_r^- W_r^-, \qquad F_a = N_a^+ W_a^+ = N_a^- W_a^- .$$

It is evident that the correctness of these definitions follows from the jumps formulae (9). In [2, 3] the continuity of the maps

$$\begin{aligned} F_r^{-1} &: H_{r;\, m,\, k}(\Sigma_T) \to H_{r;\, m+1,\, k-1}(\Sigma_T), \\ F_a^{-1} &: H_{a;\, m,\, k}(\Sigma_T) \to H_{a;\, m+1,\, k-1}(\Sigma_T) \end{aligned} \tag{10}$$

was proved for any $m \geq -1/2$, $k \in \mathbf{R}$.

Lemma 3. *The operators F_r^{-1}, F_a^{-1} perform the maps (10) which are continuous for all $m, k \in \mathbf{R}$.*

P r o o f. From the equalities $F_r = N_r^+ V_r N_r^-$, $F_a = N_a^- V_a N_a^+$, (7) and from the fact that the operators V_r, V_a are adjoint [1] it follows that the operators F_r, F_a are adjoint too: $F_r^* = F_a$, $F_a^* = F_r$. These statements and the continuity of the maps (10) for $m \geq -1/2$, $k \in \mathbf{R}$ prove the statements of the Lemma.

Now let us get down to the properties of the retarded and the advanced volume potential operators introduced in [1]. Supposing that q_r, $q_a \in C^\infty\left(\overline{\mathbf{R}_T^{d+1}}\right)$, *supp* $q_r \subset \Omega^\pm \times (0, T]$, *supp* $q_a \subset \Omega^\pm \times [0, T)$, q_r and q_a are finite in the case of domain Ω^- — we define operators $D_{r,\pm}$, $D_{a,\pm}$ on elements q_r, q_a by

$$\begin{aligned} \left(D_{r,\pm}\, q_r\right)(X) &= \left(T_\nu\, U_r\, q_r\right)^\pm(X), \\ \left(D_{a,\pm}\, q_a\right)(X) &= \left(T_\nu\, U_a\, q_a\right)^\pm(X), \qquad X \in \Sigma_T . \end{aligned}$$

Lemma 4. *For all $\beta_r \in C_r^\infty(\overline{\Sigma}_T)$, $\beta_a \in C_a^\infty(\overline{\Sigma}_T)$ and q_r, q_a from the classes presented above the equalities*

$$\begin{aligned} (W_{r,\pm}\beta_r, q_a)_{0,T} &= \langle \beta_r, D_{a,\pm}\, q_a \rangle_{0,T}, \\ (W_{a,\pm}\beta_a, q_r)_{0,T} &= \langle \beta_a, D_{r,\pm}\, q_r \rangle_{0,T} \end{aligned} \tag{11}$$

are valid.

P r o o f.

$$(W_{r,\pm}\beta_r, q_a)_{0,T} = \underset{G_T^\pm}{S}\, q_{a,i}(X) \underset{\Sigma_T}{S} \left(T_{\nu(y)}\, \Phi_i^r\right)_l (X - Y)\beta_{r,l}(Y)\, ds_y\, dX =$$

$$= \underset{\Sigma_T}{S}\, \beta_{r,l}(Y) \underset{G_T^\pm}{S} \left(T_{\nu(y)}\, \Phi_i^r\right)_l (X - Y)\, q_{a,i}(X)\, dX\, ds_Y ,$$

It follows from the definition of the operation T_ν that

$$\left(T_{\nu(y)}\, \Phi_j^r\right)_l (X - Y) = a_{ljkh} \frac{\partial}{\partial y_k} \Phi_{hi}^r (X - Y)\, \nu_j(y) = a_{ljkh} \frac{\partial}{\partial y_k} \Phi_{ih}^a (Y - X)\, \nu_j(y).$$

