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Some extensions of the Darmois–Skitovich and Kagan Theorems to classes of the generalized random variables

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The Darmois–Skitovich Theorem and its generalization that was given by Kagan are transferred on a class of complex-valued random variables. They are also extended on wide classes of generalized random variables for which functions of bounded variation are taken instead of distribution functions.

1. Introduction. Let $X_1, X_2, \ldots, X_N$ be independent random variables (r.v.) and $L_1 = a_{11} X_1 + a_{12} X_2 + \ldots + a_{1N} X_N$, $L_2 = a_{21} X_1 + a_{22} X_2 + \ldots + a_{2N} X_N$ be two linear forms. The Darmois–Skitovich Theorem [1, 2] was one of the first results concerning characterization problems of the mathematical statistics:

Theorem A. *If L_1 and L_2 are independent, then those X_j which appear in the both forms, i. e. correspond to those j for which $a_{1j} a_{2j} \neq 0$, are Gaussian.*

Theorem A may be formulated in terms of the characteristic functions (c.f.) $\varphi_j(t)$ of r.v. X_j $(j = 1, 2, \ldots, N)$ in the following form: Let $\varphi_j(t)$ $(j = 1, 2, \ldots, N)$ satisfy the functional equation

$$\prod_{j=1}^{N} \varphi_j(a_{1j} u + a_{2j} t) = \prod_{j=1}^{N} \varphi_j(a_{1j} u) \prod_{j=1}^{N} \varphi_j(a_{2j} t)$$

for all $u, t \in \mathbb{R}$. Then $\varphi_j(t)$ are Gaussian c.f. for all j such that $a_{1j} a_{2j} \neq 0$.

This Theorem was extended in various ways. Zinger [3] has considered those restrictions on c.f. of r.v. X_j $(j = 1, 2, \ldots, N)$, that were induced by independence of the polynomial $Q(X_1, X_2, \ldots, X_N)$ and the linear form $L = a_1 X_1 + a_2 X_2 + \ldots + a_N X_N$. Laha [4], Lukacs and King [5], dealt with regression conditions on the linear forms L_1, L_2 preserving the conclusion of the Darmois–Skitovich Theorem. Kagan [6] gave an analytical extension of Laha's result. Heyde [7] characterized the condition on r.v. to be

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Gaussian by symmetry of the conditional distribution function of L_2 when L_1 is given. Zinger and Kagan [8] improved Heyde's result under additional hypothesis of existence of $(2n-1)$-order moments of r.v. X_j, and Kagan [9] gave an analytical refinement of the Heyde Theorem. Zinger and Kagan [10, 11] have studied the condition of the regression constancy in a wide sense for the problem for three forms L_1, L_2, L_3. Linnik and Skitovich [12] have transferred Theorem A to classes of functions of bounded variation and further description of this approach will be given below.

Kagan [13] has developed the Darmois-Skitovich Theorem in another direction. Let

$$L_j = a_{j1}X_1 + a_{j2}X_2 + \ldots + a_{jN}X_N, \quad j = 1, 2, \ldots, m,$$

be m linear forms of independent r.v. $X_1, X_2, \ldots, X_N$. As above we suppose that the coefficients a_{jk} are real and keep this assumption valid everywhere below. We shall also say that the joint distribution P of the forms $L_1, L_2, \ldots, L_m$ belongs to the class $D_{m,k}$, $1 \le k \le m$, if its c.f. $\varphi(t_1, \ldots, t_m; P)$ admits the factorization:

$$\varphi(t_1, \ldots, t_m; P) = \prod R_{i_1, \ldots, i_k}(t_{i_1}, \ldots, t_{i_k}), \quad (t_1, \ldots, t_m) \in \mathbb{R}^m. \tag{1.1}$$

Here $R_{i_1, \ldots, i_k}$ are continious (complex-valued) functions, $R_{i_1, \ldots, i_k}(0, \ldots, 0) = 1$, and the multiplication is extended over the set I_k of all multyindeces $(i_1, \ldots, i_k)$ satisfying $1 \le i_1 < \ldots < i_k \le m$. From now on when writing the right-hand side of (1.1) we mean this kind of factorization.

Theorem B (Kagan [13]). *If the joint distribution of the forms $L_1, L_2, \ldots, L_m$ belongs to the class $D_{m,m-1}$ and, for some j, we have $a_{1j}a_{2j} \ldots a_{mj} \ne 0$, then X_j is a Gaussian r.v.*

The class $D_{2,1}$ contains all distributions on $\mathbb{R}^2$ that have independent components, so the Darmois–Skitovich Theorem is a particular case of the Kagan result.

It should be pointed out that Kagan has obtained a more general result. Following Kagan we say that $P \in D_{m,k}(loc)$ if condition (1.1) is fulfilled in some neighborhood of the origin $|(t_1, \ldots, t_m)| < \varepsilon$. Theorem B holds if we assume that $P \in D_{m,m-1}(loc)$. Besides, if we assume that $P \in D_{m,k}(loc)$, $k < m-1$, the result comes immediately from Theorem B, we omit the explanations here.

Further researches connected with the classes $D_{m,k}(loc)$ could be found in [14-16].

2. Formulations of the main results. The aim of this work is to extend Kagan's result (and, in particular, the Darmois–Skitovich Theorem) to the case of complex-valued r.v. and also to some classes of functions of bounded variation (f.b.v.).

The contemporary statistics often deals with complex-valued r.v. that may be represented in the form $W = U + iV$, where U, V are independent Gaussian r.v. (see [17]). We denote $Z = X + iY$, where X, Y are real-valued r.v. Let us require that

$$E(e^{tY}) < \infty, \quad t \in \mathbb{R}, \tag{2.1}$$

and define the characteristic function $\varphi_Z(t)$ of the complex-valued r.v. Z by the formula

$$\varphi_Z(t) = E(\exp(it(X + iY))), \quad t \in \mathbf{R}.$$

If r.v. X and Y are independent, then

$$\varphi_Z(t) = E(\exp(itX))\, E(\exp(-tY)) = \varphi_X(t)\, \varphi_Y(it), \quad t \in \mathbf{R},$$

here $\varphi_X(t)$, $\varphi_Y(it)$ are c.f. of r.v. X, Y respectively. Here we used condition (2.1) which implies that c.f. $\varphi_Y(t)$ admits analytical continuation to the whole open complex plane C which may be expressed by the same integral that defines the function $E(\exp(itY))$ on the real axis.

D e f i n i t i o n 1. *We say that a complex-valued r.v.* $Z = X + iY$ *belongs to the class* $\mathfrak{R}_c$ $(Z \in \mathfrak{R}_c)$ *if* X, Y *are independent, r.v.* Y *satisfies condition* (2.1) *and also*

$$E(\exp(itY)) \neq 0, \quad t \in \mathbf{R}. \tag{2.2}$$

Our first result is

Theorem 1. *Let* $\{X_j\}_{j=1}^N$ *and* $\{Y_j\}_{j=1}^N$ *be two sequences of independent r.v. Let the complex-valued r.v.* $Z_j = X_j + iY_j$ $(j = 1, 2, \ldots, N)$ *belong to the class* $\mathfrak{R}_c$ *and let their c.f.* $\varphi_{Z_j}(t)$ *satisfy the equation*

$$\prod_{j=1}^N \varphi_{Z_j}(a_{1j}t_1 + a_{2j}t_2 + \ldots + a_{mj}t_m) = \prod R_{i_1,\ldots,i_{m-1}}(t_{i_1},\ldots,t_{i_{m-1}}) \tag{2.3}$$

in a certain neighborhood of the origin $|(t_1,\ldots,t_m)| < \varepsilon$. *Then* X_j, Y_j *are Gaussian r.v. if, for the corresponding* j, $a_{1j}a_{2j}\ldots a_{mj} \neq 0$.

Proposition 1. *If, in the definition of the class* $\mathfrak{R}_c$, *we cancel either the condition of independence of r.v.* X, Y *or condition* (2.2), *then there exist solutions* $\varphi_{Z_1}(t)$, $\varphi_{Z_2}(t)$ *of the functional equation*

$$\varphi_{Z_1}(t_1 + t_2)\,\varphi_{Z_2}(t_1 - t_2) = \varphi_{Z_1}(t_1)\,\varphi_{Z_2}(t_1)\,\varphi_{Z_1}(t_2)\,\varphi_{Z_2}(t_2), \quad t_1, t_2 \in \mathbf{R}, \tag{2.4}$$

such that X_1, X_2 *and* Y_1, Y_2 *are pairs of independent r.v. and all these r.v. are not Gaussian.*

Thus the condition of independence of r.v. X_j, Y_j and condition (2.2) for r.v. Y_j in Theorem 1 are necessary for the statement of the Theorem to be valid.

Let us note that Theorem 1 presents a new direction of extensions of Theorem B. It differs from the generalization of the Darmois–Skitovich Theorem for the n-dimensional random vectors due to Skitovich; Gurie and Olkin (see [17]).

Our next goal is to find the widest classes of functions of bounded variation (f.b.v.) to which the Darmois–Skitovich and Kagan Theorems could be transferred. Let B be the class of f.b.v. $V(x)$ on the axis $(-\infty, \infty)$ that are normalized by the conditions

$$\lim_{x \to -\infty} V(x) = 0, \quad \lim_{x \to \infty} V(x) = 1, \quad V(x - 0) = V(x).$$

Each f.b.v. $V(x)$ may be represented in the form:

$$V(x) = \omega(x) - \sigma(x), \tag{2.5}$$

where $\omega(x)$, $\sigma(x)$ are non-decreasing f.b.v. and $\omega(-\infty) = \sigma(-\infty) = 0$. Let $M(x)$, $T(x)$ $(x > 0)$ be positive non-increasing functions. We denote by $B_{M,T}$ the subclass of the class B consisting of f.b.v. $V(x) \in B$ such that the corresponding functions ω from expression (2.5) satisfy the condition

$$\omega(+\infty) - \omega(x) + \omega(-x) = O(M(x)), \quad x \to +\infty, \tag{2.6}$$

and the functions σ satisfy the condition

$$\sigma(+\infty) - \sigma(x) + \sigma(-x) = O(T(x)), \quad x \to +\infty. \tag{2.7}$$

For each f.b.v. $V(x) \in B$ we define a c.f. $\varphi(t; V)$ by the expression

$$\varphi(t; V) = \int_{-\infty}^{\infty} \exp(itx)\, dV(x), \quad t \in \mathbf{R}. \tag{2.8}$$

Linnik and Skitovich extended Theorem A to various classes of f.b.v. Let us consider the functions $M_{1,\gamma}(x) = T_{1,\gamma}(x) = \exp(-x^{1+\gamma})$, $(x > 0)$ for each $\gamma > 0$ and introduce the class B_1 of f.b.v. as follows $B_1 = \bigcup_{\gamma > 0} B_{M_{1,\gamma}, T_{1,\gamma}}$. For each f.b.v. $V(x) \in B_1$ its c.f. $\varphi(t; V)$ admits analytical continuation as an entire function of finite order which is defined by the integral from the right-hand side of equality (2.8).

