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Global attractors for a class of retarded quasilinear partial differential equations

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We consider the system of retarded quasilinear partial differential equations which arises in aeroelasticity. The evolution operator is constructed. The existence of the compact global attractor of finite fractal dimension is proved. We study the properties of solutions on the attractor and prove the existence of the fractal exponential attractor.

In studying nonlinear oscillations of an elastic plate in the Berger approach the following equation arises [1]:

$$\ddot{u} + \varepsilon_1 \dot{u} + \Delta^2 u - f\left(\int_\Omega |\nabla u(x, t)|^2 \right) \Delta u = p(x, t), \ x \in \Omega, \ t > 0 \tag{1}$$

with the boundary conditions

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0. \tag{2}$$

Here Ω is a bounded domain in R^2, $x = (x_1, x_2)$, $\varepsilon_1 > 0$ is a constant, $\dot{u} = \frac{\partial u}{\partial t}$, Δ is the Laplace operator. Assumptions on the scalar function $f(s)$ are given below (in [1] $f(s)$ is a linear function).

For the given initial conditions and for the given external load $p(x, t)$ the existence and uniqueness theorem of solutions of (1), (2) have been obtained in [2]. Long-time behavior of solutions of one — dimensional space version of this problem have been investigated by various authors (see, e.g. [3, 4] and the literature cited there). A similar problem in abstract setting is studied in [5].

This paper deals with oscillations of a plate in a potential supersonic flow, which is moving along the x_1 axis. The influence of the flow is taken into account on the basis of linearized theory. The analysis, which is carried out in [6, 7], shows that the interaction between the flow of gas and the plate can be described by the term

$$p(x, t) = p_0(x) - \varepsilon_2 \left\{ \dot{u} + \nu \frac{\partial u}{\partial x_1} + q(u; x, t) \right\}, \tag{3}$$

where

$$q(u; x, t) = \frac{1}{2\pi k} \int_{-\infty}^{x_1} d\xi \int_0^{2\pi} d\theta \left[\left(a_\theta \frac{\partial}{\partial x_1} + b_\theta \frac{\partial}{\partial x_2} \right)^2 u \right]^*$$

$$\times \left(\xi, x_2 - \frac{x_1 - \xi}{k} \cos\theta, t - \kappa_\theta (x_1 - \xi)\right),$$

and

$$a_\theta = \frac{\nu \sin\theta - 1}{\nu - \sin\theta}, \quad b_\theta = \frac{k \cos\theta}{\nu - \sin\theta}, \quad \kappa_\theta(\xi) = \frac{\xi}{k^2}(\nu - \sin\theta).$$

Here $\Psi^*(x)$ is the extention of $\Psi(x)$ by zero from Ω to R^2, the parameter $\nu > 1$ has the meaning of gas velocity, $k = \sqrt{\nu^2 - 1}$. The parameter $\varepsilon_2 > 0$ is defined by the intensivity of the interaction between the flow and the plate.

So we have a quasilinear partial differential equation with time retardation $t_* = l(\nu - 1)^{-1}$, where l is the size of Ω along the x_1 the axis. Therefore initial conditions (cf. [8]) must be chosen in the form

$$u\big|_{t=0_+} = u_0; \quad \dot{u}\big|_{t=0_+} = u_1; \quad u\big|_{t \in (-t_*, 0)} = \varphi(x, t). \tag{4}$$

For some reason (see below) we do not suppose the continuity of the function $\varphi(x, t)$ with respect to t. Therefore we give up compatibility between $u_0(x)$, $u_1(x)$ and $\varphi(x, t)$.

This paper is devoted to the investigation of long-time behavior of solution of the problem (1)-(4). It is well known that this behavior can be studied on the basis of the concept of global attractor (see, e.g., [4, 9, 10, 11]) and the references therein). We treat problem (1)-(4) as an infinite-dimensional dynamical system and prove that the corresponding evolution operator has a global attractor of finite fractal dimension, when ν is sufficiently large. In particular, this means that a nonlinear flutter (nonregular oscillations) of a plate in potential supersonic flow is finite-dimensional phenomenon (for information on the flutter problem see, e.g., [12]). We also prove a finiteness of number of determining modes and the existence of a fractal exponential attractor (inertial set) for the system under consideration.

In this paper we use certain ideas presented in [7], where the existence of a global attractor is proved for a version of von Karman shell with the retarded term of the form (3). However, in contrast to [7], we need to study in details a retarded linear problem similar to (1)-(4). This studying relies on some ideas borrowed from [7, 8].

1. Notations and preliminary considerations

Suppose $\{ e_k \}_{k=1}^{\infty}$ is the orthonormal basis in $L^2(\Omega)$ consisting of the eigenfunctions of the operator Δ with the Dirichlet conditions of $\partial\Omega$:

$$\Delta e_k + \lambda_k e_k = 0;\ e_k(x) = 0 \ \text{if}\ x \in \partial\Omega,\ 0 < \lambda_1 \le \lambda_2 \le \ldots .$$

We will use the following scale of the spaces:

$$\mathcal{F}_s = \left\{ u = \sum_{k=1}^{\infty} u_k e_k : \| u \|_s^2 \equiv \sum_{k=1}^{\infty} \lambda_k^s u_k^2 < \infty \right\}.$$

We denote by $\| . \|$ and $(. , .)$ the norm and the inner product in $\mathcal{F}_0 = L^2(\Omega)$. It is easy to see that $\mathcal{F}_2 = H^2(\Omega) \cap H_0^1(\Omega)$ and $\| u \|_2 = \| \Delta u \|$. Here and below $H^s(\Omega)$ is the Sobolev space of order s.

If H is a Banach space and $-\infty \le a < b \le +\infty$, then $L^2(a, b; H)$ is the space of (classes of) L^2 functions from (a, b) into H which is Hilbert for the norm

$$|f|_{L^2(a, b; H)} = \left(\int_a^b |f(t)|_H^2 \right)^{1/2} .$$

In a similar manner one defines the space $L^\infty(a, b; H)$. We also denote by $C(a, b; H)$ the Banach space of continuous functions from $[a, b]$ into H with the norm

$$|f|_{C(a, b; H)} = \sup \{ |f(t)|_H : t \in [a, b] \}.$$

Below we will use the following spaces:

$$W_{s,T} = \left\{ u(x, t) \in L^2(s - t_*, T; F_2) : \dot{u}(x, t) \in L^2(s, T; F_0) \right\}, \quad T \ge s,$$

$$\mathcal{H}_\sigma = \mathcal{F}_{2+2\sigma} \times \mathcal{F}_{2\sigma} \times L^2(-t_*, 0; \mathcal{F}_{2+2\sigma}),\ \mathcal{H} = \mathcal{H}_0 .$$

We denote by P_N the orthoprojector in $\mathcal{F}_s$ onto the subspace spanned by $\{ e_1, \ldots, e_N \}$ and set $Q_N = I - P_N$. Let $\widetilde{P}_N$ be the orthoprojector in $\mathcal{H}_\sigma$ onto the subspace

$$P_N \mathcal{F}_{2+2\sigma} \times P_N \mathcal{F}_{2\sigma} \times L^2(-t_*, 0; P_N \mathcal{F}_{2+2\sigma}) \ \text{and}\ \widetilde{Q}_N = I - \widetilde{P}_N .$$

The next property of the retarded term is of importance for our considerations.

