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Meyers type regularity for bounded and almost periodic solutions to nonlinear second order parabolic equations with mixed boundary conditions

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We have obtained the conditions which garantees that an $W_0^{1,2}$-valued almost periodic solution to a nonlinear parabolic equation is really almost periodic with values in the space $W_0^{1,p}$, where $p-2>0$ is sufficiently small.

0. Introduction

It is well-known that the gradients of generalized solutions to Dirichlet problem for nonlinear elliptic equations have excited summability (Meyers estimates; see, for example, [1-3]). In the recent papers of the first author [4-6] a new approach to such estimates was proposed, which permit to include in consideration domains with Lipschitz boundary and mixed boundary value problems (in the of case of second order equations). In particular, [5] contains the results on $W^{1,p}$-regularity of solutions to mixed boundary value problems for nonlinear parabolic equations.

In the present paper a similar approach is developed to investigate $W^{1,p}$-regularity of bounded and almost periodic (a.p.) solutions to parabolic mixed boundary value problems. Here we consider boundedness and almost periodicity in the sense of Stepanov, almost periodicity in the sense of Besicovich, and the case of solutions on the whole axis which are integrable with the exponent p (the last case is closely related to Besicovich a.p. solutions).

The main idea is the following. We reduce the problem under consideration to an operator equation of the form $u=Qu=T_\lambda S_\lambda u$ in a corresponding scale of functional spaces (parametrized by the exponent $p\geq 2$). Then we look for a fixed point of Q by means of the contraction principle. So, we need to estimate the Lipschitz constant of the operator Q. Here T_λ is the inverse operator to the similar problem for the heat operator

$J_\lambda u = \lambda u' - \Delta u$. On the other hand the Lipschitz constant of S_λ may be estimated explicitly in terms of the original problem. It turns out that, with a suitable choice of λ, this Lipshitz constant is less than 1.

Therefore, we have to estimate the norm $\| T_\lambda \|_p$, which depends on the exponent p. First of all, using an interpolation techniques, we show that $\| T_\lambda \|_p$ is close to $\| T_\lambda \|_2$ as p is close to 2. We note here, that in the Stepanov case we make use a new interpolation result, while the Besicovich case may be covered by the classical Riesz-Thorin theorem. Then we estimate the norm $\| T_\lambda \|_2$. In the Besicovich case it is more or less clear (by means of the energy argument) that this norm is equal to 1. On the contrary, in the Stepanov case $\| T_\lambda \|_2$ is greater than 1. This gives rise to an additional quantitative condition (5.6), which guarantees excited regularity. However, it is still open to determine is this condition essential, or not.

Now, we sketch briefly the contents of the paper. In Section 1 we start with general definitions and notations. In particular, we define the class of regular spatial domains by pasting together corresponding local models. Section 2 deals with some general properties of the operator T_λ. We show that T_λ has the same norms in different functional spaces, e.g. in the spaces of Stepenov bounded and Stepanov almost periodic functions with the same exponent p. Then, in Section 3, we study some properties of regular domains and estimate $\| T_\lambda \|_p$ by means of $\| T_\lambda \|_2$. With this aim we prove here the new interpolation result (lemma 3.2) for linear operators acting in the spaces of bounded in the sense of Stepanov functions. We do not know is the similar result true for the spaces of Stepanov a.p. functions. Therefore, it is crucial for our approach to use the norm coincidence result stated in Section 2. In Section 4 we derive the estimate of the norm $\| T_\lambda \|_2$ in the Stepanov case. Our main results are stated and proved in Section 5. These results affirm that there is excited regularity of solutions for suitable subclasses of the class of regular domains. Then, in Section 6, we investigate these subclasses in more details. We show that in the Besicovich case the regularity result takes place, really, for any regular domain, while in the Stepanov case it is so for any domain, whose boundary is, locally, a graph of a Lipschitz function. Finally, we conclude with a few general remarks (Section 7).

As for results on existence of a.p. solutions we refer to [7,8].

1. Preliminaries

Let G be a bounded subset of $\mathbf{R}^n$. We denote by $\mathring{G}$, ∂G, and $\overline{G}$ the interior, the boundary, and the closure of G respectively. As in [5] we call G regular, if for every $y \in \partial G$ there exist open subsets U and $\tilde{U}$ of $\mathbf{R}^n$ and a bijective map $\varphi : U \to \tilde{U}$ such that

(i) φ and φ^{-1} are Lipschitzian, and the Jacobian determinant of φ is constant;

(ii) $y \in U$ and $\varphi(U \cap G)$ is one of the following sets

$$E_1 = \{ x \in \mathbf{R}^n : |x| < 1, x_n < 0 \},$$

$$E_2 = \{ x \in \mathbf{R}^n : |x| < 1, x_n \le 0 \},$$

$$E_3 = \{ x \in E_2 : x_1 > 0 \text{ or } x_n < 0 \}.$$

Here $|\cdot|$ stands for the usual Euclidian norm in $\mathbf{R}^n$. In what follows we always assume G to be regular. Moreover, for the sake of simplicity it is assumed that $(n-1)$-dimensional surface measure of $\overline{G} \setminus G$ is positive.

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We denote by $W_0^{1,p}(G)$ the closure of the set

$$\{ u_{|\overset{\circ}{G}} : u \in C^\infty(\mathbf{R}^n),\ G \cap \operatorname{supp} u \text{ is compact in } G \}.$$

(All functions we consider are assumed to be complex valued.) We can define the norm in $W_0^{1,p}(G)$ by the formula

$$\| u \|_{1,p}^p = \int_G |\nabla u|^p \, dx.$$

By $W^{-1,p}(G)$ we denote the dual space to $W_0^{1,p'}(G)$. We shall omit G in such notations, if no confusion is possible.