Theorem C **(Linnik, Skitovich [12]).** *Let c.f.* $\varphi(t; V_j)$ $(j = 1, 2, \dots, N)$ *of f.b.v.* $V_j \in B_1$ *satisfy the functional equation*

$$\prod_{j=1}^{N} \varphi(a_{1j} t_1 + a_{2j} t_2; V_j) = \prod_{j=1}^{N} \varphi(a_{1j} t_1; V_j) \prod_{j=1}^{N} \varphi(a_{2j} t_2; V_j), \quad t \in \mathbf{R}. \tag{2.9}$$

Then, for j such that $a_{1j} a_{2j} \neq 0$, V_j *are Gaussian distribution functions.*

Our second result is an extension of this Theorem. Let us consider the functions

$$M_{2,c}(x) = \exp(-cx), \quad x > 0, \quad T_{2,c}(x) = \exp(-cx \ln x), \quad x > 1$$

for each $c > 0$. Let us introduce the class of f.b.v. $B_2 = \bigcap_{c > 0} B_{M_{2,c}, T_{2,c}}$. We would like to note that c.f. of f.b.v. V from this class are entire functions as well as c.f. of f.b.v. σ from representation (2.5) and also

$$|\varphi(t; \sigma)| = O(\exp(\exp(c|\operatorname{Im} t|))), \quad |t| \to \infty, \ \forall c > 0. \tag{2.10}$$

Theorem 2. *Let f.b.v.* V_j $(j = 1, 2, \dots, N)$ *belong to the class* B_2 *and let their c.f.* $\varphi(t; V_j)$ *satisfy relation* (2.9) *in a certain neighborhood of the origin* $|(t_1, t_2)| < \varepsilon$. *If, for some j,* $a_{1j} a_{2j} \neq 0$, *then* V_j *is a Gaussian distribution function* (*d.f.*).

Proposition 2. *If in the hypothesis of Theorem 2 we change the class B_2 to the wider class $B_3 = \bigcap_{c>0} \bigcup_{r>0} B_{M_{2,c}, T_{2,r}}$, then Theorem 2 does not hold. Namely, there are f.b.v. V_j ($j = 1, 2, 3, 4$) in the class B_3 that are non-Gaussian and their c.f. satisfy the equation*

$$\prod_{j=1}^{4} \varphi(a_{1j} t_1 + a_{2j} t_2; V_j) = \prod_{j=1}^{4} \varphi(a_{1j} t_1; V_j) \prod_{j=1}^{4} \varphi(a_{2j} t_2; V_j), \quad t \in \mathbf{R}, \tag{2.11}$$

where $a_{1j} = 1$ ($j = 1, 2, 3, 4$), $a_{21} = a_{22} = 1$, $a_{23} = a_{24} = -1$.

Note that the class B_2 contains the class B_1 and therefore Theorem 2 is an extension of Theorem C. However, the class B_2 does not contain all d.f. It contains only those d.f. F for which

$$1 - F(x) + F(-x) = O(e^{-rx}), \quad x \to +\infty, \ \forall r > 0.$$

Thus our next step is to find a class $B_4^{(s)}$ of f.b.v. that contains all symmetric d.f. and such that Theorem 2 holds for this class. Let $B_4^{(s)}$ be a subclass of the class $B_5 = \bigcap_{c>0} B_{1, T_{2,c}}$ consisting of symmetric (i.e. $1 - V(x+0) = V(-x)$, $x \geq 0$) f.b.v. V satisfying

$$\int_{-\infty}^{\infty} x^{2k} \, d\omega(x) \geq \int_{-\infty}^{\infty} x^{2k} \, d\sigma(x), \quad k = 1, 2, \ldots . \tag{2.12}$$

The classes $B_{1, T_{2,c}}$ participating in the definition of B_5 are the standard classes $B_{M,T}$ with $M(x) \equiv 1$, $T(x) = T_{2,c}(x)$ being defined in the definition of the class B_2. Note, that the integral in the left-hand side of (2.12) always makes sense even though it may be equal to $+\infty$.

Theorem 3. *Theorem 2 is valid if we change the class B_2 to the class $B_4^{(s)}$.*

The condition of symmetry and condition (2.12) for f.b.v. in the statement of Theorem 3 allow us to prove

Theorem 4. *If f.b.v. $V_j \in B_4^{(s)}$ ($j = 1, 2, \ldots, N$) and their c.f. satisfy* (2.9) *in some neighborhood of the origin $|(t_1, t_2)| < \varepsilon$, then $V_j \in B_2$ ($j = 1, 2, \ldots, N$).*

Thus Theorem 3 follows from Theorems 2 and 4.

In connection with study of factorization of Gaussian d.f. into composition of f.b.v. the Cramer Theorem was transferred to different classes of f.b.v. Linnik and Skitovich [12] considered the class B_1; Grunkemeier [18] considered the subclass B_6 of the class $B_7 = \bigcap_{c>0} B_{M_{2,c}, M_{2,c}}$ (the function $M_{2,c}(x)$ was introduced in the definition of the class B_2) of f.b.v. V such that

$$1 - V(x) \geq 0, \quad V(-x) \geq 0 \tag{2.13}$$

for sufficiently large x. (Note, that the Grunkemeier class of f.b.v. is more restricted, than B_6). Chistyakov [19] dealt with the subclass B_8 of the class B_5 such that

$$I(y, V) = \int_{-\infty}^{\infty} e^{-yx}\, dV(x) \neq 0, \quad y \in \mathbb{R},$$

(the integral $I(y, V) = I(y, \omega) - I(y, \sigma)$ always makes sense for f.b.v. $V \in B_8$ due to condition (2.7) with $T(x) = T_{2,c}(x)$ and arbitrary $c > 0$). Yakovleva [20] considered the subclass B_9 of the class $B_{10} = \bigcap_{c>0} B_{1, M_{2,c}}$ of f.b.v. V such that

$$I(y, \sigma)/I(y, \omega) = 1 - e^{-Q(y)}, \text{ where } Q(y) = O(e^{c|y|}),\ y \to +\infty,\ \forall c > 0. \qquad (2.14)$$

The classes $B_{1, M_{2,c}}$ in the definition of the class B_{10} are the classes $B_{M,T}$, where $M(x) \equiv 1$, $T(x) = M_{2,c}(x)$. The division in the left-hand side of (2.14) is always possible because $0 < I(y, \omega) \leq \infty$, and $0 < I(y, \sigma) < \infty$ by condition (2.7) with $T(x) = M_{2,c}(x)$ and arbitrary $c > 0$. As we have already noted $B_1 \subset B_2$. It is obvious that $B_7 \cap B_8 \subset B_2$. Hence, Theorem 2 holds true for the classes B_1, $B_7 \cap B_8$. One can easily construct examples to show that the classes B_6, B_8, B_9 does not contain one another and also that the same is true for the classes B_2, B_6, $B_7 \cap B_9$. Nevertheless the following Proposition holds:

Proposition 3. *Let f.b.v. V_j ($j = 1, 2, \ldots, N$) belong either to the class B_6 or to the class $B_7 \cap B_9$ and let their c.f. satisfy relation (2.9) in a certain neighborhood of the origin $|(t_1, t_2)| < \varepsilon$. Then f.b.v. $V_j \in B_2$ ($j = 1, 2 \ldots, N$).*

Corollary. *Theorem 2 holds true for f.b.v. V from the classes B_6 and $B_7 \cap B_9$.*

The situation changes substantially when we try to extend the Kagan Theorem.

Proposition 4. *There are pairwise noncollinear vectors (a_{1j}, a_{2j}, a_{3j}) ($j = 1, 2, 3$) with the conditions $a_{1j} a_{2j} a_{3j} \neq 0$ and non-Gaussian f.b.v. V_j ($j = 1, 2, 3$) which all belong to either the classes B_1 or $B_7 \cap B_8$ or $B_7 \cap B_9$ and such that c.f. of f.b.v. V_j satisfy the equation*

$$\prod_{j=1}^{3} \varphi(a_{1j} t_1 + a_{2j} t_2 + a_{3j} t_3; V_j) =$$

$$= R_{1,2}(t_1, t_2)\, R_{1,3}(t_1, t_3)\, R_{2,3}(t_2, t_3), \quad (t_1, t_2, t_3) \in \mathbb{R}^3. \qquad (2.15)$$

Thus the Kagan Theorem is not true in the classes of f.b.v. that contain the classes B_1, $B_7 \cap B_8$, $B_7 \cap B_9$. But this is not the same for f.b.v. from the Grunkemier class B_6.

