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On a counterexample concerning unique continuation for elliptic equations in divergence form

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We construct a second order elliptic equation in divergence form in $\mathbf{R}^3$, with a non-zero solution which vanishes in a half-space. The coefficients are α-Hölder continuous of any order $\alpha < 1$. This improves a previous counterexample of Miller [1,2] Moreover, we obtain coefficients which belong to a finer class of smoothness, expressed in terms of the modulus of continuity.

Introduction. The aim of this paper is to improve a counterexample due to Keith Miller [1,2]. Part of the results presented here belong to the author's PhD thesis [3, § 3.4].

The first who constructed an elliptic second order equation for which the Cauchy problem does not have the uniqueness property is Pliš [4]. The first and zero order coefficients of his equation are smooth, but the leading coefficients are only α-Hölder continuous of any order $\alpha < 1$. This result is optimal, since for Lipschitz-continuous coefficients we have allways uniqueness in the Gauchy problem (and even stronger results, see [5]).

Miller was concerned with the non-uniqueness in the Gauchy problem for the elliptic equation in divergence form

$$\sum_{i,j=1}^{n} \partial_i a_{ij} \partial_j u = 0, \qquad (1)$$

and the backward non-uniqueness for the corresponding parabolic equation

$$\partial_t u = \sum_{i,j=1}^{n} \partial_i a_{ij} \partial_j u. \qquad (2)$$

Here the matrix of coefficients (a_{ij}) is supposed real, symmetric, continuous and uniformly positive, i.e.,

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j \geq C\,|x|^2,\ C>0, \text{ for any } x \in \mathbf{R}^n.$$

The interest of the equations (1) and (2) comes from the fact that the first correspond to symmetric operators in $L^2(\mathbf{R}^n)$ and the second is the evolution equation for such operators. They have also a physical meaning: (2) is the heat equation in a medium with specific heat 1 and with the termic conductivity given by the matrix a_{ij}. See [2] for further comments.

Our example is better than the one in [2] in the following ways:

1) It allows optimal regularity by a precise choice of the parameters used in the construction. We obtain Hölder continuous coefficients of any order $\alpha < 1$ whereas in [2] Miller obtained only the order $\alpha = 1/6$. We obtain also a finer result:

Suppose that $\omega : [0, \infty) \to [0, \infty)$ is concave, continuous, non-decreasing, $\omega(0) = 0$, $\omega(1) > 0$ and ω satisfies:

$$\int_0^1 \frac{dt}{\omega(t)} < \infty.$$

Then we can choose the coefficients of our equation such that their moduli of continuity are majorated by ω.

2) It makes an interesting rephrasing of the problem into a system of inequations for sequences of numbers. The inherent limits of the construction below suggest that the unique continuation property for the equation (1) might hold under the assumption that $a_{ij} \in W^{1,1}$.

3) There is a simplification in the technical part which allows us to give explicit (though complicated) expressions of the coefficients.

Theorem 1. *There exist a smooth function u, smooth functions b_{11}, b_{12}, b_{13}, and continuous functions d_1, d_2 defined on $\mathbf{R}^3 \ni (t, x, y)$, with the following properties:*

(i) *u is solution of the equation*

$$\partial_t^2 u + \partial_x((b_{11} + d_1)\,\partial_x u) + \partial_y(b_{12}\,\partial_x u) + \\ + \partial_x(b_{12}\,\partial_y u) + \partial_y((b_{22} + d_2)\,\partial_y u) = 0. \tag{3}$$

(ii) *There is a $T > 0$ such that* supp $u = (-\infty, T\,] \times \mathbf{R}^2$.