Lemma 1.1. *If* $u(t) \in L^2(-t_*, T; \mathcal{F}_{2+2\sigma})$, *then*

$$\| q(u, t) \|_{2\sigma}^2 \le C\, t_* \int_{t-t_*}^{t} \| u(\tau) \|_{2+2\sigma}^2 \, d\tau, \quad 0 \le \sigma < \frac{1}{4}, \tag{1.1}$$

and the mapping $u \to q(u, t)$ is a linear continuous operator from $L^2(-t_*, T; \mathcal{F}_{2+2\sigma})$ into $L^2(0, T; \mathcal{F}_{2\sigma})$.

P r o o f. The estimate (1.1) with $\sigma = 0$ is proved in [7]. For $\sigma > 0$ the proof is completely similar. The continuity of the operator $u \to q(u, t)$ follows from (1.1).

Now we consider an auxiliary linear problem:

$$\ddot{u} + \gamma\, \dot{u} + \Delta^2 u - b(t)\Delta u + \mu \cdot u + \varepsilon_2 q(u, t) = h(t), \tag{1.2}$$

$$u\big|_{t=s+0} = u_0\,; \quad \dot{u}\big|_{t=s+0} = u_1\,; \quad u\big|_{t \in (s-t_*,\, s)} = \varphi\,(x, t-s). \tag{1.3}$$

Here $\gamma = \varepsilon_1 + \varepsilon_2$ and $s \in R$, and $q(u, t)$ is determined as in (3). The auxiliary parameter $\mu > 0$ will be chosen below. It is assumed that

$$b(t) \in C^1[s, \infty), \quad \sup_{t \ge s} \{\, | b(t) | + | \dot{b}(t) | \,\} \le b_0\,. \tag{1.4}$$

D e f i n i t i o n 1. 1. *A weak solution of the problem* (1.2), (1.3) *on an interval* $[s, T]$ *is a vector-function $u(t) \in W_{s,T}$ satisfying* (1.2) *in the sence of distributions and such that the initial conditions* (1.3) *hold.*

Theorem 1.1. *Assume* $u_0 \in \mathcal{F}_{2+2\sigma}$, $u_1 \in \mathcal{F}_{2\sigma}$, $\varphi \in L^2(-t_*, 0; \mathcal{F}_{2+2\sigma})$, $h(x, t) \in L^\infty(s, T; \mathcal{F}_{2\sigma})$ *for any $T > 0$ and for $0 \le \sigma < \frac{1}{4}$. Then for any interval $[s, T]$ the problem* (1.2), (1.3) *has a weak solution. This solution is unique and satisfies the property*

$$u(t) \in C\,(s, T; \mathcal{F}_{2+2\sigma}), \quad \dot{u}(t) \in C\,(s, T; \mathcal{F}_{2\sigma}). \tag{1.5}$$

The proof of the theorem follows an usual plan and is based on the principle of compactness [13] and the results concerning retarded finite-dimensional system [8]. We give the principal steps of the proof.

Following the Galerkin method, we define for each m an approximate solution u_m of (1.2), (1.3):

$$u_m(x, t) = \sum_{k=1}^{m} g_k(t)\, e_k\,,$$

$$\left(\ddot{u}_m + \gamma\,\dot{u}_m + \Delta^2 u_m - b(t)\Delta u_m + \mu\cdot u_m + \varepsilon_2 q(u_m, t) - h(t), e_k\right) = 0, \tag{1.6}$$

$$(u_m(s+0), e_k) = (u_0, e_k),\ (\dot{u}_m(s+0), e_k) = (u_1, e_k) \text{ and } (u_m(s+\tau), e_k) = (\varphi(\tau), e_k) \tag{1.7}$$

for $\tau \in [-t_*, 0]$. Here $k = 1, \ldots, m$, and $g_k(t) \in C^1(s, +\infty; R)$, such that $\dot{g}_k(t)$ is an absolutely continuous function.

The local existence theorem can be proved if we rewrite the corresponding retarded system for $g_k(t)$ in the integral form and use the idea of the proof of Theorem 2.2.1 from [8].

The possibility of extention of these solutions to any interval [s, T] follows from the a priori estimate:

$$\|\dot{u}_m(t)\|^2_{2\sigma} + \|u_m(t)\|^2_{2+2\sigma} \le C_1\Big(\|\dot{u}_m(s+0)\|^2_{2\sigma} + \|u_m(s+0)\|^2_{2+2\sigma} + \int_{s-t_*}^{s} \|u_m(\tau)\|^2_{2+2\sigma}\,d\tau + \int_s^t \|h(\tau)\|^2_{2\sigma}\,d\tau\Big)\exp\{C_2(t-s)\} \tag{1.8}$$

where C_1 and C_2 are positive numbers. It is obtained by multiplying (1.6) by $\lambda_k^{4\sigma}\,\dot{g}_k(t)$ and summing these relations for $k = 1, \ldots, m$. In this case it is easy to see that

$$\frac{d}{dt}E_{0,\sigma}(t) \le C_1 E_{0,\sigma}(t) + \|q(u)\|^2_{2\sigma} + \|h(t)\|^2_{2\sigma},\ C_1 > 0$$

where $E_{0,\sigma}(t) \equiv \|\dot{u}_m(t)\|^2_{2\sigma} + \|u_m(t)\|^2_{2+2\sigma}$. Thanks to (1.1) we have

$$\frac{d}{dt}E_{0,\sigma}(t) - C_1 E_{0,\sigma}(t) \le C t_* \int_{t-t_*}^{t} E_{0,\sigma}(\tau)\,d\tau + \|h(t)\|^2_{2\sigma}.$$

Therefore as in [7] the Gronwall lemma gives (1.8).

Estimate (1.8) implies the *-weak compactness of the family $(u_m(t); \dot{u}_m(t))$ in the space $L^\infty(s, T; \mathcal{F}_{2+2\sigma} \times \mathcal{F}_{2\sigma})$. Therefore there exist $u(t) \in W_{s,T}$ and the subsequence $\{u_{m_k}(t)\}$ such that $(u_{m_k}(t); \dot{u}_{m_k}(t))$ *-weakly converges to $(u(t); \dot{u}(t))$ in $L^\infty(s, T; \mathcal{F}_{2+2\sigma} \times \mathcal{F}_{2\sigma})$. Hence, by using standard method [13, 14], we can prove the existence of the weak solution. Lemma 1.1 gives the possibility to pass to the *-weak limit in the retarded term and prove that $q(u_{m_k}, t)$ converges to $q(u, t)$ weakly in the space $L^2(0, T; \mathcal{F}_{2\sigma})$.

If we consider $u(t)$ as a solution of the linear problem

$$\ddot{u} + \gamma\,\dot{u} + \Delta^2 u = \widetilde{h}(t);\ u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0;\ u\big|_{t=0} = u_0,\ \dot{u}\big|_{t=0} = u_1 \tag{1.9}$$

with

$$\widetilde{h}(t) = \widetilde{h}(t, u) = b(t)\Delta u - \mu\cdot u - \varepsilon_2 q(u, t) + h(t) \in L^\infty(0, T; \mathcal{F}_{2\sigma}),$$

we can show (see [13,14]) that the vector-function $(u(t); \dot{u}(t))$, is strong-continuous in the space $\mathcal{F}_{2+2\sigma} \times \mathcal{F}_{2\sigma}$.