Theorem 5. *Let f.b.v.* $V_j \in B_6$ $(j = 1, 2, \ldots, N)$ *and let their c.f.* $\varphi(t; V_j)$ *satisfy the relation*

$$\prod_{j=1}^{N} \varphi(a_{1j} t_1 + a_{2j} t_2 + \ldots + a_{mj} t_m; V_j) = \prod R_{i_1, \ldots, i_{m-1}} (t_{i_1}, \ldots, t_{i_{m-1}})$$

in a certain neighborhood of the origin $|(t_1, \ldots, t_m)| < \varepsilon$. *If, for some j,* $a_{1j} a_{2j} \ldots a_{mj} \neq 0$, *then* V_j *are Gaussian d.f.*

In some sence Theorem 5 may be improved for the case of symmetric f.b.v. V_j. We denote by $B_6^{(s)}$ the subclass of all symmetric f.b.v. from the class B_{10} satisfying condition (2.13) for all $x > 0$.

Theorem 6. *Theorem 5 holds true for all f.b.v.* V_j $(j = 1, 2, \ldots, N)$ *from the class* $B_6^{(s)}$.

Although Theorem 5 is not true in the class B_2 (recall that $B_1 \subset B_2$) it may be shown that there is a subclass B_{11} for which it remains correct. Let us consider the function $T_{3,\varepsilon}(x) = \exp\left(-x^{1 + 1/(3-\varepsilon)}\right)$ $(x > 0)$ for each $0 < \varepsilon < 3$ and introduce the class of f.b.v. $B_{11} = \bigcap_{c>0} \bigcup_{0<\varepsilon<3} B_{M_{2,c}, T_{3,\varepsilon}}$.

Theorem 7. *Theorem 5 holds true for f.b.v.* $V_j \in B_{11}$ $(j = 1, 2, \ldots, N)$.

Theorem 7 is unimprovable in the following sense. If we consider the wider class $B_{12} = \bigcap_{c>0} B_{M_{2,c}, T_{3,0}}$ $(T_{3,0}(x) = \exp(-x^{4/3}), x > 0)$ instead of the class B_{11}, then the following Proposition holds:

Proposition 5. *The assertion of Proposition 4 is true for f.b.v. from the class* B_{12}.

3. The extension of Kagan's Theorem B to complex-valued r.v. We formulate here some results we need to prove Theorem 1. All of them are contained, say, in monograph [21] except Lemma K which may be found in [13].

Theorem D [21, p. 25]. *If a c.f.* $\varphi_X(t)$ *is analytic in a domain G containing an interval* (ia, ib) *of the imaginary t-axis (we assume* $a \leq 0$, $b \geq 0$, $b - a > 0$*), then it is analytic in the whole strip* $a < \operatorname{Im} t < b$ *and may be represented in this strip by the formula*

$$\varphi_X(t) = E(e^{itX}) = \int_{\Omega} e^{itX(\omega)}\, dP,$$

where the integral converges absolutely and uniformly in each strip $a < a_1 \leq \operatorname{Im} t \leq b_1 < b$.

Theorem E [21, p. 28]. *If a c.f.* $\varphi_X(t)$ *is analytic in the strip* $a < \operatorname{Im} t < b$, $a \le 0 \le b$, *the following inequality holds there*:

$$|\varphi_X(t)| \le \varphi_X(i \operatorname{Im} t).$$

The following Lemma is a little refinement of Lemma A.2.1 from [21, p. 342] and may be proved in a similar way.

Lemma F. *If* $g(t)$ *is analytic and nonvanishing in the strip* $|\operatorname{Im} t| < \frac{R}{2}$, *and, in this strip,* $|g(t)| \le H < \infty$, *then*

$$|g(t)| \ge \exp\left(-\frac{3R}{\varepsilon}\left(|\ln g(0)| + |\ln H|\right)\exp\left(\frac{\pi}{R}|t|\right)\right)$$

in the strip $|\operatorname{Im} t| < \frac{R}{2} - \varepsilon$.

Let $f(t)$ $(t \in \mathbf{C})$ be an entire function. We denote

$$M(r, f) = \max_{|t| = r} |f(t)|.$$

Lemma G [21, p. 31]. *Let* $\varphi(t)$ *be an entire ridge function. Then, for each* $r > 0$,

$$M(r, \varphi) = \max\left(|\varphi(ir)|, |\varphi(-ir)|\right).$$

Theorem H [21, p. 14]. *Let* $g(t)$ *be an entire function and* $f(t) = \exp(g(t))$. *Then*

$$M(r, g) \le 4 \ln M\left(\frac{3}{2} r, f\right) + 5|g(0)|.$$

Theorem I [21, pp. 44, 361]. *Let* $Q(t)$ *be an entire function such that*

$$\varliminf_{r \to \infty} \frac{\ln M(r, Q)}{r} = 0, \tag{3.1}$$

and $Q(0) = 0$. *If the function* $\varphi(t) = \exp(Q(t))$ *is a ridge function, then* $\varphi(t) = \exp(-\sigma^2 t^2 + i\beta t)$, *where* β, σ *are real constants.*

Theorem J (Cramér) [21, p. 59]. *Let* $X_1, X_2, \ldots, X_n$ *be independent r.v. and let the sum* $X_1 + X_2 + \ldots + X_n$ *be Gaussian r.v. Then r.v.* X_j $(j = 1, 2, \ldots, n)$ *are Gaussian.*

Lemma K [13]. *Let* $\psi_1(u), \ldots, \psi_n(u)$ *be continious functions on the real axis, the vectors* $\mathbf{a}_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$ $(j = 1, 2, \ldots, n)$ *be pairwise non-collinear, and let the relation*

$$\prod_{j=1}^{n} \psi_j(a_{1j} t_1 + a_{2j} t_2 + \ldots + a_{mj} t_m) = \prod R_{i_1, \ldots, i_{m-1}}(t_{i_1}, \ldots, t_{i_{m-1}})$$

be satisfied in a certain neighborhood of the origin $|(t_1, \ldots, t_m)| < \varepsilon$. *Then, in a certain neighborhood of the origin,* $\psi_j(u) = \exp(P_j(u))$, *where* $P_j(u)$ *are polynomials of degrees not exceeding* $n + m - 2$, $j = 1, 2, \ldots, n$.

We shall also need the following auxiliary statement:

Theorem 3.1. *Let* $\varphi_X(t)$, $\varphi_Y(t)$ *be c.f. of (real-valued) r.v.* X, Y, *and let* $\varphi_Y(t)$ *be an entire non-vanishing on the real axis function. Let also the equation*

$$\varphi_X(t)\,\varphi_Y(it) = \exp(P(t)), \quad t \in [-\varepsilon, \varepsilon], \quad \varepsilon > 0 \tag{3.2}$$

holds true, where $P(t)$ *is a polynomial. Then r.v.* X, Y *are Gaussian.*

Lemma 3.1. *If we cancel the condition* $\varphi_Y(t) \neq 0$ *for* $t \in \mathbf{R}$, *then Theorem 3.1 does not remain true.*

Proof of Theorem 3.1. The proof of the Theorem is based on one method due to I.V. Ostrovskii. He used this method in studying the D. van Dantzig class of c.f. (see [21, pp. 347–348]).

By relation (3.2) and the condition $\varphi_Y(t) \neq 0$ for $t \in \mathbf{R}$, c.f. $\varphi_X(t)$ coincides on the segment $[-\varepsilon, \varepsilon]$ with a function which is analytic in a domain G containing the imaginary axis. By Theorem D, the function $\varphi_X(t)$ is entire and hence equality (3.2) holds for all $t \in \mathbf{C}$. Thus c.f. $\varphi_X(t)$, $\varphi_Y(t)$ are non-vanishing entire functions. Let us represent c.f. $\varphi_Y(t)$ in the form

$$\varphi_Y(t) = \exp(Q(t)),$$

where $Q(t)$ is an entire function, $Q(0) = 0$. By Theorem E, in every strip $|\operatorname{Im} t| < \frac{R}{2}$ $(R > 0)$ the inequality holds

$$|\varphi_X(t)| \leq H_R = \sup_{-\frac{R}{2} < \tau < \frac{R}{2}} |\varphi_X(i\tau)| < \infty.$$

Using Lemma F we get the estimate

$$|\varphi_X(t)| \geq \exp\left(-C_R \exp\left(\frac{\pi}{R}|t|\right)\right), \quad t \in \mathbf{R},$$

where $0 < C_R < \infty$ is independent of t. By Lemma G, we have, for any fixed $R > 0$,

$$M(r, \varphi_Y) = \max(|\varphi_Y(ir)|, |\varphi_Y(-ir)|) =$$

$$= \max\left(\frac{\exp(\operatorname{Re} P(r))}{|\varphi_X(r)|}, \frac{\exp(\operatorname{Re} P(-r))}{|\varphi_X(-r)|}\right) \leq \exp\left(2C_R \exp\left(\frac{\pi}{R} r\right)\right), \quad r \geq r_0.$$

Hence, for any $\varepsilon > 0$,

$$\ln M(r, \varphi_Y) = O(e^{\varepsilon r}), \quad r \to \infty.$$

Using Theorem H we get

$$M(r, Q) \le 4 \ln M(\tfrac{3}{2} r, \varphi_Y) + O(1) = O(e^{2\varepsilon r}), \quad r \to \infty.$$

Since $\varepsilon > 0$ is arbitrary small, we write

$$\lim_{r \to \infty} \frac{\ln M(r, Q)}{r} = 0.$$

By Theorem I,

$$\varphi_Y(t) = \exp(-\alpha_Y t^2 + i\beta_Y t), \quad t \in \mathbf{R},$$

where $\alpha_Y \ge 0$, $\beta_Y \in \mathbf{R}$, i. e. Y is a Gaussian r.v. Since (3.2) holds for all $t \in \mathbf{C}$, we can represent $\varphi_X(t)$ in the form $\varphi_X(t) = \exp(P_1(t))$, where $P_1(t)$ is a polynomial and applying Theorem I again we have that X also is a Gaussian r.v.

R e m a r k. Theorem 3.1 remains true if the polynomial $P(t)$ in the right-hand side of (3.2) is replaced by an entire function $Q(t)$ which satisfies condition (3.1). This may be proved just by repeating the same arguments.