(iii) *u, b_{ij} and d_i are periodic in x and in y with period 2π.*

(iv) *For any $t \in \mathbf{R}$, $u(t, \cdot, \cdot)$ satisfies the Neumann boundary condition on* $(0, 2\pi)^2$ *with respect to the equation* (3) *(seen as an equation in the variables x and y).*

(v) *d_1 and d_2 do not depend on x and y and are Hölder continuous of order α for all $\alpha < 1$.*

(vi) $$\frac{1}{2} < \begin{pmatrix} d_1 + b_{11} & b_{12} \\ b_{12} & d_2 + b_{22} \end{pmatrix} < 2 \quad on \quad \mathbf{R}^3.$$

Furthermore, there are also functions as above, satisfying conditions (i-vi) *except that* (3) *is replaced with the parabolic equation*:

$$\partial_t u = \partial_x ((b_{11} + d_1)\, \partial_x u) + \partial_y (b_{12}\, \partial_x u) + \partial_x (b_{12}\, \partial_y u) + \partial_y ((b_{22} + d_2)\, \partial_y u). \quad (4)$$

R e m a r k. The equation (3) can be seen, given the periodicity condition (iii), as an abstract equation for an $L^2(\mathbf{R}^2/2\pi\mathbf{Z}^2)$-valued function:

$$u'' = A(t)u.$$

Here $A(t)$ is an elliptic operator on the torus, which is positive in $L^2(\mathbf{R}^2/2\pi\mathbf{Z}^2)$. Thus our theorem asserts the existence of an $A(t)$ such that the Cauchy problem for the above equation does not have the uniqueness property. The interest of the point (iv) of the theorem is that the above A can be replaced with an elliptic selfadjoint operator on $L^2((0, 2\pi)^2)$, with Neumann boundary condition.

Idea of the proof. We start from the operator $\Delta = \partial_t^2 + \Delta_{xy}$, and from its solutions $e^{-\lambda t} \cos \lambda x$ and $e^{-\lambda t} \cos \lambda y$. It is convenient to view the operator to be constructed (appearing in (3)) as a perturbation of Δ. The above solutions of Δ decay with t, the bigger is λ the faster is the decay. We will "glue" an infinite number of them, corresponding to the frequencies $\lambda = \lambda_k$, such that $\lambda_k \to \infty$ as $k \to \infty$. In this way, as $t \uparrow T$ the solution will be, for shorter and shorter intervals of time, proportional with $e^{-\lambda_k t} \cos \lambda_k x$, then with $e^{-\lambda_{k+1} t} \cos \lambda_{k+1} y$ and so on. In these intervals the equation is $\partial_t^2 u + \Delta_{xy} u = 0$. In the gaps between them, we will modify the coefficients such as to fit a prescribed solution, which passes smoothly from $e^{-\lambda_k t} \cos \lambda_k x$ to $e^{-\lambda_{k+1} t} \cos \lambda_{k+1} y$. Choosing suitable λ_k and suitable lengths of the intervals and of the gaps, we obtain a smooth solution which vanishes in finite time. In fact the solution is decaying also in the gaps and we can choose intervals of length zero.

The first part of the proof consists in constructing generic functions v, B_{ij}, D_i : $[0, 5a] \times \mathbf{R}^2 \to \mathbf{R}$, $i, j = 1, 2$, which describe the solution and the coefficients in a gap. They depend on the following parameters:

$a > 0$ gives the length (in time) of the domain of definition,

$\lambda > 1/a$ is the old frequency,

$\lambda' > \lambda$ is the new frequency, and

$\rho \in (0, \lambda/\lambda')$ is a technical parameter.

These functions satisfy the equality

$$\partial_t^2 v + \partial_x((B_{11} + D_1)\,\partial_x v) + \partial_x(B_{12}\,\partial_y v) + \partial_y(B_{12}\,\partial_x v) + \partial_y((B_{22} + D_2)\,\partial_y v) = 0 \tag{5}$$

on $[0, 5a]\times\mathbf{R}^2$, and do the required job of gluing, i.e. ,there is an $\varepsilon > 0$ such that

$$B_{ij} = \delta_{ij}\,,\ D_i = 0 \text{ for } t \in [0, \varepsilon) \cup (5a - \varepsilon, 5a],$$

$$v(t, x, y) = e^{-t\lambda} \cos \lambda x \quad \text{for } t \in [0, \varepsilon),$$

$$v(t, x, y) \text{ is proportional to } e^{-t\lambda'} \cos \lambda' y \quad \text{for } t \in (5a - \varepsilon, 5a].$$

In the second stage of the proof we will construct the functions u, b_{ij}, d_i : $\mathbf{R}^3 \to \mathbf{R}$, that satisfy the conclusions of the theorem. This is done putting together an infinite number of instances of this generic construction, with appropriate values for the parameters.