Let $u_i(t)$ be a weak solution of (1.2), (1.3) with initial conditions $\{u_{0i}, u_{1i}, \phi_i\}$ and with a right side $h_i(t)$, where $i = 1, 2$. Then $u(t) = u_1(t) - u_2(t)$ is a solution of (1.9) with $u_0 = u_{01} - u_{02}$, $u_1 = u_{11} - u_{12}$ and

$$\widetilde{h}(t) == b(t)\Delta(u_1 - u_2) - \mu(u_1 - u_2) - \varepsilon_2\cdot q(u_1 - u_2, t) + h_1(t) - h_2(t).$$

Therefore combining the energy equality (see, e.g., [14]) for the problem (1.9):

$$\frac{1}{2}\left\{\|\dot{u}(t)\|^2_{2\sigma} + \|u(t)\|^2_{2+2\sigma}\right\} = \frac{1}{2}\left(\|\dot{u}(s+0)\|^2_{2\sigma} + \|u(s+0)\|^2_{2+2\sigma}\right) + \\ + \int_s^t \left\{-\gamma\|\dot{u}(\tau)\|^2_{2\sigma} + (\widetilde{h}, \dot{u})_{2\sigma}\right\} d\tau \tag{1.10}$$

and the estimate

$$\|\widetilde{h}(t)\|^2_{2\sigma} \le C\left(\|u(t)\|^2_{2+2\sigma} + t_*\int_{t-t_*}^t \|u(\tau)\|^2_{2+2\sigma}\,d\tau + \|h_1(t) - h_2(t)\|^2_{2\sigma}\right),\quad \sigma < \frac{1}{4},$$

we obtain that

$$\|\dot{u}(t)\|^2_{2\sigma} + \|u(t)\|^2_{2+2\sigma} \le \|u_1\|^2_{2\sigma} + \|u_0\|^2_{2+2\sigma} + C\int_s^t \|h_1(\tau) - h_2(\tau)\|^2_{2\sigma}\,d\tau + \\ + C\int_s^t \left\{\|u(\tau)\|^2_{2+2\sigma} + t_*\int_{\tau-t_*}^{\tau} \|u(\xi)\|^2_{2+2\sigma}\,d\xi\right\} d\tau.$$

This inequality implies

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|^2_{2\sigma} + \|u_1(t) - u_2(t)\|^2_{2+2\sigma} \le \Big(\|u_{11} - u_{12}\|^2_{2\sigma} + \|u_{01} - u_{02}\|^2_{2+2\sigma} +$$

$$+\int_s^t \| h_1(\tau) - h_2(\tau) \|_{2\sigma}^2 \, d\tau + \int_{s-t_*}^s \| u_1(\tau) - u_2(\tau) \|_{2+2\sigma}^2 \, d\tau \Big) C \exp \{ a(t-s) \} . \quad (1.11)$$

In particular, the estimate (1.11) implies uniqueness of weak solutions of (1.2), (1.3). The proof of Theorem 1.1 is complete.

Theorem 1.1 gives the possibility to define the family of strong continuous mappings $w \to U(t, s; h, w)$ in the spaces $\mathcal{H}_\sigma$ by the formula

$$U(t, s; h, w) = (u(t); \dot{u}(t); u(t+\tau)), \ \ \tau \in (-t_*, 0), \quad t \geq s.$$

Here $u(t)$ is the solution of (1.2), (1.3) with the initial conditions $w = (u_0, u_1, \varphi(\tau))$. Using (1.11), we easily obtain

$$| U(t, s; h_1, w_1) - U(t, s; h_2, w_2) |_{\mathcal{H}_\sigma}^2 \leq$$

$$\leq C\left(| w_1 - w_2 |_{\mathcal{H}_\sigma}^2 + \int_s^t \| h_1(\xi) - h_2(\xi) \|_{2\sigma}^2 \, d\xi \right) e^{a(t-s)}. \quad (1.12)$$

Here $w_i \in \mathcal{H}_\sigma$, $h_i \in L^2(s, \infty; \mathcal{F}_{2\sigma})$. If $h(t) \equiv 0$, we write $U(t, s)w = U(t, s; h, w)$. Evidently the mapping $U(t, s)$ is linear in $\mathcal{H}_\sigma$ and satisfies the semigroup properties

$$U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3), \quad t_1 \geq t_2 \geq t_3, \quad U(t, t) = I.$$

Additional properties of the evolution family $U(t, s; h, \cdot)$ are contained in the following theorem.

Theorem 1.2. *There exist constants* μ *and* ν_0 *such that for* $\nu \geq \nu_0$ *the family* $U(t, s; h, \cdot)$ *has the properties*:

1. $| U(t, s)w |_{\mathcal{H}_\sigma} \leq C_\sigma e^{-\frac{a_\sigma}{2}(t-s)} | w |_{\mathcal{H}_\sigma}, \ a_\sigma > 0, \ 0 \leq \sigma < \frac{1}{4}. \quad (1.13)$

2. $U(t, s; h, w) = U(t, s)w + \int_s^t U(t, \tau) \bar{h}(\tau) \, d\tau. \quad (1.14)$

Here $\bar{h}(\tau) = (0; h(\tau); 0)$ *and the integral is regarded in the strong sense.*

3. $| \tilde{Q}_N U(t, 0; h, w) |_{\mathcal{H}} \leq C e^{-\frac{at}{2}} \left\{ | w |_{\mathcal{H}} + \frac{1}{\lambda_{N+1}^\sigma} \int_0^t e^{-\frac{a\tau}{2}} \| h(\tau) \|_{2\sigma} \, d\tau \right\} \quad (1.15)$

for $h(t) \in L^2(0, \infty; \mathcal{F}_{2\sigma})$ *and* $w \in \mathcal{H}$.

4. *If* $\| h(\tau) \|_{2\sigma} \le C$ *is valid for all* $\tau \ge 0$ *and for some* σ *belonging to the interval* $(0, \frac{1}{4})$, *then there exists a compact set K in* $\mathcal{H}$ *such that for any* $w \in \mathcal{H}$

$$\operatorname{dist}_{\mathcal{H}}(U(t, s; h, w), K) \le C \exp\left\{-\frac{a}{2}(t-s)\right\} | w |_{\mathcal{H}}, \quad a > 0. \tag{1.16}$$

P r o o f. For the sake of simplicity we assume that $s = 0$. Consider the function

$$V_\sigma^{(m)}(t) = \frac{1}{2} \| \dot{u}_m \|_{2\sigma}^2 + \frac{1}{2} \| u_m \|_{2+2\sigma}^2 + \frac{1}{2} b(t) \| u_m \|_{1+2\sigma}^2 + \mu \| u_m \|_{2\sigma}^2 +$$

$$+ \frac{\gamma}{2} \left\{ (\dot{u}_m, u_m)_{2\sigma} + \frac{\gamma}{2} \| u_m \|_{2\sigma}^2 \right\} \equiv \sum_{k=1}^{m} \lambda_k^{2\sigma} V_k(t),$$

where $u_m(t) = \sum_{k=1}^{m} g_k(t) e_k$ is the Galerkin approximation of the solution of (1.2), (1.3) with $s = 0$, $h = 0$ and

$$V_k(t) = \frac{1}{2} \dot{g}_k^2 + \frac{1}{2} \lambda_k^2 g_k^2 + \frac{1}{2} b(t) \lambda_k g_k^2 + \mu g_k^2 + \frac{\gamma}{2} \left\{ \dot{g}_k g_k + \frac{\gamma}{2} g_k^2 \right\}.$$