P r o o f o f L e m m a 3.1. Let us consider c.f.

$$\varphi_X(t) = \frac{1}{1 + t^2}; \quad \varphi_Y(t) = (1 - t^2) \exp\left(-\frac{t^2}{2}\right).$$

They satisfy

$$\varphi_X(t)\, \varphi_Y(it) = \exp\left(\frac{t^2}{2}\right),$$

whereas none of $\varphi_X(t)$, $\varphi_Y(t)$ is a Gaussian c.f.

P r o o f o f T h e o r e m 1. We can assume, without restriction of generality, that, for all vectors $\mathbf{a}_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$ in (2.3), $a_{1j} a_{2j} \ldots a_{mj} \ne 0$. Actually, if, for some j, $a_{1j} a_{2j} \ldots a_{mj} = 0$, then the corresponding functions $\varphi_{Z_j}(a_{1j} t_1 + a_{2j} t_2 + \ldots + a_{mj} t_m)$ do not depend on some of the variables $t_1, t_2, \ldots, t_m$. Because $\varphi_{Z_j}(0) = 1$, all functions $\varphi_{Z_j}(u)$ do not vanish in a certain neighborhood of the origin $u \in [-\varepsilon, \varepsilon]$, and by dividing the both parts of equality (2.3) by them we get the necessary assertion.

Without restriction of generality we can assume that $a_{11} = a_{12} = \ldots = a_{1N} = 1$. Actually, since $a_{1s} \ne 0$ $(s = 1, 2, \ldots, N)$, this assumption holds true for the functions $\varphi_{Z_j / a_{1j}}(u)$.

The vectors $\mathbf{a}_1, \ldots, \mathbf{a}_N$ may coincide. Suppose there are n $(n \le N)$ different vectors among them. We split all functions φ_{Z_j} into groups with equal vectors $\mathbf{a}_j$ and (renumbering vectors if need be) we assume that

$$\mathbf{a}_1 = \mathbf{a}_2 = \ldots = \mathbf{a}_{k_1}, \quad \mathbf{a}_{k_1 + 1} = \ldots = \mathbf{a}_{k_2}, \quad \ldots, \quad \mathbf{a}_{k_{n-1} + 1} = \ldots = \mathbf{a}_{k_n},$$

where $k_n = N$, $\mathbf{a}_{k_1}, \mathbf{a}_{k_2}, \ldots, \mathbf{a}_{k_n}$ are pairwise non-collinear vectors. We denote

$$h_1(u) = \prod_{j=1}^{k_1} \varphi_{Z_j}(u), \ldots, h_n(u) = \prod_{j=k_{n-1}+1}^{N} \varphi_{Z_j}(u). \tag{3.3}$$

Applying Lemma K to the functions $h_1(u), \ldots, h_n(u)$ we obtain that each of these functions admits the representation

$$h_l(u) = \exp(P_l(u)) \tag{3.4}$$

in some neighborhood of the origin, where $P_l(u)$ are polynomials. Then, by Theorem 3.1, r.v.

$$X_1 + \ldots + X_{k_1}, \quad Y_1 + \ldots + Y_{k_1}, \ \ldots, \ X_{k_{n-1}} + \ldots + X_N, \quad Y_{k_{n-1}} + \ldots + Y_N$$

are Gaussian. By Theorem J, we can say that r.v. $X_1, \ldots, X_N, Y_1, \ldots, Y_N$ are Gaussian as well.

P r o o f o f P r o p o s i t i o n 1. Let X_1, X_2, Y_1, Y_2 be independent r.v. and

$$\varphi_{X_1}(t) = \varphi_{X_2}(t) = \frac{1}{1+t^2}, \quad \varphi_{Y_1}(t) = \varphi_{Y_2}(t) = (1-t^2)\exp\left(-\frac{t^2}{2}\right).$$

Then the functions $\varphi_{Z_1}(t), \varphi_{Z_2}(t)$ satisfy equation (2.4). Thus we have demonstrated that condition (2.2) in the definition of the class $\mathfrak{R}_c$ cannot be neglected if we want to keep the statement of Theorem 1 true.

Let us show that the statement of Theorem 1 does not hold, if we neglect the condition of independence of r.v. X_j, Y_j. We consider independent r.v. X_1, X_2 and independent r.v. Y_1, Y_2 such that c.f. of the two-dimensional random vectors (X_1, Y_1) and (X_2, Y_2) may be represented in the form

$$\varphi_{X_j, Y_j}(t_1, t_2) = \frac{1}{2}\left(e^{-(a_1 t_1^2 + b_1 t_2^2)} + e^{-(a_2 t_1^2 + b_2 t_2^2)}\right), \tag{3.5}$$

where the parameters a_k, b_k $(k = 1, 2)$ are positive numbers and $a_1 \neq a_2$, $b_1 \neq b_2$, $b_1 - a_1 = b_2 - a_2 = c$. From (3.5) we see that r.v. X_1, Y_1, X_2, Y_2 are non-Gaussian. Besides r.v. X_j and Y_j are dependent. If they were independent, then for all $t_1, t_2 \in \mathbf{R}$ the equality

$$\varphi_{X_j, Y_j}(t_1, t_2) = \varphi_{X_j}(t_1)\varphi_{Y_j}(t_2) \tag{3.6}$$

would hold. But if we use (3.5) and the explicit form of the functions $\varphi_{X_j}, \varphi_{Y_j}$, then setting $t_1 = t_2 = t$ one easily can see that (3.6) does not hold on the diagonal.

Let us form the complex-valued r.v. $Z_j = X_j + iY_j$ $(j = 1, 2)$. Their c.f. may be represented in the form

$$\varphi_{Z_j}(t) = \frac{1}{2}\left(e^{t^2(b_1 - a_1)} + e^{t^2(b_2 - a_2)}\right) = e^{ct^2}$$

and hence, they satisfy equality (2.4).

4. Extensions of the Darmois–Skitovich and Kagan Theorems to functions of bounded variation. To prove Theorem 2 we need some auxiliary results.

Lemma 4.1. *Let an entire functon $f(t)$ be positive on the imaginary axis, $f(0) = 1$, and*

a) $\varlimsup_{R\to\infty} \frac{\ln \ln M(R,f)}{R} = 0;$

b) *for each fixed variable $v \in \mathbf{R}$*

$$-K_1(v) \le -\ln |f(u+iv)| \le K_2(v)(|u|^m + 1), \quad u \in \mathbf{R},$$

where the parameter $m \in \mathbf{N}$, and $K_j(v)$ ($j = 1, 2$) are finite, positive and independent of u.

Then $f(t) = e^{P(t)}$, where $P(t)$ is a polynomial of degree at most m, real-valued on the imaginary axis, and $P(0) = 0$.

For the case $m = 2$ Lemma 4.1 was already proved in [19, Lemma 1]. The proof of the general case is similar, so we skip it here.

Lemma 4.2. *Let a function $f(t)$, $f(0) = 0$, be analytic in the half-plane* $\operatorname{Im} t \ge 0$ *and*

$$\max_{0\le\theta\le\pi} \max(Re\, f(R_k e^{i\theta}), 0) + \max_{-R_k \le t \le R_k} \operatorname{Re} f(t) = O(e^{cR_k}), \quad R_k \to \infty, \ \forall c > 0,$$

for some sequence of positive numbers $\{R_k\} \uparrow \infty$. Then

$$\left|f\left(\frac{1}{2} iR_k\right)\right| = O(e^{cR_k}), \quad R_k \to \infty, \ \forall c > 0.$$

This Lemma is close to Lemma 2 from [19] and may be proved in a similar way. Here we omit the proof.

Let $G_R = \{t \in \mathbf{C} : |t| < R, \operatorname{Im} t > 0\}$, and $f(t)$ be a function analytic in G_R and continuous up to the boundary. Denote

$$M^+(R,f) = \max_{t\in G_R} |f(t)|.$$

Lemma L. *Let a function $f(t)$ be analytic in the half-plane* $\operatorname{Im} t > 0$, *continuous up to the real axis, and*

a) $|f(t)| \le C(|t|^a + 1)$, $t \in \mathbf{R}$, *where C, a are positive constants*;

b) $\varlimsup_{R\to\infty} \frac{\ln M^+(R,f)}{R} = 0.$

Then

$$\lim_{R \to \infty} \frac{\ln M^+(R,f)}{R} = 0.$$

This Lemma could be proved by the standard methods using the Theorem on two constants (see [22, p. 296]) so we skip the proof.

Theorem 4.1. *Let f.b.v.* $V_j \in B_2$ $(j = 1, 2,\ldots, n)$ *and*

$$\varphi(t;\, V_1)\, \varphi(t;\, V_2) \ldots \varphi(t;\, V_n) = e^{P(t)}, \quad t \in \mathbf{R}, \tag{4.1}$$

where $P(t)$ *is a polynomial of degree* m $(m \geq 1)$. *Then* $\varphi(t;\, V_j) = e^{P_j(t)}$ $(j = 1, 2,\ldots, n)$, *for all* $t \in \mathbf{R}$, *here* $P_j(t)$ *are polynomials of degree at most* m, *real-valued on the imaginary axis.*

Corollary. *Let f.b.v.* $V_j \in B_2$ $(j = 1, 2,\ldots, n)$ *and*

$$V_1 * V_2 * \ldots * V_n = \Phi_{a,\sigma^2}, \tag{4.2}$$

where Φ_{a,σ^2} *is the Gaussian d.f. with mathematical expectation* a *and variance* σ^2. *Then f.b.v.* V_j $(j = 1, 2,\ldots, n)$ *are Gaussian d. f.*

Remark. This Corollary was already proved for the case $n = 2$ in [19, Theorem 1]. The class B_2 is not closed with respect to convolution, therefore, for $n \geq 3$, the above statement does not follow from Theorem 1, [19]. To check that the class B_2 is not closed with respect to convolution we consider f.b.v. V_1, $V_2 \in B_2$ with c.f. of the form

$$\varphi(t;\, V_1) = e^{e^{it} - 1} - e^{i\alpha t}, \quad \varphi(t;\, V_2) = e^{e^{it} - 1} - e^{-i\alpha t},$$

where $0 < \alpha < \frac{1}{2}$. It is easy to see that $V_1 * V_2 \notin B_2$.