Now, we introduce the operator $J \in L(W_0^{1,2}, W^{-1,2})$ – the space of bounded linear operators acting from $W_0^{1,2}$ into $W^{-1,2}$ – by the formula

$$\langle Ju, v \rangle = \int_G \nabla u \cdot \nabla v \, dx \quad \text{for } u, v \in W_0^{1,2},$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing. It is obvious that J maps $W_0^{1,p}$ continuously into $W^{-1,p}$. In what follows we shall consider the parabolic operator J_λ, $\lambda > 0$, defined by the formula

$$J_\lambda u = \lambda u' + Ju, \tag{1.1}$$

where "'" stands for the time derivative. Corresponding function spaces will be justified.

Recall definitions of some function spaces (see [8] for more details). Let E be a Banach space over $\mathbf{C}$. The space $BS^p(\mathbf{R}; E)$ consists of all measurable functions f: $\mathbf{R} \to E$ such that the norm

$$\|f\|_{BS^p} = \sup_{t \in \mathbf{R}} \Big(\int_t^{t+1} \|f(\tau)\|_E^p \, d\tau \Big)^{\frac{1}{p}} \tag{1.2}$$

is finite. The space $S^p(\mathbf{R}; E)$ of Stepanov almost periodic (a.p.) functions of exponent p may be defined as the closure of the set of all E-valued trigonometrical polynomials, i.e. functions of the form

$$f(t) = \sum_{k=1}^{N} a_k \exp(i \lambda_k t), \; a_k \in E, \; \lambda_k \in \mathbf{R}, \; k = 1, \dots, N = N(f).$$

Equivalently, $f \in S^p(\mathbf{R}; E)$ iff the family of its shifts $\{ f(\cdot + \tau) \}_{\tau \in \mathbf{R}}$ is precompact in $BS^p(\mathbf{R}; E)$. Sometimes we shall use another equivalent norm in these spaces:

$$\|f\|_{BS^p, \theta} = \sup_{t \in \mathbf{R}} \Big(\int_t^{t+\theta} \|f(\tau)\|_E^p \, d\tau \Big)^{\frac{1}{p}}, \; \theta > 0. \tag{1.3}$$

The space $CAP(\mathbf{R}; E)$ of Bohr a.p. functions (or uniformly a.p. functions) may be defined as the closure of all E-valued trigonometrical polynomials in the space $C_b(\mathbf{R}; E)$ of all bounded and continuous functions f: $\mathbf{R} \to E$ endowed with the usual supremum norm. Another characterization of this space is the following. A function $f \in C_b(\mathbf{R}; E)$ is Bohr a.p. if and only if its shift family $\{ f(\cdot + \tau) \}_{\tau \in \mathbf{R}}$ is precompact with respect to the supremum norm.

Now, for $f \in CAP(\mathbf{R}; E)$ we set

$$\|f\|_{B^p}^p = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \|f(\tau)\|_E^p \, d\tau.$$

The completion of $CAP(\mathbf{R}; E)$ with respect to this norm will be denoted by $B^p(\mathbf{R}; E)$. This space consists of all Besicovich a. p. functions of exponent p. We remark that a Besicovich a. p. function is, really, a class of equivalent functions, not a single function on $\mathbf{R}$. However, there is a natural embedding $S^p(\mathbf{R}; E) \subset B^p(\mathbf{R}; E)$, which is dense and continuous.

We define the following spaces

$$X_1^p = BS^p(\mathbf{R}; W_0^{1,p}),$$

$$X_1^{-p} = BS^p(\mathbf{R}; W^{-1,p}),$$

$$W_1^p = \{ u \in X_1^p : u' \in X_1^{-p} \},$$

where u' is the time derivative of u in the sense of $W^{-1,p}$-valued distributions. We introduce also the spaces X_i^p, X_i^{-p} and W_i^p for $i = 2$, 3 and 4 replacing in the previous definitions BS^p by S^p if $i = 2$, by L^p if $i = 3$, and by B^p if $i = 4$. In the case $i = 4$ the time derivative is regarded in the sense of Besicovich a.p. distributions (see [8]). The norms in X_i^p and X_i^{-p} will be denoted by $\| \cdot \|_{p,i}$ and $\| \cdot \|_{-p,i}$ respectively. We shall omit the index 'i' here, if no misunderstanding is possible. We shall write $X_{i,G}^p$, etc., if we need to indicate the set G explicitly.

2. Special linear problem – general properties

We shall consider the operator J_λ in the spaces we have just defined. More precisely, the formal expression (1.1) defines a closed linear operator from X_i^p into X_i^{-p} with the domain W_i^p, $i = 1, 2, 3$ and 4. We still denote this operator by J_λ.

Now, for $p \geq 2$ and $\lambda > 0$ let

$$M_{p,\lambda}^{(i)} = \sup \{ \| u \|_{p,i} : u \in W_i^p, \| J_\lambda u \|_{-p,i} \leq 1 \}, \tag{2.1}$$

where $i = 1, 2, 3, 4$. It is well-known, that there exists a bounded linear operator $J_\lambda^{-1} : X_i^{-2} \to X_i^2$, $\lambda > 0$ (see, for example, [8] for less familiar cases of bounded and a. p. functions). Hence, $M_{2,\lambda}^{(i)} < \infty$. Moreover,

$$M_{2,\lambda}^{(3)} = M_{2,\lambda}^{(4)} = 1, \tag{2.2}$$

as it follows from the standard energy argument. In general, it may happen that $M_{p,\lambda}^{(i)} = +\infty$. However, if $J_\lambda W_i^p = X_i^{-p}$, then there exists a bounded inverse operator $J_\lambda^{-1} : X_i^{-p} \to X_i^p$ and $M_{p,\lambda}^{(i)} < \infty$. The converse is not true in general. We shall use the quantity $M_{p,\lambda}^{(i)}$ only in the case, when $J_\lambda W_i^p = X_i^{-p}$. In this case $M_{p,\lambda}^{(i)}$ coincides with the norm of the inverse operator.