From (1.4), (1.6) one easily obtains that there exists μ_0 such that for $\mu \ge \mu_0$

$$\alpha_1(\dot{g}_k^2(t) + g_k^2(t) \lambda_k^2) \le V_k(t) \le \alpha_2(\dot{g}_k^2(t) + g_k^2(t) \lambda_k^2),$$

$$\frac{dV_k(t)}{dt} + \beta V_k(t) \le C \mid (q(u_m, t), e_k) \mid^2 \lambda_k^{2\sigma}.$$

Here positive constants α_1, α_2, β, C do not depend on k and m. Using (1.1) we have

$$\alpha_1\left(\| u_m(t) \|_{2+2\sigma}^2 + \| \dot{u}_m(t) \|_{2\sigma}^2\right) \le V_\sigma(t) \le \alpha_2\left(\| u_m(t) \|_{2+2\sigma}^2 + \| \dot{u}_m(t) \|_{2\sigma}^2\right), \tag{1.17}$$

$$\frac{d}{dt} V_\sigma^{(m)}(t) + \beta V_\sigma^{(m)}(t) \le C_0 t_* \int_{t-t_*}^{t} V_\sigma^{(m)}(\tau) \, d\tau.$$

As in [7] one easily obtains the inequality

$$V_\sigma^{(m)}(t) \le C_1 V_\sigma^{(m)}(0) \exp\{-a_1 t\}; \; a_1 > 0, \; t \ge t_*, \tag{1.18}$$

when $t_* = l(\nu - 1)^{-1}$ is chosen such that $C_0 t_* e^{\beta t_*} < \beta$. Owing to the fact of *-weak convergence $(\dot{u}_m(t); u_m(t))$ to $(\dot{u}(t); u(t))$ in the space $L^\infty(0, T; \mathcal{F}_{2\sigma} \times \mathcal{F}_{2+2\sigma})$ and using (1.17), (1.18), we obtain (1.13).

Let us prove (1.14). It is sufficient to prove (1.14) for $w = 0$. Let $u^*_m(t)$ be a solution of (1.6), (1.7) for the initial conditions $u_0 = u_1 = 0$, $\varphi(\xi) \equiv 0$.

Passing in a standard way from (1.6) to the corresponding system of first order in time, we can obtain, see [8, Th. 6.2.1]:

$$\begin{pmatrix} u^*_m(t) \\ \dot{u}^*_m(t) \end{pmatrix} = \int_s^t \begin{pmatrix} v_m(t,\tau) \\ \partial_t v_m(t,\tau) \end{pmatrix} d\tau, \qquad (1.19)$$

where $v(t,\tau)$ is a solution of (1.6), (1.7) for the initial conditions at the moment $s = \tau$: $\bar{h}(\tau) = \left(u_0 = 0;\ u_1 = h(\tau);\ \varphi(\xi) \equiv 0\right)$. Using the compactness of the Galerkin approximations, we can pass to the limit $m \to 0$ in (1.19) and finish the proof of (1.14) for $w = 0$.

Let us prove (1.15) and (1.16). From (1.13) it follows

$$\left| \int_s^t U(t,\tau; 0, \bar{h}(\tau))\, d\tau \right|_{\mathcal{H}_\sigma} \le C \int_s^t e^{-\frac{a_\sigma}{2}(t-\tau)} \| h(\tau) \|_{2\sigma}\, d\tau. \qquad (1.20)$$

Using the obvious inequality $| \tilde{Q}_N w |_{\mathcal{H}_0} \le \frac{1}{\lambda^\sigma_{N+1}} | w |_{\mathcal{H}_\sigma}$, from (1.13), (1.14) and (1.20) we obtain (1.15). From (1.20) it also follows, that the integral term in (1.14) belongs to the set

$$K = \left\{ (u_0; u_1; \varphi) : \| u_0 \|^2_{2+2\sigma} + \| u_1 \|^2_{2\sigma} + \operatorname*{ess\ sup}_{s \in [-t_*, 0]} \left(\| \varphi(s) \|^2_{2+2\sigma} + \| \dot{\varphi}(s) \|^2_{2\sigma} \right) \le A \right\} \qquad (1.21)$$

for some $A > 0$, when $\| h(t) \|_{2\sigma} \le C$, $t > 0$. Since the set K is compact in $\mathcal{H}$ (see, e.g., [13]) (1.14) and (1.13) imply (1.16).

2. The construction of the evolution operator

In this section we prove the theorem on the existence and uniqueness of weak solutions of the problem (1)-(4) and construct the corresponding evolution operator.

Definition 2.1. *A weak solution of the problem* (1)-(4) *on the interval* $[0, T]$ *is a vector-function* $u(t) \in W_{0,T}$ *satisfying* (1) *and* (3) *in the sence of distributions and such that the initial conditions* (4) *hold.*

Theorem 2.1. *Let* $u_0 \in \mathcal{F}_2$, $u_1 \in \mathcal{F}_0$, $\varphi \in L^2(-t_*, 0; \mathcal{F}_2)$, $p_0 \in \mathcal{F}_0$. *Assume* (i) *the function* $f(s)$ *is a local Lipschitz one i.e. for any* $A > 0$ *there exists a constant* L_A *such that for any* $s_1, s_2 \in [0, A]$

$$|f(s_1) - f(s_2)| \le L_A |s_1 - s_2| \tag{2.1}$$

and (ii) $f(s)$ *satisfies the property*

$$\int_0^s f(\tau)\, d\tau \ge -C_0 - \alpha s, \tag{2.2}$$

where $0 \le \alpha < \lambda_1$, λ_1 *is the first eigenvalue of* $-\Delta$ *and* C_0 *is a constant. Then the problem* (1)-(4) *has a weak solution on any interval* $[0, T]$. *The solution is unique and satisfies the property*

$$u(t) \in C(0, T; \mathcal{F}_2), \ \dot{u}(t) \in C(0, T; \mathcal{F}_0). \tag{2.3}$$

Besides the energy equality

$$E(u(t); \dot{u}(t)) = E(u_0; u_1) + \int_0^t \left(-\gamma \| \dot{u}(\tau) \|^2 - \rho \left(\frac{\partial u}{\partial x_1}, \dot{u}\right) - \varepsilon_2 (q(u), \dot{u}) + (p_0, \dot{u})\right) d\tau \tag{2.4}$$

holds. Here $\gamma = \varepsilon_1 + \varepsilon_2$, $\rho = \varepsilon_2 \nu$ *and*

$$E(u; \dot{u}) = \frac{1}{2}\left(\| \dot{u} \|^2 + \| \Delta u \|^2 + F(\| \nabla u \|^2)\right), \ F(r) \equiv \int_0^r f(s) ds. \tag{2.5}$$

P r o o f. We follow the line of argument given in the proof of Theorem 1.1. Define an approximate solution of (1)-(4)

$$u_m(x, t) = \sum_{k=1}^{m} g_k(t)\, e_k .$$

$$\left(\ddot{u}_m + \gamma \dot{u}_m + \Delta^2 u_m - f(\| \nabla u_m \|^2) \Delta u_m + \rho \frac{\partial u_m}{\partial x_1} + \varepsilon_2 q(u_m, t) - p_0, e_k\right) = 0, \tag{2.6}$$

$$(u_m(0_+), e_k) = (u_0, e_k), \ (\dot{u}_m(0_+), e_k) = (u_1, e_k) \ \text{and} \ (u_m(\tau), e_k) = (\varphi(\tau), e_k). \tag{2.7}$$

for $\tau \in [-t_*, 0]$, where $k = 1, \ldots, m$ and $g_k(t) \in C^1(0, \infty, R)$ such that $\dot{g}_k(t)$ is an absolutely continuous function.