Construction of the generic v, B_{ij}, D_i. Let $\chi : \mathbf{R} \to [0, 1]$ be a smooth function with $\chi(t) = 1$ in a neighbourhood of $[1, \infty)$ aud $\chi(t) = 0$ in a neighbourhood of $(-\infty, 0]$. Each of the intervals $[(i-1)a, ia]$, with $i = 1,\ldots, 5$ (henceforth called steps) will have a precise job. We will describe the functions v, B_{ij} and D_i in each of them.

The first step serves to a smooth decay of $B_{22} + D_2$ from 1 to ρ^2:

$$v = e^{-\lambda t} \cos \lambda x,\ B_{11} = B_{22} = 1,\ B_{12} = D_1 = 0,\ D_2 = \chi\left(\frac{t}{a}\right)(\rho^2 - 1). \tag{6}$$

Since v does not depend on y, the last term in the l.h.s. of (5) vanishes and therefore (5) is satisfied for arbitrary D_2.

The second step is the "seed" step and serves to introduce a tiny component of the solution oscillating in y.

$$v = e^{-\lambda t} \cos \lambda x + \tilde{c}\chi\left(\frac{t-a}{a}\right) e^{-\rho\lambda' t} \cos \lambda' y. \tag{7}$$

The constant factor

$$\tilde{c} \overset{\text{def}}{=} e^{\frac{5a}{2}(\rho\lambda' - \lambda)} \tag{8}$$

serves to make the two components of the solution (one oscillating in x and one in y) of equal amplitude at $t = \frac{5a}{2}$, the middle of the third step. We put

$$B_{22} = 1, \quad D_1 = 0, \quad D_2 = \rho^2 - 1,$$

and we construct below $B_{11} \overset{\text{def}}{=} 1 + \tilde{B}$ and B_{12}. The equation (5) reads:

$$\lambda^2 e^{-\lambda t} \cos \lambda y + \tilde{c}\left(\frac{1}{a^2}\chi''\left(\frac{t-a}{a}\right) - \frac{2}{a}\chi'\left(\frac{t-a}{a}\right)\rho\lambda' + \chi\left(\frac{t-a}{a}\right)\rho^2\lambda'^2\right)e^{-\rho\lambda' t}\cos \lambda' y +$$

$$+ \partial_x\left((1+\tilde{B})\,\partial_x e^{-\lambda t}\cos\lambda x\right) + \partial_x\left(B_{12}\,\partial_y \tilde{c}\chi\left(\frac{t-a}{a}\right)e^{-\rho\lambda' t}\cos\lambda' y\right) +$$

$$+ \partial_y\left(B_{12}\,\partial_x e^{-\lambda t}\cos\lambda x\right) + \partial_y\left(\rho^2\,\partial_y \tilde{c}\chi\left(\frac{t-a}{a}\right)e^{-\rho\lambda' t}\cos\lambda' y\right) = 0.$$

After reductions:

$$\tilde{c}\left(\frac{1}{a^2}\chi''\left(\frac{t-a}{a}\right) - \frac{2}{a}\rho\lambda'\chi'\left(\frac{t-a}{a}\right)\right)e^{-\rho\lambda' t}\cos\lambda' y + \partial_x(-\tilde{B}e^{-\lambda t}\sin\lambda x) +$$

$$+ \partial_x\left(-B_{12}\tilde{c}\chi\left(\frac{t-a}{a}\right)e^{-\rho\lambda' t}\lambda'\sin\lambda' y\right) + \partial_y(-B_{12}e^{-\lambda t}\lambda\sin\lambda x) = 0.$$