R e m a r k 2.1. Scaling $\tau = \lambda t$ turns the operator J_λ into the operator J_1. Hence, $J_1 W_i^p = X_i^{-p}$ implies $J_\lambda W_i^p = X_i^{-p}$ for any $\lambda > 0$ $(i = 1, 2, 3, 4)$. Moreover, $M_{p,\lambda}^{(3)}(\mathbf{R})$ and $M_{p,\lambda}^{(4)}(\mathbf{R})$ do not depend on $\lambda > 0$. This follows from the scaling invariance of the corresponding norms. In the sequel we shall omit the index 'λ' in $M_{p,\lambda}^{(i)}$, when $i = 3$ and 4.

Proposition 2.1. $M_p^4 = M_p^3$. *Moreover,* $J_\lambda W_3^p = X_3^{-p}$ *iff* $J_\lambda W_4^p = X_4^{-p}$.

P r o o f. We note that the definition of the quantity $M_{p,\lambda}^{(i)}(\mathbf{R})$ makes sense for any $p > 1$. If we prove the first statement of the proposition for this extended range of p's, we can derive the second one by means of a simple duality argument.

Now, we fix $\sigma \in (0, 1)$ and consider a smooth function ψ_T on $\mathbf{R}$ such that $0 \le \psi_T(t) \le 1$, $|\psi_T'(t)| \le C \cdot T^{-\sigma}$ for $t \in \mathbf{R}$, $\psi_T(t) = 1$ for $|t| \le T$, and $\psi_T(t) = 0$ for $|t| \ge T + T^\sigma$, where $C > 0$ does not depend on T. Evidently, such the function exists for any $T > 0$. For any $W_0^{1,p}$-valued trigonometrical polynomial $v(t)$ we have

$$\frac{1}{2T} \| J_\lambda (\psi_T v) \|_{-p,3}^p =$$

$$= \frac{1}{2T} \int\limits_{|t| \le T} \| J_\lambda v \|_{-1,p}^p \, dt + \frac{1}{2T} \int\limits_{T \le |t| \le T + T^\sigma} \psi_T^p \| J_\lambda v \|_{-1,p}^p \, dt +$$

$$+ \frac{1}{2T} \int\limits_{T \le |t| \le T + T^\sigma} |\psi_T'|^p \| v' \|_{-1,p}^p \, dt = \frac{1}{2T} \int\limits_{|t| \le T} \| J_\lambda v \|_{-1,p}^p \, dt + J_1 + J_2.$$

It is easy to see that $J_1 \le C \cdot T^{\sigma - 1}$, where $C > 0$ does not depend on T (but depends on v). Similarly, $J_2 \le C \cdot T^{\sigma(1-p)-1}$. Hence,

$$\lim_{T \to \infty} \frac{1}{2T} \| J_\lambda (\psi_T v) \|_{-p,3}^p = \| J_\lambda v \|_{-p,4}^p .$$

In a similar way

$$\lim_{T \to \infty} \frac{1}{2T} \| \psi_T v \|_{-p,3}^p = \| v \|_{-p,4}^p .$$

The last two equations imply $M_p^{(4)} \le M_p^{(3)}$, since trigonometrical polynomials form a dense subset in W_4^p.

Now, let u be a smooth compactly supported $W_0^{1,p}$-valued function on $\mathbf{R}$. Assume that supp $u \subset [-T, T]$ and denote by u_T the $2T$-periodic extension of $u_{|[-T,T]}$ to $\mathbf{R}$. Then

$$\frac{1}{2T} \| u \|_{p,3}^p = \| u_T \|_{p,4}^p ,$$

$$\frac{1}{2T} \| J_\lambda u \|_{-p,3}^p = \| J_\lambda u_T \|_{-p,4}^p .$$

If we combine these equations with a simple density argument, we see that $M_p^{(3)} \le M_p^{(4)}$. Hence, $M_p^{(3)} = M_p^{(4)}$ for any $p > 1$, and the proof is complete.

R e m a r k 2.2. The proof of Proposition 2.1 is an adaptation of the arguments used in [9] and [8, Ch. 5] to our more simple situation.

Proposition 2.2. $M_{p,\lambda}^{(1)} = M_{p,\lambda}^{(2)}$. *Moreover,* $J_\lambda W_1^p = X_1^{-p}$ *if and only if* $J_\lambda W_2^p = X_2^{-p}$.

P r o o f. Since W_2^p and X_2^{-p} are closed subspaces in W_1^p and X_1^{-p} respectively, we have $M_{p,\lambda}^{(2)} \le M_{p,\lambda}^{(1)}$. Conversely, let $u \in W_1^p$ and $T > 0$. Denote by u_T the $2T$-periodic extension of $u_{|[-T,T]}$ to $\mathbf{R}$. It is easy to see that $u_T \in W_2^p$ and

$$\| u \|_{p,1} = \lim_{T \to \infty} \| u_T \|_{p,2} ,$$

$$\| J_\lambda u \|_{-p,1} = \lim_{T \to \infty} \| J_\lambda u_T \|_{-p,2} .$$

(We remark that $\| \cdot \|_{r,2} = \| \cdot \|_{r,1}$.) These equations imply $M_{p,\lambda}^{(1)} \le M_{p,\lambda}^{(2)}$ and, hence, $M_{p,\lambda}^{(1)} = M_{p,\lambda}^{(2)}$.