The local existence theorem can be proved if we rewrite the corresponding retarded system for $g_k(t)$ in the integral form and use the idea of the proof of Theorem 2.2.1 from [8].

Multiplying (2.6) by $\dot{g}_k(t)$ and summing these relations for $k = 1,\dots, m$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\| \dot{u}_m \|^2 + \| \Delta u_m \|^2 + F(\| \nabla u_m \|^2)\right\} + \gamma \| \dot{u}_m \|^2 +$$
$$+ \varepsilon_2 (q(u_m), \dot{u}_m) + \rho \left(\frac{\partial u_m}{\partial x_1}, \dot{u}_m \right) = (p_0, \dot{u}_m).$$

From (2.2) it follows that $E(u; \dot{u}) \geq -\frac{C_0}{2} + \frac{1}{2}(1 - \frac{\alpha}{\lambda_1})(\| \dot{u} \|^2 + \| \Delta u \|^2)$. Hence

$$\frac{d}{dt} E(u_m(t); \dot{u}_m(t)) \leq C_1 E(u_m(t); \dot{u}_m(t)) + \| q(u_m) \|^2 + C_2 ; \; C_1, \; C_2 \geq 0.$$

As above (see (1.8)), we have

$$\| \dot{u}_m(t) \|^2 + \| \Delta u_m \|^2 \leq C_1\Big(\| \dot{u}_m(0) \|^2 + \| \Delta u_m(0) \|^2 + \int_{-t_*}^{0} \| \Delta u_m(s) \|^2 ds +$$
$$+ F(\| \nabla u_m(0) \|^2) + 1\Big) \exp \{ \beta t \}; \; C_1, \; \beta > 0. \tag{2.8}$$

This estimate allows us to choose a weak convergent subsequence $\{ u_{m_k}(t) \}$ and to pass to the limit in the same way as in Sect. 1. The Lipschitz property of f and the compactness of the embedding $W_{0,T}$ into $L^2(0, T; \mathcal{F}_1)$ make it possible to pass to the limit in the nonlinear term.

The proof of the properties (2.3) is also similar to the linear case, if we consider the solution of (1)-(4) as a solution of (1.9) with

$$\tilde{h}(t) = f(\| \nabla u \|^2) \Delta u - \rho \frac{\partial u}{\partial x_1} - \varepsilon_2 q(u, t) + p_0 .$$

The law of conservation of energy (2.4) is obtained from (1.10) for $\sigma = 0$. Here we also use the equality

$$-2\int_0^t \Big(f(\| \nabla u(\tau) \|^2) \Delta u(\tau), \dot{u}(\tau)\Big) d\tau = F(\| \nabla u(t) \|^2) - F(\| \nabla u(0) \|^2)$$

which is correct for $u \in W_{0,T}$, $T > t$. It follows from a similar equality for $u \in C^1(0, T; \mathcal{F}_2)$ and from denseness of the space $C^1(0, T; \mathcal{F}_2)$ in $W_{0,T}$.

Let us prove uniqueness of solutions. Let u_1 and u_2 be solutions of (1)-(4). Then $u = u_1 - u_2$ is a solution of (1.2), (1.3) for $b(t) \equiv 0$ and

$$h(t) = f(\| \nabla u_1 \|^2)\, \Delta u_1 - f(\| \nabla u_2 \|^2)\, \Delta u_2 - \rho \left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_1} \right) + \mu\,(u_1 - u_2).$$

From (2.1) and (2.3) it follows that $\| h(t) \| \le C_T \| \Delta u(t) \|$ for $t \in [0, T]$, where the constant C_T also depends on $\| \Delta u_1(t) \|$ and $\| \Delta u_2\,(t) \|$.
Therefore from (1.11) and the Gronwall lemma we obtain

$$\| \dot u_1(t) - \dot u_2(t) \|^2 + \| \Delta(u_1(t) - u_2(t)) \|^2 \le$$
$$\le C_T \left(\| \dot u_1(0) - \dot u_2(0) \|^2 + \| \Delta(u_1(0) - u_2(0)) \|^2 + \int_{-t_*}^{0} \| \Delta(u_1(\tau) - u_2(\tau)) \|^2 d\tau \right) \tag{2.9}$$

for any $t \in [0, T]$. The estimate (2.9) implies uniqueness of solutions.

The proof of Theorem 2.1 is complete.

Relying on the Theorem 2.1 we are now in a position to construct the evolution operator S_t in the space

$$\mathcal{H} = \mathcal{F}_2 \times \mathcal{F}_0 \times L^2\,(-t_*, 0; \mathcal{F}_2)$$

by the formula

$$S_t(u_0; u_1; \varphi\,(s)) = (u(t); \dot u(t); u(t+s)), \ \ s \in (-t_*, 0), \ \ t \ge 0, \tag{2.10}$$

where $u(t)$ is the weak solution of the problem (1)-(4).

The operator S_t has properties:

a) $S_t S_\tau w = S_{t+\tau} w$ for $t, \tau > 0$ and $S_0 w = w$;

b) $S_t w$ is strong continuous with respect to t for any $w \in \mathcal{H}$ (see (2.3));

c) $S_t w$ depends on $w \in \mathcal{H}$ continuously (see (2.9)) and there exists a constant $C_{T,R}$ such that

$$| S_t y_1 - S_t y_2 |_{\mathcal{H}} \le C_{T,R} | y_1 - y_2 |_{\mathcal{H}}$$

for any $t \in [0, T]$ and $y_1, y_2 \in \mathcal{H}$ possessing the property $| y_i |_{\mathcal{H}} \le R$.

Thus we have constructed the infinite — dimensional dynamical system $(S_t, \mathcal{H})$ corresponding to the problem (1)-(4). Its asymptotical properties are studied in the next section.

3. Long-time behavior

This section deals with the investigation of the long-time behavior of the evolution semigroup S_t. As we know (see, e.g. [3,4,9,10] and the references given there), in studying such questions an important role is played by attractors, whose investigation allows us to answer the question of possible limit regimes of the system under consideration. In this section we prove our main theorem on the existence of a global attractor of finite fractal dimension for the dynamical system $(S_t, \mathcal{H})$ associated with the problem (1)-(4). We also prove the existence of finite number of determining modes (for definition see, e.g., [10,15]) and the existence of a fractal exponential attractor. The notion of exponential attractors (inertial sets) was introduced in [16]. They are compact positive invariant sets with finite fractal dimension which attract exponentially trajectories of the system. In this section the assumptions on the function $f(s)$ are stronger than those in Sect. 2.