Proof of the Corollary. By Theorem 4.1, we have $\varphi(t;\, V_j) = \exp\,(ia_j t + b_j t^2)$, $(j = 1, 2,\ldots, n)$, where a_j, b_j are real constants. Since the absolute values of all $\varphi(t;\, V_j)$ are bounded on the real axis we have $b_j \leq 0$ $(j = 1, 2,\ldots, n)$. Therefore f.b.v. V_j $(j = 1, 2,\ldots, n)$ are Gaussian.

Proof of Theorem 4.1. C.f. $\varphi(t;\, V)$ of f.b.v. $V \in B_2$ may be analytically continued from the real axis to $\mathbf{C}$, this continuation is given by the same integral from the right-hand side of (2.8). Therefore, equation (4.1) also holds for all $t \in \mathbf{C}$ and below we use it just in this sense. The functions $\varphi(t;\, V_j)$ are real-valued on the imaginary axis and equal one at the origin. Using (4.1) we can see that they do not vanish in the open complex plain $\mathbf{C}$. Therefore they are positive on the imaginary axis. Let us show that the functions

$\varphi(t;\,V_j)$ satisfy condition b) of Lemma 4.1. The left-hand inequality follows immediately from the relation

$$|\,\varphi(u+iv;\,V_j)\,| \le \varphi(iv;\,\omega_j) + \varphi(iv;\,\sigma_j), \quad u,v \in \mathbb{R}, \tag{4.3}$$

where ω_j, σ_j are non-decreasing f.b.v. from representation (2.5) for f.b.v. V_j and ω_j satisfies condition (2.6) with $M(x) = M_{2,c}(x)$, for an arbitrary $c>0$, σ_j satisfies (2.7) with $T(x) = T_{2,c}(x)$, for an arbitrary $c>0$. Let us check the right-hand inequality in condition b). Using the relation

$$c_j(v) = \sum_{\substack{k=1\\k\neq j}}^{n} \ln \sup_{u\in\mathbb{R}} |\,\varphi(u+iv;\,V_k)\,| < \infty, \quad v\in\mathbb{R},\ (j=1,2,\ldots,n),$$

and (4.1) we get

$$\ln|\,\varphi(u+iv;\,V_j)\,| \ge -c_j(v) + \operatorname{Re} P(u+iv), \quad (j=1,2,\ldots,n).$$

Hence, for any fixed $v\in\mathbb{R}$, there exists $K_{2,j}(v)>0$ such that

$$-\ln|\,\varphi(u+iv;\,V_j)\,| \le K_{2,j}(v)(|u|^m+1), \quad u\in\mathbb{R}\ (j=1,2,\ldots,n). \tag{4.4}$$

To get the assertion of the Theorem it now suffices to prove that because of (4.1) with arbitrary polynomial $P(t)$ (not only of degree m), at least one function $\varphi(t;\,\omega_j)$ satisfies condition a) of Lemma 4.1. Indeed, assume that it is true for $\varphi(t;\,\omega_1)$. From (2.10) we estimate the functions $\varphi(t;\,\sigma_j)$ $(j=1,2,\ldots,n)$

$$\varphi(t;\,\sigma_j) = O\left(\exp\left(\exp\left(c|t|\right)\right)\right), \quad |t|\to\infty, \quad t\in\mathbb{C},\ \forall c>0. \tag{4.5}$$

Hence, the inequalities

$$M(R,\varphi(t;\,V_j)) \le M(R,\varphi(t;\,\omega_j)) + M(R,\varphi(t;\,\sigma_j)), \quad (j=1,2,\ldots,n)$$

yield condition a) for the function $\varphi(t;\,V_1)$. But then the function $\varphi(t;\,V_1)$ satisfies all hypothesis of Lemma 4.1 and, therefore, has the form $\varphi(t;\,V_1) = \exp(P_1(t))$, $t\in\mathbb{R}$, where $P_1(t)$ is a polynomial of degree at most m, real-valued on the imaginary axis. Dividing the both sides of (4.1) by $\varphi(t;\,V_1)$ we arrive to the equation

$$\varphi(t;\,V_2)\,\varphi(t;\,V_3)\ldots\varphi(t;\,V_n) = e^{\widetilde{P}(t)}, \quad t\in\mathbb{C},$$

where $\widetilde{P}(t)$ is a polynomial of degree at most m. Repeating the above arguments $n-1$ times we come to the conclusion of Theorem 4.1.

Thus, we have only to check that there exists at least one function $\varphi(t;\,\omega_j)$ satisfying condition a) of Lemma 4.1. Suppose it is not true. Then, for $j=1,2,\ldots,n$, there exists $\delta>0$ such that

$$M(R,\varphi(t;\,\omega_j)) = \max\{\varphi(-iR;\,\omega_j),\varphi(iR;\,\omega_j)\} > \exp(e^{\delta R}), \quad R>R_\delta. \tag{4.6}$$

Show that these inequalities yield a contradiction. There are two choices. The first one is that there exists sequence $\{R_k\}$, $R_k\uparrow\infty$, such that, for all $j=1,2,\ldots,n$, we have either

$$M(R_k,\varphi(t;\,\omega_j)) = \varphi(iR_k;\,\omega_j) \ \text{ or } \ M(R_k,\varphi(t;\,\omega_j)) = \varphi(-iR_k;\,\omega_j).$$

The second choise is that for each sufficiently large R there exist $j_1 = j_1(R)$ and $j_2 = j_2(R)$ $(j_1 \neq j_2)$, such that $M(R, \varphi(t; \omega_{j_1})) = \varphi(iR; \omega_{j_1})$, $M(R, \varphi(t; \omega_{j_2})) = \varphi(-iR; \omega_{j_2})$. The first choice is impossible because of relations (4.1), (4.5), (4.6). Now we are going to show that the second one is also impossible. Indeed, let us assume that it takes place. Then there exist a sequence $\{R_k\}$, $R_k \uparrow \infty$, and a set, of indeces $j_1, j_2, \dots, j_l, j_{l+1}, \dots, j_n$ $(l < n)$ independent of R_k such that

$$M(R_k, \varphi(t; \omega_{j_s})) = \varphi(iR_k; \omega_{j_s}), \quad s = 1, 2, \dots, l$$

and

$$M(R_k, \varphi(t; \omega_{j_s})) = \varphi(-iR_k; \omega_{j_s}), \quad s = l+1, \dots, n.$$

Without loss of generality, we can renumber the functions $\varphi(t; \omega_j)$ such that $j_1 = 1$, $j_2 = 2, \dots, j_l = l, j_{l+1} = l+1, \dots, j_n = n$. By conditions (4.5), (4.6), we get the estimates

$$\varphi(iR_k; V_j) > \frac{1}{2} \exp(\exp(\delta R_k)), \quad R_k > R_{1\delta}, \quad j = 1, 2, \dots, l.$$

Together with (4.1) this yields, for $R_k > R_{2\delta}$,

$$\varphi(iR_k; V_{l+1}) \dots \varphi(iR_k; V_n) \le 2^l \exp\left(|P(iR_k)| - l e^{\delta R_k}\right) \le \exp\left(-\frac{1}{2} e^{\delta R_k}\right). \quad (4.7)$$

Therefore at least one of the functions from the left-hand side of (4.7), say for definiteness $\varphi(t; V_{l+1})$, satisfies

$$\varphi(iR_k; V_{l+1}) \le \exp\left(-\frac{1}{2(n-l)} e^{\delta R_k}\right), \quad R_k > R_{2\delta}. \quad (4.8)$$

Then, by (4.5), we conclude that

$$\varphi(iR_k; \omega_{l+1}) = O\left(\exp(\exp(cR_k))\right), \quad R_k \to \infty, \ \forall c > 0,$$

and taking into account estimate (4.5) we, for any constant $c > 0$, get the inequality

$$|\varphi(t; V_{l+1})| \le H(R_k) = A(c) \exp(\exp(cR_k)) \quad (4.9)$$

in each strip $-1 \le \operatorname{Im} t \le R_k$. Here and below in the proof of this theorem we denote by $A(c)$ various positive constants. Since $\varphi(t; V_{l+1})$ does not vanish in $\mathbf{C}$ Lemma F with $g(t) = \varphi(t + i(R_k - 1)/2; V_{l+1})$ yields the estimate

$$|\varphi(t; V_{l+1})| \ge \exp\left(-10R_k\left(\left|\ln \varphi\left(\frac{1}{2} i(R_k - 1); V_{l+1}\right)\right| + |\ln H(R_k)|\right) e^{\pi |t| / R_k}\right)$$

for $0 \le \operatorname{Im} t \le R_k - 1$. Let us consider the entire function $\psi_{l+1}(t) = \ln \varphi(t; V_{l+1})$ (we fix the logarithm branch $\ln \varphi(0; V_{l+1}) = 0$). Denote $U_{l+1}(t) = \operatorname{Re} \psi_{l+1}(t)$. Using (4.3), (4.4) we obtain

$$|U_{l+1}(t)| = O\left(|t|^m\right), \quad |t| \to \infty, \quad (4.10)$$

for $|\mathrm{Im}\, t| \le 1$. Because of estimates (4.9), (4.10) the function $\psi_{l+1}(t)$ satisfies the hypothesis of Lemma 4.2. By this Lemma, the relation

$$\left| \psi_{l+1}\left(\frac{1}{2} i (R_k - 1)\right) \right| = O\left(e^{cR_k}\right), \quad R_k \to \infty, \ \forall c > 0$$

holds. Therefore, recalling the representation of $H(R_k)$, we immediately obtain the estimate

$$|\varphi(t; V_{l-1})| \ge \exp\left(-A(c) R_k \exp\left(cR_k + \frac{\pi |t|}{R_k}\right)\right) \tag{4.11}$$

for $0 \le \mathrm{Im}\, t \le R_k - 1$. By estimates (4.9) and (4.11) we, for any fixed $c > 0$, get the inequality

$$|U_{l+1}(t)| \le A(c) e^{cR_k} \tag{4.12}$$

in the half-disk $G_{R_k - 1} = \{ t \in \mathbf{C} : |t| < R_k - 1, \mathrm{Im}\, t > 0 \}$. Applying the Schwarts formula (see [22, p. 288]) to the function $\psi_{l+1}(\zeta) = \psi_{l+1}(t + \zeta)$, $|\zeta| < 1$, we write

$$\psi_{l+1}(t+\zeta) = \frac{1}{2\pi} \int_0^{2\pi} U_{l+1}(\mathrm{Re}\, t + \cos\theta, \mathrm{Im}\, t + \sin\theta) \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\theta + i\, \mathrm{Im}\, \psi_{l+1}(t).$$