We note now that any duality argument is inapplicable in this situation. Therefore, we prove the second assertion directly. Assume, that $J_\lambda W_1^p = X_1^{-p}$. Then for any $f \in X_2^{-p} \subset X_1^p$ there is a unique solution $u \in W_1^p$ of the equation $J_\lambda u = f$ and $\| u \|_{W_1^p} \le C \cdot \| u \|_{-p,1}$, where $C > 0$ does not depend on f. Since the set

$\{f(\cdot+\tau)\}_{\tau\in\mathbf{R}}$ is precompact in X_1^{-p}, the last inequality and the translation invariance of J_λ give rise to the precompactness of $\{u(\cdot+\tau)\}_{\tau\in\mathbf{R}}$ in the space W_1^p. Hence, $u\in W_2^p$.

Now, we assume, that $J_\lambda W_2^p = X_2^{-p}$. Let $f\in X_1^{-p}$. The $2T$-periodic extension, f_T, of $f_{|[-T,T]}$ belongs to X_2^{-p} and $\|f_T\|_{-p,2}\le\|f\|_{-p,1}$. Then there exists a unique solution $u_T\in W_2^p$ of the equation $J_\lambda u_T = f_T$ and $\|u_T\|_{W_2^p}\le C\cdot\|f\|_{-p,1}$. Hence, there exists a sequence $T_k\to\infty$ such that $u_{T_k}\to u$ weakly in $L^p_{\mathrm{loc}}(\mathbf{R};W_0^{1,p})$ and $u'_{T_k}\to u'$ weakly in $L^p_{\mathrm{loc}}(\mathbf{R};W^{-1,p})$. It is evident that $u\in W_1^p$ and $J_\lambda u=f$. The proof is complete.

3. Special linear problems and regular domains

For $p\ge 2$ we denote by $\mathcal{R}^p_{(s)}$ the class of all regular subsets of $\mathbf{R}^n$ such that $J_\lambda W_1^p = X_1^{-p}$, $\lambda>0$ (or, equivalently, $J_\lambda W_2^p = X_2^{-p}$). Denote also by $\mathcal{R}^p_{(b)}$ the class of all regular subsets such that $J_\lambda W_3^p = X_3^{-p}$, $\lambda>0$ (or, equivalently, $J_\lambda W_4^p = X_4^{-p}$). Now we modify slightly some of our notations as follows

$$M^{(s)}_{p,\lambda} = M^{(1)}_{p,\lambda} = M^{(2)}_{p,\lambda},$$

$$M^{(b)}_{p} = M^{(3)}_{p} = M^{(4)}_{p}.$$

Recall, that $M^{(i)}_{p,\lambda}$, $i=3,4$, does not depend on $\lambda>0$ (see Remark 2.1) and $M^{(b)}_2=1$. Here 's' and 'b' correspond to "Stepanov" and "Besicovich" respectively.

Lemma 3.1. *Let $G\in\mathcal{R}^q_{(b)}$ (resp. $G\in\mathcal{R}^q_{(s)}$)) for some $q>2$. Then $G\in\mathcal{R}^p_{(b)}$ (resp. $G\in\mathcal{R}^p_{(s)}$) for $p\in[2,q]$ and*

$$M^{(b)}_p \le (M^{(b)}_q)^\theta,$$

resp.

$$M^{(s)}_{p,\lambda} \le (M^{(s)}_{2,\lambda})^{1-\theta}(M^{(s)}_{q,\lambda})^\theta,$$

where $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$.

P r o o f is almost identical to that of Lemma 1 [5]. Only one remark is in order. In the 's'-case we need to use the following interpolation result of Lemma 3.2 instead of classical Riesz-Thorin theorem.

Lemma 3.2. *Let* $E_r = BS^r(\mathbf{R}; L^r(G))$ *and* A *be a linear operator acting continuously on* E_r *and on* E_q, $q \in (1, \infty)$, $r \in (1, \infty)$. *Then* $A \in L(E_p)$ *and*

$$\| A \|_{L(E_p)} \le \| A \|_{L(E_r)}^{1-\theta} \| A \|_{L(E_q)}^{\theta},$$

where $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{q}$.

P r o o f. We define an operator

$$\Phi_\alpha : E_r \to F_r = l^\infty(L^r((0, 1) \times G))$$

by the formula

$$\Phi_\alpha u = \{ \varphi_k(\cdot) \}_{k \in \mathbf{Z}},$$

where

$$\varphi_k(\tau, x) = u(\alpha + \tau + k, x), \ \ \alpha \in (0, 1), \ \ \tau \in (0, 1), \ \ x \in G.$$

It is obvious that Φ_α is an isomorphism of E_r onto F_r and

$$\| u \|_{E_r} = \sup_{\alpha \in (0, 1)} \| \Phi_\alpha u \|_{F_r}.$$

Let $A^{\alpha, \beta} = \Phi_\alpha \circ A \circ \Phi_\beta^{-1}$. Then

$$\| A \|_{L(E_r)} = \sup_{\alpha \in (0, 1)} \inf_{\beta \in (0, 1)} \| A^{\alpha, \beta} \|_{L(F_r)}. \tag{3.1}$$

The operator $A^{\alpha, \beta}$ may be represented by an infinite matrix (A_{ij}) and

$$\| A^{\alpha, \beta} \|_{L(F_r)} = \sup_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} \| A_{ij} \|_{L(L^r)}.$$

By classical Riesz-Thorin theorem we have

$$\| A^{\alpha, \beta} \|_{L(F_p)} \le \sup_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} \| A_{ij} \|_{L(L^r)}^{1-\theta} \cdot \| A_{ij} \|_{L(L^q)}^{\theta} \le$$

$$\le \sup_{i \in \mathbf{Z}} [(\sum_{j \in \mathbf{Z}} \| A_{ij} \|_{L(L^r)})^{1-\theta} \cdot (\sum_{j \in \mathbf{Z}} \| A_{ij} \|_{L(L^q)})^{\theta}] \le$$

$$\le \| A^{\alpha, \beta} \|^{1-\theta}_{L(F_r)} \cdot \| A^{\alpha, \beta} \|^{\theta}_{L(F_q)}.$$

Using (3.1) we complete the proof of Lemma 3.2.