D e f i n i t i o n 3.1. *A bounded closed set* $\mathfrak{A}$ *in* $\mathcal{H}$ *is said to be a global attractor of the semigroup* S_t *if it is strictly invariant* ($S_t \mathfrak{A} = \mathfrak{A}$ *for any* $t \geq 0$) *and* $\lim_{t\to\infty} h(S_t B, \mathfrak{A})$ *for any bounded set B in* $\mathcal{H}$. *Here*

$$h(B, A) = \sup \{ \operatorname{dist}_{\mathcal{H}}(y, A) \mid y \in B \} \tag{3.1}$$

Our main result is

Theorem 3.1. *Assume that* $p_0(x) \in \mathcal{F}_1$ *and* $f(s) \in C^1(R_+)$ *satisfies the properties*

$$sf(s) - \alpha \int_0^s f(\tau)\, d\tau \geq -C \tag{3.2}$$

for some $\alpha > 0$ *and* $C > 0$

$$\liminf_{s\to+\infty} f(s) > \frac{\rho^2}{\gamma^2}, \tag{3.3}$$

where $\rho = \varepsilon_2 \nu$ *and* $\gamma = \varepsilon_1 + \varepsilon_2$. *Then there exists* ν_0 *such that for any* $\nu > \nu_0$ *the semigroup* S_t *has a compact global attractor* $\mathfrak{A}$ *in* $\mathcal{H}$. *The attractor* $\mathfrak{A}$ *is of finite* *fractal dimension (for the definition, see., e.g., [10]) and consists of elements* $w = (\upsilon(0); \dot{\upsilon}(0); \upsilon(\xi))$, $\xi \in [-t_*, 0]$, *where* $\upsilon(\xi)$ *belongs to the space*

$$\mathcal{C}_\sigma = C^1(-t_*, 0; \mathcal{F}_{2\sigma}) \cap C(-t_*, 0; \mathcal{F}_{2+2\sigma}),\ 0 \le \sigma < \frac{1}{4}. \tag{3.4}$$

The proof uses the results on existence of attractors for abstract asymptotically compact dynamical system [4,10] and the Ladyzhenskaya theorem on finite dimensionality of invariant sets. We also rely on the following lemmas.

Lemma 3.1. *There exists* ν_0 *such that for any* $\nu > \nu_0$ *the semigroup* S_t *is dissipative,, i.e. there exists* $R > 0$ *such that* $|S_t h|_{\mathcal{H}} \le R$ *for any* $h \in B$, $t \ge t_0(B)$. *Here* B *is any bounded set in* $\mathcal{H}$.

P r o o f. The arguments presented below have a formal character. They can be made rigorous if the Galerkin approximation is considered.

Let $u(t)$ be a weak solution of the problem (1)-(4). We define the function

$$V(t) = E(t) + \frac{\gamma}{2}\varphi(t)$$

where $E(t)$ is given by (2.5) and

$$\varphi(t) = (u(t), \dot{u}(t)) + \frac{\gamma}{2}\|u(t)\|^2.$$

Since

$$\frac{d}{dt}\varphi(t) = \|\dot{u}\|^2 - \left(u, \Delta^2 u - f(\|\nabla u\|^2)\Delta u + \rho\frac{\partial u}{\partial x_1} + \varepsilon_2 q(u) - p_0\right)$$

and $(u, \frac{\partial u}{\partial x_1}) = 0$ we have

$$\frac{d}{dt}\varphi(t) \le \|\dot{u}\|^2 - \frac{1}{2}\|\Delta u\|^2 - \|\nabla u\|^2 f(\|\nabla u\|^2) + \|q(u)\|^2 + C \tag{3.5}$$

Using (2.4) we obtain

$$\frac{d}{dt}E(t) \le -(\gamma - \omega_1 - \omega_2)\|\dot{u}\|^2 + \frac{\rho^2}{4\omega_1}\left\|\frac{\partial u}{\partial x_1}\right\|^2 + \frac{C}{2\omega_2}(\|q(u)\|^2 + \|p_0\|^2) \tag{3.6}$$

for any $\omega_1, \omega_2 > 0$. If we set $\omega_1 = \frac{1}{2}\kappa\gamma$ and $\omega_2 = \frac{1}{4}(1-\kappa)\gamma$, where $0 < \kappa < 1$, then (3.5) and (3.6) imply the estimate

$$\frac{d}{dt}V(t) \le -\frac{1}{4}\gamma(1-\kappa)\|\dot{u}\|^2 - \frac{\gamma}{4}\|\Delta u\|^2 +$$

$$+\frac{\gamma}{2}\|\nabla u\|^2\left(-f(\|\nabla u\|^2)+\frac{\rho^2}{\kappa\gamma^2}\right)+C_1\|q(u)\|^2+C_2 .$$

From (3.3) it follows that there exists κ, $0<\kappa<1$, such that

$$\|\nabla u(t)\|^2\left(-\kappa f(\|\nabla u\|^2)+\frac{\rho^2}{\kappa\gamma^2}\right)\le C.$$

Therefore, using (3.2), we have

$$\frac{d}{dt}V(t)\le-\frac{1}{4}\gamma(1-\kappa)\|\dot u\|^2-\frac{\gamma}{4}\|\Delta u\|^2-\frac{\gamma}{2}\alpha(1-\kappa)F(\|\nabla u\|^2)+$$

$$+C_1\|q(u)|^2+C_2$$

Since (3.3) implies that $F(s)\ge\frac{\rho^2}{\gamma^2}s-C$, it is easy to see that

$$\frac{d}{dt}V(t)\le-\alpha_1E(t)+C_1\|q(u,t)\|^2+C_2,\quad \alpha_1>0,$$

$$V(t)\ge\alpha_2E(t)-C_3,\quad \alpha_2>0.$$

Therefore, using (1.1), for $\sigma=0$ we can conclude that

$$\frac{d}{dt}V(t)+\beta V(t)\le C_1t_*\int_{t-t_*}^{t}V(\tau)\,d\tau+C_2$$

for some $\beta>0$. As in [7] (see also Sect. 1) it implies the required assertion, when $t_*=l(\nu-1)^{-1}$ is chosen such that $C_1\,t_*e^{\beta t_*}<\beta$.

Lemma 3.2. *If* $|S_tw_i|_{\mathcal{H}}\le R$, $i=1,2$ *for any* $t>0$, *then there exist positive constants* C_R *and* ξ_R *such that*

$$|S_tw_1-S_tw_2|_{\mathcal{H}}\le C_Re^{\xi_Rt}|w_1-w_2|_{\mathcal{H}} \tag{3.7}$$

P r o o f. Let u_1 and u_2 be two solutions of (1)-(4). Then the function $u_j(t)$ is a solution of (1.2), (1.3) for $b(t)\equiv0$ and

$$h_j(t)=f(\|\nabla u_j\|^2)\,\Delta u_j+\rho\frac{\partial u_j}{\partial x_1}+p_0,\quad j=1,2.$$

It is obvious that if $\|u_j(t)\|_2\le R$, then $\|h_1(t)-h_2(t)\|\le C_R\|u_1-u_2\|_2$. Using the Gronwall lemma and (1.11), we obtain (3.7).