Now, differentiating this equation with respect to ζ and setting $\zeta = 0$, we have

$$\psi'_{l+1}(t) = \frac{1}{\pi} \int_0^{2\pi} U_{l+1}(\mathrm{Re}\, t + \cos\theta, \mathrm{Im}\, t + \sin\theta)\, e^{-i\theta}\, d\theta.$$

Using estimates (4.12) and (4.10), we obtain

$$|\psi'_{l+1}(t)| \le A(c) e^{cR_k}, \quad t \in G_{R_k - 2},$$

for any fixed $c > 0$, and

$$|\psi'_{l+1}(t)| = O\left(|t|^m\right), \quad |t| \to \infty, \quad |\mathrm{Im}\, t| \le \frac{1}{2}.$$

By

$$|\psi_{l+1}(t)| \le \left| t \int_0^1 \psi'_{l+1}(tu)\, du \right| \le |t| \max_{0 \le u \le 1} |\psi'_{l+1}(tu)|,$$

we finally get

$$|\psi_{l+1}(t)| \le A(c) e^{2cR_k}, \quad t \in G_{R_k - 2},$$

for any fixed $c > 0$, and

$$|\psi_{l+1}(t)| = O\left(|t|^{m+1}\right), \quad |t| \to \infty, \quad |\mathrm{Im}\, t| \le \frac{1}{2}.$$

These estimates show us that Lemma L can be applied to the function $\psi_{l+1}(t)$. By this Lemma,

$$\lim_{R\to\infty} \frac{1}{R} \ln M^+(R, \psi_{l+1}) = 0$$

which easily yields

$$\left| \ln \varphi(iR, V_{l+1}) \right| = O(e^{cR}), \quad R \to \infty, \ \forall c > 0.$$

But this is a contradiction to estimate (4.8). Theorem 4.1 is proved.

Proof of Theorem 2. As when proving Theorem 1, without loss of generality we may assume that $a_{1j}a_{2j} \neq 0$ and $a_{11} = a_{12} = \ldots = a_{1N} = 1$, for all $\mathbf{a}_j = (a_{1j}, a_{2j})$ in (2.9). Some of the numbers $a_{21}, \ldots, a_{2N}$ may coincide. Let there be n ($n \le N$) different ones among them. Let, for the sake of definiteness,

$$a_{21} = \ldots = a_{2k_1}, \quad a_{2,k_1+1} = \ldots = a_{2k_2}, \ \ldots, \quad a_{2,k_{n-1}+1} = \ldots = a_{2k_n}, \tag{4.13}$$

where $k_n = N$, and $a_{2k_1}, a_{2k_2}, \ldots, a_{2k_n}$ are pairwise different numbers. We denote

$$H_1(u) = \prod_{j=1}^{k_1} \varphi(u; V_j), \ \ldots, \quad H_n(u) = \prod_{j=k_{n-1}+1}^{k_n} \varphi(u; V_j). \tag{4.14}$$

Applying Lemma K to the functions $H_1(u), \ldots, H_n(u)$, we see that each function $H_l(u)$ ($l = 1, \ldots, n$) may be represented in the form

$$H_l(u) = \exp\{ b_{l1} u + b_{l2} u^2 + \ldots + b_{ln} u^n \} \tag{4.15}$$

in a certain neighborhood of the origin. By the Hermite condition on the functions $\varphi(u; V_j)$, the coefficients b_{lp} are real for even p and purely imaginary for odd p. Since c.f. $\varphi(u; V_j)$ ($j = 1, 2, \ldots, N$) are entire, equality (4.15) fulfills for all $u \in \mathbb{C}$. Thus all functions in (2.9) do not vanish in C. Denoting $\ln \varphi(u; V_j) = \psi_j(u)$ ($\psi_j(0) = 0$), we have

$$\sum_{j=1}^{N} \psi_j(t_1 + a_{2j} t_2) = \sum_{j=1}^{N} \psi_j(t_1) + \sum_{j=1}^{N} \psi_j(a_{2j} t_2)$$

for all $t_1, t_2 \in \mathbb{C}$. Now, if we differentiate this equation with respect to t_2 and put $t_2 = 0$, we have

$$\sum_{j=1}^{N} \alpha_j \psi_j''(t_1) = c_1$$

for $t_1 \in \mathbb{C}$, where $\alpha_j = a_{2j}^2 > 0$, c_1 is a constant. Integrating two times with respect to t_1 and exponentiating the obtained equality, we get

$$\prod_{j=1}^{N} (\varphi(t_1; V_j))^{\alpha_j} = \exp(P(t_1)), \quad t_1 \in \mathbb{C},$$

where $P(t)$ is a polynomial of degree not exceeding two. This equation may be written in the form

$$\prod_{j=1}^{n} \left(H_l(t_1)\right)^{\alpha_l} = \exp\left(P(t_1)\right), \quad t_1 \in \mathbf{C}.$$

Now let us show that $b_{lp} = 0$ for $l = 1, 2, \ldots, n$, $p = 3, \ldots, n$. By the previous equality, we can write

$$\alpha_1 b_{1p} + \alpha_2 b_{2p} + \ldots + \alpha_n b_{np} = 0, \quad p = 3, \ldots, n. \tag{4.16}$$

Let $p = n$ and let it be an even number. The absolute values of the functions $H_l(u)$ are bounded on the real axis. Therefore all $b_{ln} \le 0$ and, by (4.16), we have $b_{lp} = 0$ $(l = 1, 2, \ldots, n)$, when $p = n$. Let n be an odd number. In this case it is easy to see that, for each fixed $y \in \mathbf{R}$,

$$| H_l(x + iy) | = \exp\left((i\, b_{ln} C_n^1 y + b_{l,n-1}) x^{n-1} + O(|x|^{n-2})\right), \quad |x| \to \infty, \quad x \in \mathbf{R}.$$

Since, for any fixed $y \in \mathbf{R}$, $| H_l(x + iy) |$ is bounded with respect to $x \in \mathbf{R}$, then

$$i\, b_{ln} C_n^1 y + b_{l,n-1} \le 0$$

for all $y \in \mathbf{R}$ and therefore $b_{ln} = 0$ $(l = 1, 2, \ldots, n)$. The similar arguments for the cases $p = n - 1, n - 2, \ldots, 3$ yield the desired result.

Notice that, in representation (4.15) of the functions $H_l(u)$, the coefficients b_{2l} are non-positive and the coefficients b_{1l} are purely imaginary. It means that

$$V_1 * \ldots * V_{k_1} = \Phi_{a_1, \sigma_1^2}, \quad \ldots, \quad V_{k_{n-1}+1} * \ldots * V_{k_n} = \Phi_{a_n, \sigma_n^2},$$

where Φ_{a_l, σ_l^2} are Gaussian d.f. Then, by the Corollary of Theorem 4.1, all f.b.v. V_j are Gaussian and Theorem 2 is proved.

P r o o f o f P r o p o s i t i o n 2. The assertion of Proposition 2 comes immediately from the Linnik–Skitovich example [12]. Actually, let us consider two functions W_1, W_2 from the class B with c.f.

$$\varphi(t; W_1) = \exp\left(-\frac{t^2}{2} + \lambda\,(e^{i\mu t} - 1)\right), \tag{4.17}$$

$$\varphi(t; W_2) = \exp\left(-\frac{t^2}{2} - \lambda\,(e^{i\mu t} - 1)\right), \tag{4.18}$$

λ being a positive parameter. The functions W_1, W_2 satisfy

$$\operatorname{Var} W_j(x)\Big|_{|x| > y} = O\left(\exp\left(-y \frac{\ln y}{2}\right)\right), \quad j = 1, 2, \quad y > 0,$$

and hence, belong to the class B_3. Now set $W_3 = W_1$, $W_4 = W_2$. It is easy to see, that the functions $\varphi(t; W_j)$, $j = 1, 2, 3, 4$, satisfy equality (2.11).

P r o o f o f T h e o r e m 4. This proof is based on two auxiliary Lemmas.

Lemma 4.3. *Let $V_1, \dots, V_n$ be symmetric f.b.v. and belong to the class B and let each of them be represented in form* (2.5) ($V_j = \omega_j - \sigma_j$), *where ω_j, σ_j are non-decreasing f.b.v., and let*

$$\int_{-\infty}^{\infty} x^{2l}\, d\sigma_j(x) < \infty, \quad (j = 1, 2, \dots, n)$$

for some positive integer l. If their c.f. $\varphi(t; V_1), \dots, \varphi(t; V_n)$ satisfy the relation

$$\varphi(t_k; V_1) \dots \varphi(t_k; V_n) = \varphi(t_k) \tag{4.19}$$

on a sequence of real numbers $\{t_k\}$, $t_k \downarrow 0$, where $\varphi(t)$ is an even 2l-times differentiable real-valued function, then

$$\int_{-\infty}^{\infty} x^{2l}\, d\omega_j(x) < \infty, \quad (j = 1, 2, \dots, n).$$

In the case $n = 2$ this Lemma was already proved in [23]. If $n > 2$, this conclusion may be obtained in a similar way, therefore here we omit the proof.