Lemma 3.3. *Let* $\{ U_1, \dots, U_N \}$ *be an open covering of* $\overline{G}$ *and* $U_i \cap G \in \mathcal{R}^q_{(s)}$ (*resp.* $U_i \cap G \in \mathcal{R}^q_{(b)}$), $i = 1, \dots, N$ *for some* $q > 2$. *Then* $G \in \mathcal{R}^q_{(s)}$ (*resp.* $G \in \mathcal{R}^q_{(b)}$).

P r o o f is similar to that of Lemma 5 [5].

Lemma 3.4. *Let* $E = \{ x \in \mathbf{R}^n : |x| < 1 \}$. *Then for every* $f \in BS^p(\mathbf{R}; L^p(G))$ (*resp.* $f \in L^p(\mathbf{R} \times E)$) *there exists a unique solution of the equation*

$$u' - \Delta u = f \tag{3.2}$$

such that

$$u \in BS^p(\mathbf{R}; W^{2,p}(E) \cap W^{1,p}_0(E))$$

resp.

$$u \in L^p(\mathbf{R}; W^{2,p}(E) \cap W^{1,p}_0(E)).$$

P r o o f. Let $Z(x, \xi, t)$ be the classical Green function of the Dirichlet initial boundary value problem for equation (3.2) on E. Let

$$u_k(x, t) = \int_k^{k+1} d\tau \int_E Z(x, \xi, \tau) f(\xi, t - \tau)\, d\xi,$$

and

$$u(x, t) = \sum_{k=0}^{\infty} u_k(x, t). \tag{3.3.}$$

As in [10], standard properties of the Green function and Mikhlin theorem on Fourier multipliers imply that

$$\| u_0 \|_{L^p(s, s+1; W^{2,p})} \le C \cdot \| f \|_{L^p(s-1, s+1; L^p)} \le 2C \cdot \| f \|_{BS^p(\mathbf{R}; L^p)}.$$

Moreover, it is well-known that

$$|\partial_x^\alpha Z(x, \xi, t)| \le C_\alpha e^{-\sigma t}, \quad t \ge 1,$$

for some $\sigma > 0$. Hence, for $k \ge 1$

$$\int_s^{s+1} \| u_k(\cdot, t)\|^p_{W^{2,p}} dt = \sum_{|\alpha| \le 2} \int_s^{s+1} dt \, \| \int_k^{k+1} d\tau \int_E \partial_x^\alpha Z(x, \xi, \tau) f(\xi, t-\tau)\, d\xi \|^p_{L^p} \le$$

$$\le C \cdot e^{-\sigma k} \int_s^{s+1} dt \int_k^{k+1} d\tau \, \| f(\cdot, t-\tau)\|^p_{L^p} \le C \cdot e^{-\sigma k} \cdot \| f \|^p_{BS^p(\mathbf{R};\, L^p(E))}.$$

Together with (3.3) this implies that $u \in BS^p(\mathbf{R};\, W^{2,p}(E))$. It is evident that $u(x, t)$ is a solution of (3.2) and $u \in BS^p(\mathbf{R};\, W_0^{1,p}(E))$. The L^p-statement may be proved in a similar way. The proof is complete.

Now, exactly as in [5], Lemmas 4 and 5, we have

Lemma 3.5. *For every $p \in [2, \infty)$ the sets E, E_1, E_2 and E_3 belong to the classes $\mathcal{R}^p_{(s)}$ and $\mathcal{R}^p_{(b)}$.*

4. The key estimate

In this section we have to estimate the quantity $M^{(s)}_{2,\lambda}$. Let κ be the embedding constant for $W_0^{1,2}(G) \subset L^2(G)$, i.e. $\|u\|_{L^2} \le \kappa \cdot \|u\|_{W^{1,2}}$, $u \in W^{1,2}(G)$.

Lemma 4.1. *The following inequality*

$$M^{(s)}_{2,\lambda} \le \sqrt{\frac{3}{2}} \, \frac{1 + 2\lambda\kappa^2}{\sqrt{4 + 6\lambda\kappa^2} - 1}, \quad \lambda > 0, \tag{4.1}$$

is valid.

Proof. We apply here a more refined version of the arguments we have used in the proof of Lemma 1.1 [8, Ch. 2]. Let $u \in W_1^2(\mathbf{R})$ and $f = J_\lambda u \in X_1^{-2}(\mathbf{R})$. To get the required estimate we may assume that supp $u \subset \mathbf{R}_+$ (see, e.g., [8], Section 2 of Ch. 3). We introduce the following notations: $\varphi(t) = \lambda^{1/2} \cdot \|u(t)\|_{L^2}$, $h(t) = \|u(t)\|_{W^{1,2}}$,

$g(t) = \|f(t)\|_{W^{-1,2}}$, and $\Theta = \|f\|_{X_1^{-2}}$. We have $\varphi(t) \le k \cdot h(t)$, where $k = \lambda^{1/2} \cdot \kappa$, and $\varphi(0) = 0$.

Multiplying the equation $\lambda u' + \Delta u = f$ by u and integrating, we obtain the inequality

$$\frac{1}{2}\varphi^2(t_2) - \frac{1}{2}\varphi^2(t_1) + \int_{t_1}^{t_2} h^2(t)\,dt \le \int_{t_1}^{t_2} h(t)g(t)\,dt, \tag{4.2}$$

where $0 \le t_1 < t_2$.