Let us consider the solution $u(t)$ of (1)-(4) as a solution of linear problem (1.2), (1.3) with $b(t) = f(\| \nabla u(t) \|^2)$,

$$h(t) = -\rho \frac{\partial u(t)}{\partial x_1} + \mu\, u(t) + p_0 .$$

One can easily verify that $b(t)$ has the property (1.4) when the solution $u(t)$ belongs to the ball of dissipation (see Lemma 3.1). The constant b_0 depends on R. Since $H^{2\sigma}(\Omega) = H_0^{2\sigma}(\Omega) = \mathcal{F}_{2\sigma}$ for $\sigma < \frac{1}{4}$ we have that $\| h(t) \|_{2\sigma} \le C_R$, when $\| \Delta u(t) \| \le R$ and $\sigma < \frac{1}{4}$. That is why we can use the results of Theorem 1.2 and obtain that the compact set K defined by (1.21) for some constant $A = A(R)$ satisfies the property

$$h(S_t B, K) \le C_B \exp\{-at\},\ a > 0 \tag{3.8}$$

for any bounded set B in $\mathcal{H}$. Here $h(A, B)$ is defined by (3.1). In particular (3.8) means that the semigroup S_t is asymptotically compact in the space $\mathcal{H}$. Therefore (see, e.g., [4,10] the semigroup S_t possesses a global attractor $\mathfrak{A}$.

It is clear that any element lying in $\mathfrak{A}$ has a form $(\upsilon(0), \dot{\upsilon}(0), \upsilon(\tau))$, where $\upsilon(\tau) \in \mathcal{C}$ for $\sigma < \frac{1}{4}$.

Let us prove that the attractor $\mathfrak{A}$ has a finite dimension in $\mathcal{H}$. Let $\{ \varphi_j(s) \}_{j=0}^{\infty}$ be the trigonometric basis in $L^2(-t_*, 0; R)$: $\varphi_0(s) = 1/\sqrt{t_*}$;

$$\varphi_{2k-1}(s) = \sqrt{\frac{2}{t_*}} \sin 2\frac{\pi k}{t_*} s,\ k = 1, 2, \ldots;\ \varphi_{2k}(s) = \sqrt{\frac{2}{t_*}} \cos 2\frac{\pi k}{t_*} s,\ k = 1, 2, \ldots$$

We consider in $\mathcal{H}$ the finite — dimensional subspace:

$$\mathcal{H}_{N,M} = P_N \mathcal{F}_2 \times P_N \mathcal{F}_0 \times \mathrm{Lin}\{ \varphi_j(s) e_k(x) \mid k = 1, \ldots, N; j = 0, \ldots, M \}.$$

Denote by $\mathcal{P}_{N,M}$ the orthoprojector in $\mathcal{H}$ on this subspace.

Lemma 3.3. *Let $w_1, w_2 \in K_* = \bigcup_{t \ge 0} S_t K$ where K is the same as in (3.8). Then*

$$| (1 - \mathcal{P}_{N,M})(S_t w_1 - S_t w_2) |_{\mathcal{H}} \le$$

$$\le \left\{ C_1 e^{-\frac{at}{2}} \left(1 + \frac{1}{\lambda_{N+1}^{\sigma}} e^{bt}\right) + C_2 \frac{\lambda_N}{M+1} e^{\xi t} \right\} | w_1 - w_2 |_{\mathcal{H}} \tag{3.9}$$

Proof. Let $S_t w_l = (u_l(t), \dot u_l(t), u_l(t+\tau))$, $l = 1, 2$. Evidently

$$(1-\mathscr{P}_{N,M})\, S_t\, w_l = \Big\{ Q_N u_l(t);\; Q_N \dot u_l(t);\; Q_N u_l(t+s) + \sum_{j=M+1}^{\infty} \int_{-t_*}^{0} \varphi_j(\tau)\, P_N u_l(t+\tau)\, d\tau\, \varphi_j(s) \Big\}$$

and

$$| (1-\mathscr{P}_{N,M})(S_t w_1 - S_t w_2) |^2_{\mathscr{H}} = | \widetilde{Q}_N (S_t w_1 - S_t w_2) |^2_{\mathscr{H}} + \sum_{k=1}^{N} \lambda_k^2 \sum_{j=M+1}^{\infty} \left(\int_{-t_*}^{0} \varphi_j(\tau)(u_1(t+\tau) - u_2(t+\tau), e_k)\, d\tau \right)^2 \tag{3.10}$$

where $\widetilde{Q}_N$ is defined as in Sect. 1. Denote the last item $\sum_1$. Taking into account that

$$\sum_{j=M+1}^{\infty} (\varphi_j, g)^2_{L^2(-t_*, 0)} \le \frac{C}{(M+1)^2} \sum_{k \ge (M+1)/2} \left\{ k^2 (\varphi_{2k-1}, g)^2 + k^2 (\varphi_{2k}, g)^2 \right\} \le \frac{C}{(M+1)^2} \int_{-t_*}^{0} | \dot g(\tau) |^2\, d\tau,$$

we have

$$\sum\nolimits_1 \le \frac{C \lambda_N^2}{(M+1)^2} \int_{-t_*}^{0} \| \dot u_1(t+\tau) - \dot u_2(t+\tau) \|^2 d\tau \le \frac{C \lambda_N^2}{(M+1)^2} \int_{-t_*}^{0} | S_{t+\tau} w_1 - S_{t+\tau} w_2 |^2_{\mathscr{H}}\, d\tau .$$

Since $K_* \subset \{ w \in \mathscr{H} : | w |_{\mathscr{H}} \le R \}$ for some $R > 0$, Lemma 3.2 implies

$$\sum\nolimits_1 \le C \frac{\lambda_N^2}{(M+1)^2} e^{2 \xi t} | w_1 - w_2 |^2_{\mathscr{H}} \tag{3.11}$$

In the same way as this is done in the proof of (3.8) one can easily verify that $| S_t w |_{\mathscr{H}_\sigma} \le R_\sigma$ for $w \in K$ and $\sigma < \frac{1}{4}$. Consequently

$$| S_t w |_{\mathscr{H}_\sigma} \le R_\sigma ,\quad w \in K_* ,\quad \sigma < \frac{1}{4} . \tag{3.12}$$

We now consider the function $u_1(t)-u_2(t)$ as a solution of (1.2), (1.3) with $b(t)=f(\|\nabla u_1\|^2)$ and

$$h(t)=\mu(u_1-u_2)+\rho\frac{\partial}{\partial x_1}(u_1-u_2)-\left(f(\|\nabla u_1\|^2)-f(\|\nabla u_2\|^2)\right)\Delta u_2 .$$

It is clear from (3.12) that $b(t)$ possesses the property (1.4) and

$$\|h(t)\|_{2\sigma}\le C_{R_\sigma}\|\Delta(u_1-u_2)\|,\ \sigma<\frac{1}{4}. \tag{3.13}$$

Since $S_t w_1 - S_t w_2 = U(t,0;h,w_1-w_2)$ it follows from (1.15), (3.7) and (3.13) that

$$|\tilde{Q}_N(S_t w_1 - S_t w_2)|^2_{\mathcal{H}}\le C_1 e^{-\frac{at}{2}}\left(1+\frac{1}{\lambda^{\sigma}_{N+1}}e^{bt}\right)|w_1-w_2|_{\mathcal{H}}.$$

Combining this with (3.10) and (3.11), we obtain (3.9).

If we choose t_0 and N and M such that

$$C_1 e^{-\frac{at_0}{2}}\left(1+\frac{1}{\lambda^{\sigma}_{N+1}}e^{bt_0}\right)<\frac{\delta}{2}\ \text{ and }\ C_2\frac{\lambda_N}{M+1}e^{\xi t_0}<\frac{\delta}{2}\ \text{ for a given } 0<\delta<1,$$

then Lemma 3.3 implies

$$|(1-\mathcal{P}_{N,M})(S_{t_0}w_1-S_{t_0}w_2)|_{\mathcal{H}}\le\delta|w_1-w_2|_{\mathcal{H}} \tag{3.14}$$

for all $w_1, w_2\in K_*$. Since the global attractor $\mathfrak{A}$ lies in K_* the properties (3.7) and (3.14) allow us to apply the Ladyzhenskaya theorem (see, e.g., [3,15]) on finite dimensionality of invariant sets.