Simplifying this equation by $e^{-\lambda t}$ and using the notation

$$\tilde{\chi}(t) = \tilde{c}e^{(\lambda - \rho\lambda')t}\left(\frac{1}{a^2}\chi''\left(\frac{t-a}{a}\right) - \frac{2\rho\lambda'}{a}\chi'\left(\frac{t-a}{a}\right)\right), \tag{9}$$

we obtain the equivalent relation

$$\tilde{\chi}(t)\cos\lambda' y = \lambda\,\partial_x(\tilde{B}\sin\lambda x) +$$

$$+ \lambda'\tilde{c}\chi\left(\frac{t-a}{a}\right)e^{(\lambda-\rho\lambda')t}\sin\lambda' y\,\partial_x B_{12} + \lambda\sin\lambda x\,\partial_y B_{12}.$$

If we choose first B_{12} then from the above relation, $\tilde{B}\lambda\sin\lambda x$ has to be the primitive of some function (depending on y and t as parameters). But this is possible only if that function has zero integral from $k\pi/\lambda$ to $(k+1)\pi/\lambda$, in order to allow the primitive to have zeros at $x = k\pi/\lambda$. To this end we take

$$B_{12}(t, x, y) = \tilde{\chi}(t)\frac{2\sin\lambda x\sin\lambda' y}{\lambda\lambda'}. \tag{10}$$

Then the above relation becomes:

$$\lambda\,\partial_x(\tilde{B}\sin\lambda x) = \tilde{\chi}(t)\cos\lambda' y - \lambda\sin\lambda x\,\tilde{\chi}(t)\,\frac{2\sin\lambda x\,\lambda'\cos\lambda' y}{\lambda\lambda'} -$$

$$-\lambda'\,\tilde{c}\chi\left(\frac{t-a}{a}\right)e^{(\lambda-\rho\lambda')t}\sin\lambda' y\,\tilde{\chi}(t)\,\frac{2\lambda\cos\lambda x\sin\lambda' y}{\lambda\lambda'},$$

and this yields further simplifying by λ:

$$\partial_x(\tilde{B}\sin\lambda x) = \tilde{\chi}(t)\cos\lambda' y\,\frac{1-2\sin^2\lambda x}{\lambda} -$$

$$-\tilde{c}\chi\left(\frac{t-a}{a}\right)e^{(\lambda-\rho\lambda')t}\tilde{\chi}(t)\,\frac{2\cos\lambda x\sin^2\lambda' y}{\lambda},$$

and since $\int(1-2\sin^2\lambda x)\,dx = \sin\lambda x\cos\lambda x/\lambda + C$, we obtain integrating from 0 to x with respect to x and then simplifying by $\sin\lambda x$:

$$\tilde{B}(t,x,y) = \tilde{\chi}(t)\left(\frac{\cos\lambda' y\cos\lambda x}{\lambda^2} - \tilde{c}\,e^{(\lambda-\lambda'\rho)t}\chi\left(\frac{t-a}{a}\right)\frac{2\sin^2\lambda' y}{\lambda^2}\right). \tag{11}$$

The third step has the coefficients

$$B_{11}=1,\ B_{12}=D_1=0,\ B_{22}=1,\ D_2=\rho^2-1,$$

and the solution is

$$v = e^{-\lambda t}\cos\lambda x + \tilde{c}\,e^{-\rho\lambda' t}\cos\lambda' y.$$

This step serves to propagate the two components with different speeds. Although the second term (depending on y) has a space frequency $\lambda' > \lambda$, its decay rate is smaller than that of the term depending on x, since $\rho\lambda' < \lambda$.