Lemma 4.4. *Let f.b.v. $V_1, \dots, V_n$ belong to the class $B_4^{(s)}$ and their c.f. $\varphi(t; V_1), \dots, \varphi(t; V_n)$ satisfy relation* (4.19), *where the function $\varphi(t)$ is even, real-valued on the real axis, and analytic in the strip $|\operatorname{Im} t| < H$. Then the functions $\varphi(t; V_1), \dots, \varphi(t; V_n)$ are analytic in the same strip and equation* (4.19) *is satisfied at each point t of this strip.*

Proof of Lemma 4.4. By Lemma 4.3 c.f. $\varphi(t; V_1), \dots, \varphi(t; V_n)$ are infinitely differentiable and, by the Rolle Theorem, relation (4.19) implies the equation

$$\left.\left(\prod_{j=1}^{n} \varphi(t; V_j)\right)^{(q)}\right|_{t=0} = \varphi^{(q)}(0) \tag{4.20}$$

for all $q = 0, 1, 2, \dots$. From Cauchy inequalities we obtain that the estimates

$$|\varphi^{(q)}(0)| \le M(H - \varepsilon, \varphi)(H - \varepsilon)^{-q} q!, \quad q = 0, 1, \dots$$

hold for any $\varepsilon \in (0, H)$. Let us write equation (4.20) in the form

$$\sum_{j=1}^{n} \varphi^{(q)}(0; V_j) \prod_{p \neq j} \varphi(0; V_p) + \\ + \sum_{q_1 + \dots + q_n = q} \frac{q!}{q_1! \dots q_n!} \varphi^{(q_1)}(0; V_1) \dots \varphi^{(q_n)}(0; V_n) = \varphi^{(q)}(0). \tag{4.21}$$

All functions $\varphi(t; V_j)$ are even, thus their odd derivatives vanish at the origin. Therefore it suffices to consider the case of even q. Because of (2.12) it is easy to see that the summands in the left-hand side of (4.21) have the same argument. This implies the estimates

$$\max_j |\varphi^{(q)}(0; V_j)| \le |\varphi^{(q)}(0)| \le M(H-\varepsilon, \varphi)(H-\varepsilon)^{-q} q!, \quad q = 0, 2, 4, \ldots$$

that yields the assertion of Lemma 4.4.

Let us form the functions $H_1, \ldots, H_n$ as we did it in the proof of Theorem 2. These functions are of form (4.15) and $b_{l, 2m+1} = 0$ $(l = 1, 2, \ldots, n;\ m = 0, 1, \ldots, \left[\frac{n-1}{2}\right])$. Then, by Lemma 4.4, all functions $\varphi(t; V_j)$ $(j = 1, \ldots, N)$ are entire. Hence, f.b.v. V_j $(j = 1, \ldots, N)$ belong to the class B_2. Theorem 4 is proved.

We need two Lemmas.

Lemma 4.5. *Let real coefficients $d_1, \ldots, d_{2m}$, $m \ge 1$, be given, and $(-1)^m d_{2m} < 0$. Then the function $f(t) = \exp\left\{d_1 it + d_2(it)^2 + \ldots + d_{2m}(it)^{2m}\right\}$ is c.f. of certain f.b.v. V satisfying*

$$\mathrm{Var}\, V(x)\Big|_{|x| > y} \le C(V) \exp\left\{-\frac{1}{20m} |d_{2m}|^{-1/(2m-1)} y^{1 + 1/(2m-1)}\right\}$$

with some positive constant $C(V)$.

We confine ourselves to short outline of the proof of the Lemma. It can be done by estimating the Fourier transform of the function $f(t)$ by mean of replacing the line of integration to the corresponding horizontal line in the complex t-plane.

Let V be f.b.v. and let M be a real number. Consider the function

$$f(t; V, M) = \int_{-\infty}^{M} e^{itx}\, dV(x), \quad t \in \mathbf{R}.$$

The integral in the right-hand side defines the analytical continuation of $f(t; V, M)$ to the lower half-plane.

Lemma M. *Let $f(t; V, M)$ be a non-vanishing function in the lower half-plane $\mathrm{Im}\, t < 0$. Then, for sufficiently large y,*

$$|f(-iy; V, M)| \ge e^{-cy},$$

where $c = c(V, M)$ is a positive constant.

This statement is a direct consequence of the Carathéodory Theorem (see [21, p. 342]).

P r o o f o f P r o p o s i t i o n 3. First, we consider the case $V_1, V_2, \ldots, V_N \in B_0$. C.f. $\varphi(t; V_j)$ $(j = 1, 2, \ldots, N)$ are entire functions satisfying

$$\varphi(t;V_j)=1+it\int\limits_{+0}^{\infty}(1-V_j(x))e^{itx}\,dx-it\int\limits_{-\infty}^{0}V_j(x)e^{itx}\,dx \tag{4.22}$$

for all $t\in\mathbf{C}$. One can obtain this relation if integrating by parts. This formula gives us the estimate which is similar to the ridge condition for c.f. (see Theorem E):

$$|\varphi(t;V_j)|\le\frac{|t|}{|\mathrm{Im}\,t|}\left(|\varphi(i\,\mathrm{Im}\,t;V_j)|+e^{c_j|\mathrm{Im}\,t|}+2c_j\right),\quad \mathrm{Im}\,t\neq 0, \tag{4.23}$$

here the constant $c_j>0$ depends on f.b.v. V_j.

Construct the functions $H_1,\dots,H_n$ as when proving Theorem 2. These functions are entire and admit representation (4. 15) for all complex t. Since $V_j(x)\ge 0$, $1-V_j(x)\ge 0$ for sufficiently large x, then, using Lemma M, we have the estimates

$$|\varphi(iy;V_j)|\ge e^{-c|y|},\ (j=1,2,\dots,N) \tag{4.24}$$

for sufficiently large $|y|$. We can now note that formula (4.15) yields the inequalities

$$\left|\prod_{j=k_{l-1}+1}^{k_l}\varphi(iy;V_j)\right|=|H_l(iy)|\le e^{C|y|^n}\quad |y|\ge y_0\quad (l=1,2,\dots,n),$$

where $C>0$ is a constant. Together with (4.24), this yields the estimates

$$|\varphi(iy;V_j)|\le e^{C|y|^n+Nc|y|},\quad |y|\ge y_1,\quad (j=1,2,\dots,N). \tag{4.25}$$

By (4.23) the order of entire functions $\varphi(t;V_j)$ ($j=1,2,\dots,N$) does not exceed n and by (4. 15) they are non-vanishing in the whole complex t-plane. Now the Hadamard Theorem (see [21, p. 13]) gives

$$\varphi(t;V_j)=\exp\left\{d_{j1}t+d_{j2}t^2+\dots+d_{jm_j}t^{m_j}\right\}\ (m_j\le n,\ \ d_{jm_j}\neq 0), \tag{4.26}$$

where d_{jp} are purely imaginary for odd p and real for even p. Besides, as we could see in the proof of Theorem 2, if $m_j>1$, then m_j is an even number and $d_{jm_j}<0$. By Lemma 4.5 f.b.v. V_j with such c.f. belong to the class $B_1\subset B_2$.

Now let us consider the case when $V_1,\dots,V_N$ belong to the class $B_7\cap B_9$. For each f.b.v. V_j we write representation (2.5), i.e. $V_j=\omega_j-\sigma_j$ where ω_j, σ_j are non-decreasing f.b.v. and

$$\mathrm{Var}\,V_j(x)\Big|_{|x|>y}=O(e^{-cy}),\quad y\to+\infty,\ \forall c>0.$$

Besides, by condition (2.14) we have

$$I(y,\omega_j)-I(y,\sigma_j)=I(y,\omega_j)e^{-Q_j(y)}, \tag{4.27}$$

where $Q_j(y)=O(e^{c|y|})$, $y\to\infty$, $\forall c>0$. Let us come back again to the functions $H_1,\dots,H_n$. In this case they are also entire and (4.15) fulfills for all $t\in\mathbf{C}$. Therefore the equations

$$\prod_{j=k_{l-1}+1}^{k_l} \left(I(y;\omega_j)e^{-Q_j(y)}\right) = \prod_{j=k_{l-1}+1}^{k_l} \varphi(iy; V_j) =$$
$$= e^{b_{l1} iy + b_{l2}(iy)^2 + \ldots + b_{ln}(iy)^n} \quad (l = 1, 2, \ldots, n)$$

are satisfied for $y \in \mathbf{R}$. Hence

$$\varphi(iy;\omega_j) = O\left(e^{e^{c|y|}}\right), \quad y \to \infty, \quad \forall c > 0 \quad (j = 1, 2, \ldots, N). \tag{4.28}$$

From (4.27) we may obtain the similar estimates for $\varphi(iy;\sigma_j)$ $(j = 1, 2, \ldots, N)$. It is easy to see that the inequality

$$e^{My}(\omega_j(+\infty) - \omega_j(M) + \omega_j(-M)) \le \int_{|x| \ge M} e^{y|x|}\, d\omega_j(x) \le \varphi(iy;\omega_j) + \varphi(-iy;\omega_j)$$

is satisfied for any $y, M > 0$. Now let $c > 0$, $y = \frac{1}{c}\ln M$. By (4.28) we have

$$\omega_j(+\infty) - \omega_j(M) + \omega_j(-M) = O\left(e^{-\frac{1}{c}M\ln M}\right),$$
$$M \to +\infty, \ \forall c > 0 \quad (j = 1, 2, \ldots, N).$$

The same estimate for the quantity $\sigma_j(+\infty) - \sigma_j(M) + \sigma_j(-M)$ $(j = 1, 2, \ldots, N)$ may be obtained similary, so f.b.v. V_j $(j = 1, 2, \ldots, N)$ belong to the class B_2.