The proof of Theorem 3.1 is complete.

R e m a r k 3.1. The assertion similar to Lemma 3.3 can be also obtained for the modified von Karman system. Therefore the method presented above allows us to prove the finite dimensionality of the global attractor constructed in [7] for the problem of nonlinear oscillations of an elastic shell in potential supersonic flow.

We also note that Theorems 2.1 and 3.1 can be applied for the investigation of the problem (1)-(4) in more smooth spaces. In particular, Theorem 2.1 gives the possibility to define the evolution operator S_t of the problem (1)-(4) in the Banach space $\mathcal{C}=\mathcal{C}_0$ defined by (3.4) and endowed with the norm

$$|\psi|_{\mathcal{C}}=\max_{[-t_*,0]}\|\psi(\tau)\|_2+\max_{[-t_*,0]}\|\dot{\psi}(\tau)\|$$

by the formula $[\hat{S}_t\psi](\tau)=u(t+\tau),\ \tau\in[-t_*,0]$, where $u(t)$ is the solution of the problem (1)-(4) for initial conditions $u_0=\psi(0)$, $u_1=\dot{\psi}(0)$, $\varphi(\tau)=\psi(\tau)$. Theorem 1.2 and 2.1 and

Lemma 3.1 allow us to obtain the properties of dissipativity and strong continuity of S_t in the space $\mathcal{C}$. Theorem 3.1 implies the following assertion.

Corollary 3.1. *The dynamical system* $(\hat{S}_t, \mathcal{C})$ *has a global attractor* $\mathfrak{A}_{\mathcal{C}}$ *which is a bounded set in the spaces* $\mathcal{C}_\sigma$ *defined by (3.4),* $\sigma < \frac{1}{4}$. *The attractor* $\mathfrak{A}_{\mathcal{C}}$ *has a finite fractal dimension as a compact subset of the space* $L^2(-t_*, 0; \mathcal{F}_2)$.

P r o o f. By virtue of Theorem 3.1 it suffices to prove that the set

$$\mathfrak{A}_{\mathcal{C}} \equiv \{\psi(\tau) : (\psi(0); \dot{\psi}(0); \psi(\tau)) \equiv w \in \mathfrak{A}\}$$

possesses the following property of attraction

$$\lim_{t\to\infty} \sup \{\operatorname{dist}_{\mathcal{C}}(\hat{S}_t y, \mathfrak{A}_{\mathcal{C}}) : y \in B\} = 0$$

for any bounded set B in $\mathcal{C}$. We can obtain it from the attraction property of $\mathfrak{A}$ in $\mathcal{H}$ and the estimate (3.7).

Using the method applied in [3,4] we can also prove the following assertion on finiteness of number of essential mode.

Theorem 3.2. *There exist constants* ν_0 *and* N_0 *such that for any* $\nu > \nu_0$ *and* $N > N_0$ *and for any trajectories* $S_t w_1$ *and* $S_t w_2$ *belonging to the attractor* $\mathfrak{A}$, *the following assertions are true*:

1. *If* $\lim\limits_{t\to\infty} \{\| P_N(\dot{u}_1(t) - \dot{u}_2(t)) \| + \| P_N(u_1(t) - u_2(t)) \|_2\} = 0$,

then

$$\lim_{t\to\infty} \sup_{\tau\in[-t_*,0]} \{\| \dot{u}_1(t+\tau) - \dot{u}_2(t+\tau) \| + \| u_1(t+\tau) - u_2(t+\tau) \|_2\} = 0.$$

2. *If* $\tilde{P}_N S_t w_1 = \tilde{P}_N S_t w_2$ *for any* $t \in R$, *then* $S_t w_1 = S_t w_2$.

The proof relies on Theorem 1.2 and can be obtained in exactly the same way as in [3,7]. Similar arguments are also applied in the proof of Lemma 3.3.

Unfortunately, the structure of the global attractor of the problem (1)-(4) is unknown. We are also unable to estimate the rate of attraction of solution to the attractor. However, relying on Lemmas 3.2 and 3.3, we can establish the following theorem on the existence of a fractal exponential attractor (inertial set). This concept was introduced in [16] (see also [17]) for a class of compact dynamical systems.

Theorem 3.3. *The dynamical system* $(S_t, \mathscr{H})$ *governed the problem* (1)-(4) *and defined by* (2.10) *has a fractal exponential attractor, i.e. there exists a compact positive invariant set* $\mathfrak{A}_{\exp}$ *in* $\mathscr{H}$ *of finite fractal dimension possessing the property*

$$h(S_t B, \mathfrak{A}_{\exp}) \le C_B \exp(-\beta t),$$

for any bounded set B *in* $\mathscr{H}$. *Here* β *is a positive constant, and* $h(A, B)$ *is defined by* (3.1).

P r o o f. Let K_* be as in Lemma 3.3. Since $S_t K_* \subset K_*$, we can consider a restriction (S_t, K_*) of the dynamical system $(S_t, \mathscr{H})$ on the compact K_*. Lemmas 3.2 and 3.3 imply that for any $\delta > 0$ there exists an ortoprojector $\mathscr{P}$ in $\mathscr{H}$ of finite dimension and a $t_0 > 0$ such that

$$|(I - \mathscr{P}) S_{t_0} w_1 - S_{t_0} w_2|_{\mathscr{H}} \le \delta |w_1 - w_2|_{\mathscr{H}}.$$

and

$$|S_{t_0} w_1 - S_{t_0} w_2|_{\mathscr{H}} \le L |w_1 - w_2|_{\mathscr{H}}, \quad L > 0$$

for any w_1 and w_2 lying in K_*. These estimates allow us to verify the assumptions of the main theorem from [16] and to prove the existence of a fractal exponential attractor $\mathfrak{A}_{\exp}$ for the compact dynamical system (S_t, K_*). Since $K_* \supset K$ estimate (3.8) implies that $h(S_t B, K_*) \le C_B \exp(-at)$, $a > 0$. Therefore one can easily verify that $\mathfrak{A}_{\exp}$ is a fractal exponential attractor for the system $(S_t, \mathscr{H})$.

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Глобальные аттракторы для одного класса квазилинейных дифференциальных уравнений в частных производных с запаздыванием

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В статье исследуется система квазилинейных дифференциальных уравнений в частных производных с запаздыванием, возникающая в аэроупругости. Строится эволюционный оператор. Доказано существование компактного глобального аттрактора, который имеет конечную фрактальную размерность. Получены некоторые свойства решений на аттракторе, доказано существование фрактального экспоненциального аттрактора.

Глобальні аттрактори для одного класу квазілінійних диференційних рівнянь в часткових похідних з запізнюванням

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Досліджено систему квазілінійних диференційних рівнянь в часткових похідних з запізнюванням, яка виникає в аеропружності. Будується еволюційний оператор. Доведено існування компактного глобального аттрактора, який має скінченну фрактальну розмірність. Отримано деякі властивості рішень на аттракторі та доведено існування фрактального експоненційного аттрактора.