The fourth step is symmetric to the second one and the construction is similar. Its purpose is to remove the component of v oscillating in x, which has become small relatively to the other component.

$$v = \chi\left(\frac{4a-t}{a}\right)e^{-\lambda t}\cos\lambda x + \tilde{c}\,e^{-\rho\lambda' t}\cos\lambda' y.$$

Here the roles of x and y have changed. We have

$$B_{11}=1,\quad D_1=0,\quad D_2=\rho^2-1,$$

and B_{12}, $B_{22} \overset{\text{def}}{=} 1 + \tilde{\tilde{B}}$ are computed below.

The equation (5) gives

$$\partial_t^2\left(\chi\left(\frac{4a-t}{a}\right)e^{-\lambda t}\cos\lambda x+\tilde{c}\,e^{-\rho\lambda' t}\cos\lambda' y\right)+\partial_x\left(1\cdot\partial_x\chi\left(\frac{4a-t}{a}\right)e^{-\lambda t}\cos\lambda x\right)+$$

$$+\partial_y(B_{12}\partial_x v)+\partial_x(B_{12}\partial_y v)+\partial_y((\rho^2+\tilde{\tilde{B}})\,\partial_y\tilde{c}\,e^{-\rho\lambda' t}\cos\lambda' y)=0,$$

(we substituted the actual value of v only in the terms which are subject to reductions) and we obtain after reduction

$$\left(\frac{1}{a^2}\chi''\left(\frac{4a-t}{a}\right)+\frac{2\lambda}{a}\chi'\left(\frac{4a-t}{a}\right)\right)e^{-\lambda t}\cos\lambda x+\partial_y\left(B_{12}\chi\left(\frac{4a-t}{a}\right)e^{-\lambda t}\partial_x\cos\lambda x\right)+$$

$$+\partial_x\left(B_{12}\tilde{c}\,e^{-\rho\lambda' t}\partial_y\cos\lambda' y\right)+\partial_y\left(\tilde{\tilde{B}}\,\tilde{c}\,e^{-\rho\lambda' t}\partial_y\cos\lambda' y\right)=0.$$

Simplifying by $\tilde{c}\,e^{-\rho\lambda' t}$ and using the notation

$$\tilde{\tilde{\chi}}(t)=\frac{e^{(\rho\lambda'-\lambda)t}}{\tilde{c}}\left(\frac{1}{a^2}\chi''\left(\frac{4a-t}{a}\right)+\frac{2\lambda}{a}\chi'\left(\frac{4a-t}{a}\right)\right),\tag{12}$$

the relation becomes

$$\tilde{\tilde{\chi}}(t)\cos\lambda x=\partial_y\left(B_{12}\chi\left(\frac{4a-t}{a}\right)\frac{e^{(\rho\lambda'-\lambda)t}}{\tilde{c}}\lambda\sin\lambda x\right)+$$

$$+\lambda'\sin\lambda' y\,\partial_x B_{12}+\partial_y\left(\tilde{\tilde{B}}\,\lambda'\sin\lambda' y\right).$$

We choose

$$B_{12}=\tilde{\tilde{\chi}}(t)\frac{\sin\lambda x}{\lambda}\frac{2\sin\lambda' y}{\lambda'},\tag{13}$$

and taking the second term in the r.h.s. to the left in the above relation we obtain the equivalent relation

$$\tilde{\tilde{\chi}}\cos\lambda x(1-2\sin^2\lambda' y)=$$

$$=\partial_y\left(\tilde{\tilde{B}}\,\lambda'\sin\lambda' y+\tilde{\tilde{\chi}}(t)\frac{\sin\lambda x}{\lambda}\frac{2\sin\lambda' y}{\lambda'}\chi\left(\frac{4a-t}{a}\right)\frac{e^{(\rho\lambda'-\lambda)t}}{\tilde{c}}\lambda\sin\lambda x\right).$$