P r o o f o f P r o p o s i t i o n 4. Let us consider the function $f(t) = e^{-10^{-6}t^4}$. By Lemma 4.5 this is c.f. of some f.b.v. $V \in B_{M_{1,\gamma}, M_{1,\gamma}}$ with $\gamma = 1/3$, i.e. f.b.v. V from the class B and also

$$\operatorname{Var} V(x)\Big|_{|x|>y} = O\left(e^{-y^{4/3}}\right), \quad y \to +\infty.$$

It is obvious that $V \in B_7 \bigcap B_8$. Let us show that f.b.v. V also belongs to the class $B_7 \bigcap B_9$. It is enough to establish condition (2.14) with

$$\omega(x) = \frac{1}{2}\left\{\operatorname{Var} V(x)\Big|_{-\infty}^{x} + V(x)\right\}, \quad \sigma(x) = \frac{1}{2}\left\{\operatorname{Var} V(x)\Big|_{-\infty}^{x} + V(x)\right\}.$$

We note that $\omega(x) \not\equiv 0$ and therefore $I(y,\omega) \ge e^{-c|y|}$ $(\forall\, y \in \mathbf{R})$, with some positive constant c. Now we mention that

$$\frac{I(y,\omega) - I(y,\sigma)}{I(y,\omega)} = \frac{e^{-10^{-6}y^4}}{I(y,\omega)} = O\left(e^{-\frac{1}{2}10^{-6}y^4}\right), \quad y \to \infty,$$

which implies the fulfillment of condition (2.14) for f.b.v. V.

Now let us take $V_1 = V_2 = V_3 = V$. It may be immediately checked that their c.f. $\varphi(t; V_j)$ $(j = 1, 2, 3)$ satisfy equation (2.15), where non-collinear vectors (a_{1j}, a_{2j}, a_{3j}) $(j = 1, 2, 3)$ are the following

$$\left(1, 2+\sqrt{6}, \frac{2+\sqrt{6}}{5+2\sqrt{6}}\right), \quad \left(1, 5+2\sqrt{6}, -\frac{1}{5+2\sqrt{6}}\right), \quad (1, -1, 1).$$

Thus the Proposition is proved.

Below we need some kind of sharpening of the Marcinkievicz Theorem.

Lemma 4.6. *Let V be a f.b.v. from the class B_7 and let its c.f. $\varphi(t; V)$ satisfy the relation*

$$|\varphi(t; V)| = O\left(e^{c|t|}(1 + |\varphi(i \operatorname{Im} t; V)|)\right), \quad |t| \to \infty, \tag{4.29}$$

outside the strip $|\operatorname{Im} t| < 1$, c being a positive constant. Let also its c.f. $\varphi(t; V)$ may be represented in the form $\varphi(t; V) = e^{P(t)}$, where $P(t)$ is a polynomial. Then V is a Gaussian d.f.

P r o o f o f L e m m a 4.6 follows the pattern of that in [21, p. 41]. Let $P(t) = d_1 t + \ldots + d_m t^m$ and $d_m \neq 0$. As we could see above, the coefficients d_l are purely imaginary for odd l's and real for even l's. Moreover, if $m \geq 2$, then m is an even number and $d_m < 0$. Thus if $m \leq 2$, then V is a Gaussian d.f. Let us show that the assumption $m > 2$ yields a contradiction. We consider the ray $\arg t = \theta_m = -\frac{\pi}{m}$ in the t-plane. It is obvious that

$$\ln |\varphi(re^{i\theta_m}; V)| = |d_m| r^m (1 + o(1)), \quad r \to \infty. \tag{4.30}$$

On the other hand, by relation (4.29) c.f. $\varphi(t; V)$ can be estimated as

$$\ln |\varphi(re^{i\theta_m}; V)| \leq \ln |(1 + \varphi(ir \sin \theta_m; V))| + O(r), \quad r \to \infty. \tag{4.31}$$

Comparing (4.30) and (4.31) we see that

$$|d_m| r^m (1 + o(1)) \leq |d_m| r^m |\sin \theta_m|^m + o(r^m), \quad r \to \infty.$$

If $m > 2$, then $|\sin \theta_m| < 1$ and we get a contradiction. This completes the proof of Lemma 4.6.

P r o o f o f T h e o r e m 5. As when proving Theorem 1 we assume that $a_{1j} \ldots a_{mj} \neq 0$ and $a_{1j} = 1$ for all vectors $\mathbf{a}_j = (a_{1j}, \ldots, a_{mj})$, $j = 1, \ldots, N$. This involves no loss of generality. Also we form the functions $h_j(u)$ ($j = 1, 2, \ldots, n$) by formula (3.3) with the only difference that we use the functions $\varphi(u; V_j)$ instead of the functions $\varphi_{Z_j}(u)$. Since $\varphi(u; V_j)$ are entire functions, then, according to Lemma K, the functions $h_1(u), \ldots, h_n(u)$ satisfy relation (3.4) for all complex u. Now by repeating the arguments from the proof of Proposition 3 for the class B_6 ($B_6 \subset B_7$) we see that the functions $\varphi(u; V_j)$ ($j = 1, 2, \ldots, N$) admit representation (4.26). From estimate (4.23) for c.f. $\varphi(u; V_j)$ ($j = 1, 2, \ldots, N$) it follows that these functions satisfy inequality (4.29). Thus f.b.v. V_j satisfy all the hypothesis of Lemma 4.6 and therefore are Gaussian.

Proof of Theorem 6. As in the proof of Theorem 5 we see that the functions $h_1(u),\dots, h_n(u)$ satisfy relation (3.4) in a certain neighborhood of the origin. Since all the functions $\varphi(u; V_j)$ are even and real-valued, the functions $h_1(u),\dots, h_n(u)$ also satisfy these relations. Integrating by parts we note that

$$\int_{-\infty}^{\infty} x^{2k}\, dV_j(x) = 2k\left(\int_0^{\infty} x^{2k-1}(1 - V_j(x))dx - \int_{-\infty}^{0} x^{2k-1}\, V_j(x)dx\right) \geq 0,$$

$$k = 1, 2,\dots, \quad j = 1, 2,\dots, N.$$

Thus f.b.v. $V_j(x)$ satisfy condition (2.12). Therefore f.b.v. V_j ($j = 1, 2,\dots, N$) satisfy the hypothesis of Lemmas 4.3 and 4.4 for any $H > 0$. By these Lemmas, c.f. $\varphi(u; V_j)$ ($j = 1, 2,\dots, N$) are entire functions and hence our f.b.v. V_j ($j = 1, 2,\dots, N$) belong to the class B_6. By Theorem 5, they are Gaussian.

Proof of Theorem 7. We start as when proving Theorem 5. Using the same arguments we can see that relation (3.4) is satisfied for each function $h_l(u)$, $l = 1, 2,\dots, n$ and all complex u. We note that the class $B_{11} \subset B_2$. By Theorem 4.1, c.f. $\varphi(u; V_j)$ ($j = 1, 2,\dots, N$) admit the representation

$$\varphi(u; V_j) = \exp\left\{d_{j1}\, u + d_{j2}\, u^2 + \dots + d_{js_j}\, u^{s_j}\right\}, \quad s_j \leq m + N - 2, \quad d_{js_j} \neq 0. \tag{4.32}$$

Suppose $s_j > 1$. We have already mentioned that, in this case, the numbers d_{jp} are purely imaginary for odd p and real for even p. Besides, s_j is an even number and $d_{js_j} < 0$.

Now let us show that all $s_j \leq 2$. Indeed if s_j is divisible by 4 (we denote it as $4\,|\,s_j$), then

$$\varphi(iy; \omega_j) \leq \varphi(iy; \sigma_j) + O(1) \leq e^{|y|^{4-\varepsilon}}, \quad |y| \to +\infty,$$

for some $\varepsilon \in (0, 4)$. Hence, in this case, c.f. $\varphi(u; V_j)$ is an entire function of order less than 4. This contradicts to the assumption $4\,|\,s_j$.

If s_j is not divisible by 4 and $s_j \geq 6$, then

$$\varphi(iy; \omega_j) \geq \exp\left\{\frac{1}{2}\,|\,d_{js_j}\,|\,y^{s_j}\right\} \geq \exp\left\{\frac{1}{2}\,|\,d_{js_j}\,|\,y^6\right\}, \quad |y| \geq y_0,$$

and hence, for $|y| \geq y_0$,

$$\varphi(iy; \sigma_j) \leq \frac{1}{2}\varphi(iy; \omega_j).$$

This gives us the inequality

$$1 \leq \frac{\varphi(iy; \omega_j)}{\varphi(iy; V_j)} \leq 2, \quad |y| \geq y_0.$$

The two last estimates yield

$$|\varphi(u+iy;V_j)| \le \varphi(iy;\omega_j)+\varphi(iy;\sigma_j) \le \frac{3}{2}\varphi(iy;\omega_j) \le 3\varphi(iy;V_j)$$

for $|y| \ge y_0$ and $u \in \mathbf{R}$. This inequality shows that, in this case, c.f. $\varphi(u;V_j)$ satisfies relation (4.29). Since $B_{11} \subset B_7$ and (4.32) holds, V_j satisfies hypothesis of Lemma 4.6. By this Lemma, V_j is a Gaussian d.f. It contradicts to the assumption $s_j \ge 6$.

So we have showed that all of f.b.v. V_j $(j = 1,\dots,N)$ are Gaussian d.f. and thus Theorem 7 is proved.

P r o o f o f P r o p o s i t i o n 5. It suffices to repeat the final part of the proof of Proposition 4, if mention that f.b.v. $V_1 = V_2 = V_3$ with c.f. $\varphi(t;V_j) = e^{-10^{-6}t^4}$ belong to the class B_{12}.

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Перенесение теорем Дармуа–Скитовича и Кагана на классы обобщенных случайных величин

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Теорема Дармуа–Скитовича о характеризации нормальности случайных величин (с.в.) свойством независимости линейных форм и ее обобщение, данное А.М. Каганом, переносятся на классы комплекснозначных с.в., а также на широкие классы обобщенных с.в., которым вместо функций распределения отвечают функции ограниченной вариации.

Перенесення теорем Дармуа–Скитовича та Кагана на класи узагальнених випадкових величин

Г.П. Чистяков, С.Ю. Пергаменцев

Теорема Дармуа–Скитовича про характеризацію нормальності випадкових величин (в.в.) умовою незалежності лінійних форм та її узагальнення, здобуте А.М. Каганом, перенесено на класи комплекснозначних в.в. і на широкі класи узагальнених в.в., замість функцій розподілу яких розглядаються функції обмеженої варіації.