Therefore,

$$(W_{r,\pm}\beta_r, q_a)_{0,T} = \underset{\Sigma_T}{S}\beta_{r,l}(X)\underset{G_T^\pm}{S} a_{ljkh}\frac{\partial}{\partial x_k}\Phi^a_{ih}(X-Y)\, q_{a,i}(Y)\, \nu_j(x)\, dY\, ds_X =$$

$$= \underset{\Sigma_T}{S}\beta_{r,l}(X)\, a_{ljkh}\frac{\partial}{\partial x_k}\left(\underset{G_T^\pm}{S}\Phi^a_{ih}(X-Y)\, q_{a,i}(Y)\right)\nu_j(x)\, ds_X = \langle \beta_r, D_{a,\pm}\, q_a\rangle_{0,T}$$

The second equality in (11) is checked similarly.

The results of Lemma 4 give us an opportunity to extend the operators $D_{r,\pm}$, $D_{a,\pm}$ by continuity onto spaces $\mathring{H}_{r;\,m,\,k}(G_T^\pm)$, $\mathring{H}_{a;\,m,\,k}(G_T^\pm)$, respectively, by

$$D_{r,\pm} = W^*_{a,\pm}, \qquad D_{a,\pm} = W^*_{r,\pm} \tag{12}$$

From (12) and from the continuity of the maps (8) the validity of the next statements follows immediately.

The operators $D_{r,\pm}$, $D_{a,\pm}$ perform the maps

$$\begin{aligned} D_{r,\pm} &: \mathring{H}_{r;\,m,\,k}(G_T^\pm) \to H_{r;\,m+1/2,\,k-1}(\Sigma_T),\\ D_{a,\pm} &: \mathring{H}_{a;\,m,\,k}(G_T^\pm) \to H_{a;\,m+1/2,\,k-1}(\Sigma_T) \end{aligned} \tag{13}$$

which are continuous for all $m, k \in \mathbf{R}$.

In conclusion of this section let us introduce operators $D^\pm_{r,\pm}$, $D^\pm_{a,\pm}$ by the formulas $\left(D^\pm_{r,\pm} q_r\right)(X) = \left(T_\nu U_r^\pm q_r\right)(X)$, $\left(D^\pm_{a,\pm} q_a\right)(X) = \left(T_\nu U_a^\pm q_a\right)(X)$, $X \in \Sigma_T$ where $U_r^\pm$, $U_a^\pm$ are the operators introduced in [1]. It is evident that for all $m > -1/2$, $k \in \mathbf{R}$ these operators perform the continuous maps

$$\begin{aligned} D^\pm_{r,\pm} &: H_{r;\,m,\,k}(G_T^\pm) \to H_{r;\,m+1/2,\,k-1}(\Sigma_T),\\ D^\pm_{a,\pm} &: H_{a;\,m,\,k}(G_T^\pm) \to H_{a;\,m+1/2,\,k-1}(\Sigma_T). \end{aligned} \tag{14}$$

4. The solvability of the second main mixed problem

Let us suppose that the right-hand side of the equation (1) $q \in C^\infty(\overline{G_T^\pm})$, *supp* $q \subset \Omega^\pm \times (0, T]$ and q is finite in the case of domain G_T^-. We also suppose that the boundary value $g \in C_r^\infty(\overline{\Sigma}_T)$. The solutions of the problems $\mathrm{II}_r^\pm$ can be represented in the form

$$u(X) = (U_r q)(X) + (W_{r,\pm} F_r^{-1}(g - D_{r,\pm} q))(X), \ X \in G_T^\pm.$$

In the case *supp* $q \subset \overline{\Omega^\pm} \times (0, T]$ we represent the solutions of these problems in the form

$$u(X) = (U_r^\pm q)(X) + (W_{r,\pm} F_r^{-1}(g - D^\pm_{r,\pm} q))(X), \ X \in G_T^\pm. \tag{15}$$

Let us introduce solving operators of the problems $\mathrm{II}_r^\pm$ $R^\pm_{2,r}$ which set the correspondence between the pair $\{q, g\}$ and the solutions $u(X)$, $X \in G_T^\pm$ of these problems. It is shown below that the operators $R^{\pm}_{2,r}$ can be extended by continuity onto products of Sobolev spaces including spaces with the negative norm.