First of all we prove the inequality

$$\varphi^2(t) \le 3 \cdot \left(\frac{1}{2} + k^2\right)\Theta^2, \quad t \in \mathbf{R}_+. \tag{4.3}$$

Inequality (4.2) with $t_1 = 0$, $t_2 = t$ implies that $\varphi^2(t) \le \frac{1}{2}\Theta^2$, $t \in [0, 1]$. Hence, (4.3) is valid for $t \in [0, 1]$.

Now, we assume, that (4.3) is valid for $t \in [0, n]$. We have to prove that it is also valid for $t \in [0, n+1]$. Let $\tau_j \in [j-1, j]$ be a point such that

$$\varphi(\tau_j) = \max_{[j-1,j]} \varphi(t).$$

If $\varphi(\tau_{n+1}) \le \varphi(\tau_{n-1})$, then the required assertion is trivially valid. So, we assume, that $\varphi(\tau_{n+1}) > \varphi(\tau_{n-1})$. Inequality (4.2) and the well-known inequality

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon} \tag{4.4}$$

imply

$$\int_{\tau_{n-1}}^{\tau_{n+1}} h^2(t)\,dt \le \frac{3}{4\varepsilon(1-\varepsilon)}\Theta^2.$$

It is obvious that here the optimal choice of ε is $\varepsilon = \frac{1}{2}$. Hence,

$$\int_{\tau_{n-1}}^{\tau_{n+1}} h^2(t)\,dt \le 3 \cdot \Theta^2.$$

Since $\varphi(t) \le k \cdot h(t)$, we have

$$\int_{\tau_{n-1}}^{\tau_{n+1}} \varphi^2(t)\,dt \le 3k^2\,\Theta^2 .$$

Hence, there exists $t_0 \in [\tau_{n-1}, \tau_{n+1}]$ such that

$$\varphi^2(t_0) \le \frac{3k^2\,\Theta^2}{\tau_{n+1} - \tau_{n-1}} \le 3k^2\,\Theta^2. \tag{4.5}$$

Now, inequality (4.2) with $t_1 = t_0$ and $t_2 = \tau_{n+1}$ implies

$$\frac{1}{2}\,\varphi^2(\tau_{n+1}) - \frac{1}{2}\,\varphi^2(t_0) + \int_{t_0}^{\tau_{n+1}} h^2(t)\,dt \le$$

$$\le [\int_{t_0}^{\tau_{n+1}} h^2(t)\,dt]^{1/2} \cdot [\int_{t_0}^{\tau_{n+1}} g^2(t)\,dt]^{1/2} \le \int_{t_0}^{\tau_{n+1}} h^2(t)\,dt + \frac{1}{2}\int_{t_0}^{\tau_{n+1}} g^2(t)\,dt.$$

Since $\tau_{n+1} - t_0 \le 3$ we have (using (4.5))

$$\varphi^2(\tau_{n+1}) \le \varphi^2(t_0) + \frac{1}{2}\int_{t_0}^{\tau_{n+1}} g^2(t)\,dt \le 3\cdot(\frac{1}{2} + k^2)\Theta^2,$$

and (4.3) is proved.

Now, (4.2), (4.3) and (4.4) imply

$$\int_{\tau}^{\tau+1} h^2(t)\,dt \le \frac{1}{2}\,\varphi^2(\tau) + \varepsilon \int_{\tau}^{\tau+1} h^2(t)\,dt + \frac{1}{4\,\varepsilon}\int_{\tau}^{\tau+1} g^2(t)\,dt \le$$

$$\le [\,\frac{3}{2}\cdot(\frac{1}{2} + k^2) + \frac{1}{4\varepsilon}\,]\,\Theta^2 + \varepsilon \int_{\tau}^{\tau+1} g^2(t)\,dt.$$

Hence

$$\int_{\tau}^{\tau+1} h^2(t)\,dt \le \frac{1}{4}\cdot\frac{3(1+2k^2)\,\varepsilon + 1}{\varepsilon\,(1-\varepsilon)}\cdot\Theta^2.$$

It is not hard to see that

$$\min_{\varepsilon > 0} \frac{3(1+2k^2)\varepsilon + 1}{\varepsilon(1-\varepsilon)} = \frac{6(1+2k^2)^2}{\left(\sqrt{4+6k^2}-1\right)^2}.$$

This implies (4.1) and the proof is complete.

R e m a r k 4.1. We know nothing about exactness of estimate (4.1). However, there is an example, which shows that $M^{(s)}_{2,\lambda} > 1$.

R e m a r k 4.2. With the norm $\|\cdot\|_{S^p,\theta}$ (see (1.3)) as a background we can introduce the quantity $M^{(s)}_{p,\lambda,\theta} = M^{(1)}_{p,\lambda,\theta} = M^{(2)}_{p,\lambda,\theta}$ using (2.1) with $\|\cdot\|_{S^p}$ replaced by $\|\cdot\|_{S^p,\theta}$. Scaling $\tau = \theta s$ turns $\|\cdot\|_{S^p,\theta}$ into $\theta\|\cdot\|_{S^p}$ and the operator J_λ into $J_{\lambda/\theta}$. Hence, we have immediately

$$M^{(s)}_{p,\lambda,\theta} = M^{(s)}_{p,\lambda/\theta}. \tag{4.6}$$

5. Main results

Now, we are going to prove regularity results for bounded and a.p. solutions to rather general nonlinear second order parabolic operators under the assumption that $G \in \mathcal{R}^p_{(s)}$ or $G \in \mathcal{R}^p_{(b)}$.

Let $b : \mathbf{R} \times G \times \mathbf{C}^n \to \mathbf{C}^n$ be a function such that $b(\cdot,\cdot,0) = 0$, satisfying the following hypotheses:

$$\operatorname{Re}(b(t,x,\xi) - b(t,x,\eta)) \cdot (\bar{\xi} - \bar{\eta}) \geq m|\xi-\eta|^2, \quad m > 0, \tag{5.1}$$

$$|b(t,x,\xi) - b(t,x,\eta)| \leq M|\xi - \eta|, \quad M > 0, \tag{5.2}$$

for $(t,x) \in \mathbf{R} \times G$ and $\xi, \eta \in \mathbf{C}^n$,

$$b(\cdot,\cdot,\xi) \text{ is measurable for every } \xi \in \mathbf{C}^n. \tag{5.3}$$

Here $\bar{\xi}$ denotes vector whose components are complex conjugate to those of ξ.

We consider the equation

$$u' + A(t)u = f, \tag{5.4}$$

where the operator $A(t) : W^{1,2}_0(G) \to W^{-1,2}(G)$ is given by the formula

$$\langle A(t)u, v \rangle = \int_G b(t, x, \nabla u) \cdot \nabla v \, dx, \quad u, \ v \in W_0^{1,2}(G). \tag{5.5}$$

It is well-known that under hypotheses (5.1)-(5.3) for any $f \in X_i^{-2}$ equation (5.4) has a unique solution $u \in W_1^2$, $i = 1, 3$ (see, for example, [8]). Moreover, if additionally the hypothesis

(ap) $|\xi| b(t, x, \xi)$ is a. p. on $t \in \mathbf{R}$ uniformly with respect to $(x, \xi) \in G \times \mathbf{C}^n$

is valid, then the previous statement holds for $i = 2, 4$ too.

Theorem 5.1. *There is $p_0 > 2$ such that for any $G \in \mathcal{R}^p_{(s)}$, $p \in [2, p_0]$, the following statement holds true. Assume* (5.1)-(5.3) *to be valid and*

$$M < \sqrt{3} \cdot m. \tag{5.6}$$

Then for any $f \in BS^p(\mathbf{R}; W^{-1,p})$ there exists a unique solution $u \in BS^p(\mathbf{R}; W_0^{1,p})$ of equation (5.4). *If, additionally, (ap) is valid, then for any $f \in S^p(\mathbf{R}; W^{-1,p})$ we have $u \in S^p(\mathbf{R}; W_0^{-1,p})$.*

Theorem 5.2. *There is $p_0 > 2$ such that for any $G \in \mathcal{R}^p_{(b)}$, $p \in [2, p_0]$, the following statement holds true. Assume that* (5.1)-(5.3) *are valid. Then for any $f \in L^p(\mathbf{R}; W^{-1,p})$ there exists a unique solution $u \in L^p(\mathbf{R}; W_0^{1,p})$ of equation* (5.4). *If, additionally, (ap) is valid, then for any $f \in B^p(\mathbf{R}; W^{-1,p})$ equation* (5.4) *has a unique solution $u \in B^p(\mathbf{R}; W_0^{-1,p})$.*

P r o o f s of Theorems 5.1 and 5.2. As in [5] we define the operator

$$Q_f u = J_\lambda^{-1}((J - \lambda A)u + \lambda f).$$

It is not hard to see that for $f \in X_i^{-p}$ the operator Q_f is a Lipschitz map from X_i^p into itself ($i = 1, 2, 3, 4$). Moreover, any fixed point of Q_f is a solution of (5.4). Therefore, we shall prove that under appropriate conditions Q_f is a contraction. We consider the case of Theorem 5.1 only (the case of Theorem 5.2 is similar). We shall use the norm in BS^p defined by (1.3) (instead of (1.2)).

Set $\lambda = m/M^2$. A simple calculation shows us that the Lipschitz constant of Q_f is equal to $k \cdot M^{(s)}_{p,\lambda,\theta}$, where $k = (1 - \frac{m^2}{M^2})^{1/2}$ and $M^{(s)}_{p,\lambda,\theta}$ is defined in Remark 4.1.

Now (4.6), Lemma 4.1 and Lemma 3.1 imply that we can choose $\theta > 0$ being sufficiently large so that $k \cdot M^{(s)}_{p, \lambda, \theta} < 1$ for $p > 2$ being sufficiently close to 2. Hence, Q_f is a contraction and the proof is complete.

R e m a r k 5.1. Under the assumption of Theorems 5.1 and 5.2 the map $f \mapsto u$ is a Lipschitz map between corresponding spaces.

6. Lipschitz diffeomorphisms, Lipschitz sets, and regular sets

Now we present more details about set classes $\mathcal{R}^p_{(b)}$ and $\mathcal{R}^p_{(s)}$. First of all, we have

Theorem 6.1. *Any regular set belongs to* $\mathcal{R}^p_{(b)}$ *for some* $p > 2$.

The result follows immediately from Lemma 3.3 and

Theorem 6.2. *Let* $G \in \mathcal{R}^q_{(b)}$, $q > 2$, *and* $\varphi : G \to \tilde{G}$ *is a Lipschitz diffeomorphism with a constant Jacobian determinant. Then* $\tilde{G} \in \mathcal{R}^p_{(b)}$ *for some* $p > 2$.