Theorem 1. *For all $k \in \mathbf{R}$ the operators $R^{\pm}_{2,r}$ perform the continuous maps*

$$R^{\pm}_{2,r} : \begin{cases} H_{r;m,k}(G^{\pm}_T) \times H_{r;m+1/2,k-1}(\Sigma_T), & m \geq -1 \\ \mathring{H}_{r;m,k}(G^{\pm}_T) \times H_{r;m+1/2,k-1}(\Sigma_T), & m \leq -1 \end{cases} \to H_{r;m+2,k-2}(G^{\pm}_T).$$

P r o o f. First we consider the case $m \geq -1$. From (15) and from the continuity of the maps (14) and (18) in [1] it follows that

$$U^{\pm}_r q \in H_{r;m+2,k-1}(G^{\pm}_T), \qquad g - D^{\pm}_{r,\pm} q \in H_{r;m+1/2,k-1}(\Sigma_T)$$

for any $q \in H_{r;m,k}(G^{\pm}_T)$, $g \in H_{r;m+1/2,k-1}(\Sigma_T)$. The reguired statement in the case $m \geq -1$ follows immediately from the equalities $W_{r,\pm} F^{-1}_r = \mathring{R}^{\pm}_{2,r}$ and from the continuity of the maps (3).

Now let us suppose that $m \leq -1$, $q \in \mathring{H}_{r;m,k}(G^{\pm}_T)$, $g \in H_{r;m+1/2,k}(\Sigma_T)$ and write the solutions of the problems $\mathrm{II}^{\pm}_r$ in the form

$$u = U_r q + W_{r,\pm}\left(F^{-1}_r g - F^{-1}_r D_{r,\pm} q\right).$$

Noticing that $F^{-1}_r D_{r,\pm} = \left(W_{a,\pm} F^{-1}_a\right)^* = \left(\mathring{R}^{\pm}_{2,a}\right)^*$ and using continuity of the second of the maps (3) we see that

$$F^{-1}_r D_{r,\pm} : \mathring{H}_{r;m,k}(G^{\pm}_T) \to H_{r;m+3/2,k-1}(\Sigma_T) \tag{16}$$

are continious for all $m \leq -1$, $k \in \mathbf{R}$. From (16) and from Lemma 3 it follows that $F^{-1}_r g - F^{-1}_r D_{r,\pm} q \in H_{r;m+3/2,k-1}(\Sigma_T)$. The statement of the theorem for $m \leq -1$ follows from the continuity of the first map in (8).

R e m a r k. Solving operators $R^{\pm}_{2,a}$ of the problems $\mathrm{II}^{\pm}_{2,a}$ are introduced just like the operators $R^{\pm}_{2,r}$. It is evident that these operators perform the continuous maps

$$R^{\pm}_{2,a} : \begin{cases} H_{a;m,k}(G^{\pm}_T) \times H_{a;m+1/2,k-1}(\Sigma_T), & m \geq -1 \\ \mathring{H}_{a;m,k}(G^{\pm}_T) \times H_{a;m+1/2,k}(\Sigma_T), & m \leq -1 \end{cases} \to H_{a;m+2,k-2}(G^{\pm}_T)$$

for all $k \in \mathbf{R}$.

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Разрешимость второй основной смешанной задачи теории упругости в полной шкале соболевских пространств

И.Ю. Чудинович

Доказана однозначная разрешимость второй основной задачи динамической теории упругости в полной шкале соболевских пространств, включающей пространства с негативными нормами.

Розв'язуваність другої основної змішаної задачі теорії пружності у повній шкалі соболєвських просторів

І.Ю. Чудінович

Доведено однозначну розв'язуваність другої основної задачі динамічної теорії пружності у повній шкалі соболєвських просторів, що містить простори з негативними нормами.