P r o o f. Essentially, we repeat here the arguments that were used in the proof of Theorem 2 [5]. Let $Lu = u \circ \varphi^{-1}$. It is easy to see that L maps $W^{1,p}_0(G)$ onto $W^{1,p}_0(\tilde{G})$. Denote by L^* the adjoint operator acting from $W^{-1,p}(\tilde{G})$ onto $W^{-1,p}(G)$ for any $p > 1$. It is not difficult to check that $LL^* u = \lambda u$, where λ denotes the constant Jacobian determinant of φ. We shall also consider L (resp. L^*) as a linear bounded operator from $X^2_{i,G}$ onto $X^2_{i,\tilde{G}}$ (resp. $X^{-2}_{i,\tilde{G}}$ onto $X^{-2}_{i,G}$), $i = 1, 2, 3, 4$. It is evident that the operator $A = \lambda^{-1} L^* J_{\tilde{G}} L$ satisfies the assumptions of Theorem 5.2 for some $p > 2$. Hence, $X^{-p}_{i,G} = \{ u' + Au : u \in W^p_{i,G} \}$ for some $p > 2$ ($i = 3, 4$). Let $\tilde{f} \in X^{-p}_{i,\tilde{G}}$. Then $L^* \tilde{f} \in X^{-p}_{i,G}$, and there exists $u \in W^p_{i,G}$ such that

$$L^* \tilde{f} = \lambda(u' + Au) = L^* (Lu)' + L^* J_{\tilde{G}} Lu.$$

This implies that $(Lu)' + J_{\tilde{G}} Lu = \tilde{f}$. It is easy to check that $Lu \in W^p_{i,\tilde{G}}$. Hence, $\tilde{G} \in \mathcal{R}^p_{(b)}$. The proof is complete.

R e m a r k 6.1. In Theorem 6.2 we cannot assert that $\tilde{G} \in \mathcal{R}^q_{(b)}$.

The arguments of the proof of Theorem 6.2 do not work in the Stepanov case ($i = 1, 2$) because of strong assumption (5.6) in Theorem 5.1. Nevertheless, we have

Theorem 6.3 *Let $\partial G = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\overline{G} \setminus G = \Gamma_1$. Assume, that ∂G is locally a graph of a Lipschitz function of $(n-1)$ variables, and $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$ is locally a graph of a Lipschitz function of $(n-2)$ variables (in the case $n = 2$ consist of a finite number of points). Then $G \in \mathcal{R}^p_{(s)}$ for some $p > 2$.*

P r o o f. In the case under consideration there exists a collection of Lipschitz diffeomorphisms $\varphi_1, \dots, \varphi_N$, having Jacobian determinants equal to 1, such that
(i) $\varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_N$ maps one of the sets E_1, E_2 or E_3 onto a part of G containing a given point $y \in \partial G$;
(ii) the Jacobian matrix of φ_k, $k = 1, \dots, N$, has sufficiently small out-of-diagonal elements.

We describe briefly how to construct such diffeomorphisms. For example, let $0 \in \Gamma_1 \setminus \overline{\Gamma}_2$. Then in a small neighbourhood of 0 the set G is of the form

$$\{ x : x_n < h(x_1, \dots, x_{n-1}) \},$$

where h is a Lipschitz function. Let $0 = t_0 < t_1 < \dots < t_N = 1$. We define the Lipschitz diffeomorphisms ψ_k, $k = 1, \dots, N$, by the formula

$$\psi_k(x) = (x_1, \dots, x_{n-1}, x_n - t_k h(x_1, \dots, x_{n-1})).$$

Then we set $\varphi_k = \psi_{k-1} \circ \psi_k^{-1}$, $k = 1, \dots, N$. If $t_k - t_{k-1}$, $k = 1, \dots, N$, is sufficiently small, then statement (ii) is valid.

The composition $\varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_N$ maps the set

$$\tilde{E}_1 = \{ x \in \mathbf{R}^n : |x| < \varepsilon, x_n < 0 \}$$

onto a corresponding part of G. It is obvious how to complete this set of diffeomorphisms so that the statements (i) and (ii) are valid. All other cases of location of the point $y \in \partial G$ may be treated in a similar way.

Now, we can use the argument of the proof of Theorem 6.3 in the following way. Let $G_N = E_1, E_2$ or E_3, and $G_k = \varphi_k G_{k+1}$, $i = 1, \dots, N-1$. Statement (ii) implies that the operator $A_k = L_k^* J_{G_{k-1}} L_k$, where $L_k u = u \circ \varphi_k^{-1}$, satisfies all the conditions of

Theorem 5.1. Hence, G_1, a small part of G around any given point $y \in \partial G$, belongs to $\mathcal{R}^p_{(s)}$. Using compactness of $\overline{G}$ and Lemma 3.3, we conclude.

7. Concluding remarks

It must be pointed out that the results of Theorems 5.1 and 5.2 may be extended in an evident way to cover the case of operators of the form

$$Au = -\nabla b(t, x, u, \nabla u) + b_0(t, x, u, \nabla u),$$

and the case, when $n-1$-dimensional surface measure of $\overline{G} \setminus D$ is equal to zero (this corresponds to the Neumann boundary value problem). Moreover, BS^p and L^p versions of Theorems 5.1 and 5.2 are still valid for equation (5.4) on a half-axis with zero initial condition.

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Регулярность типа Мейера для ограниченных и почти периодических решений нелинейных параболических уравнений второго порядка со смешанными краевыми условиями

К. Грегер, А. Панков

Получены условия, при которых почти периодическое решение нелинейного параболического уравнения, принадлежащее априори $\overset{0}{W}{}^{1,2}$, будет почти периодической функцией со значениями в $\overset{0}{W}{}^{1,p}$ с достаточно малым $p-2>0$.

Регулярність типу Мейера для обмежених і майже періодичних розв'язків нелінійних параболічних рівнянь другого порядку із мішаними крайовими умовами

К. Грьогер, А. Панков

Отримано умови, які гарантують, що майже періодичний розв'язок нелінійного параболічного рівняння є в дійсності періодичною функцією із значеннями у просторі $\overset{0}{W}{}^{1,p} \subset \overset{0}{W}{}^{1,2}$, де $p-2>0$ є достатньо малим.