Since $\partial_y\dfrac{\sin\lambda' y\cos\lambda' y}{\lambda'}=1-2\sin^2\lambda' y$, the following relation ensures that (5) is fulfilled for $t\in[3a,4a]$ (after simplification by $\sin\lambda' y$):

$$\tilde{\tilde{\chi}} \cos \lambda x \frac{\cos \lambda' y}{\lambda'} = \tilde{\tilde{B}}\, \lambda' + \tilde{\tilde{\chi}}(t) \frac{2 \sin^2 \lambda x}{\lambda'} \chi \left(\frac{4a - t}{a}\right) \frac{e^{(\rho\lambda' - \lambda)t}}{\tilde{c}},$$

that is,

$$\tilde{\tilde{B}} = \frac{\tilde{\tilde{\chi}}(t)}{\lambda'^2} \left(\cos \lambda x \cos \lambda' y - 2\chi \left(\frac{4a - t}{a}\right) \frac{e^{(\rho\lambda' - \lambda)t}}{\tilde{c}} \sin^2 \lambda x\right). \tag{14}$$

The aim of the fifth step is to increase the coefficient $B_{22} + D_2$ from the value ρ^2 to 1, in order to get back to the values $B_{11} = B_{22} = 1$ and $B_{12} = D_1 = D_2 = 0$ (this ensure the continuity of coefficients in the final construction). As in the previous steps, it is simpler to choose first v and then construct the coefficients accordingly.

Let us define $\chi_1(t) = \int_0^t \chi(s)\, ds$. Then we have $\chi_1(t) = t + \chi_1(1) - 1$ in a neighborhood of $[1, \infty)$ and $\chi_1(t) = 0$ in a neighborhood of $(-\infty, 0]$. The solution is

$$v = \tilde{c} \cos \lambda' y \exp\left(-\lambda' \rho t - \lambda'(1 - \rho) a \chi_1 \left(\frac{t - 4a}{a}\right)\right).$$

The coefficients are

$$B_{11} = B_{22} = 1, B_{12} = D_1 = 0 \text{ and}$$

$$D_2 = \frac{\partial_t^2 v}{\partial_y^2 v} - 1 = \frac{\partial_t^2 v}{\lambda'^2 v} - 1 =$$

$$= \left(\rho + (1 - \rho)\chi \left(\frac{t - 4a}{a}\right)\right)^2 - \frac{1 - \rho}{a\lambda'} \chi' \left(\frac{t - 4a}{a}\right) - 1. \tag{15}$$

We will eliminate now one of our parameters. The constant ρ is very sensitive in our construction; in fact $1 - \rho^2$ is the order of magnitude of the coefficient D_2. In the steps 2 and 4 there is an exponential factor in $\tilde{\chi}(t)$ and in $\tilde{\tilde{\chi}}(t)$ which will manage to make the coefficients B_{ij} (more precisely, $B_{ij} - \delta_{ij}$) small at little expense. Therefore, since we have the restriction $\rho < \frac{\lambda}{\lambda'}$, which gives $1 - \rho^2 > 1 - (\lambda/\lambda')^2$, we cannot do better (modulo a multiplicative constant) than choosing $\rho = (\lambda/\lambda')^2$. We have then $1 - \rho^2 \approx 2(1 - (\lambda/\lambda')^2)$ for λ/λ' close to 1. In order to keep formulas to a reasonable complexity we will continue to use the constant ρ, substituting λ^2/λ'^2 to it when needed.

We can express the solution in a single formula:

$$v_{a,\lambda,\lambda'}(t,x,y)=\chi\left(\frac{4a-t}{a}\right)e^{-\lambda t}\cos\lambda x+$$

$$+\tilde{c}\chi\left(\frac{t-a}{a}\right)e^{-\frac{\lambda^2}{\lambda'}t-\left(1-\left(\frac{\lambda}{\lambda'}\right)^2\right)\lambda' a\chi_1\left(\frac{t-4a}{a}\right)}\cos\lambda' y. \tag{16}$$

Let us notice here that

$$v_{a,\lambda,\lambda'}(t,x,y)=\begin{cases}e^{-\lambda t}\cos\lambda x & \text{in a neighborhood of } 0,\\ \alpha(a,\lambda,\lambda')\,e^{-\lambda'(t-5a)}\cos\lambda' y & \text{in a neighborhood of } 5a.\end{cases} \tag{17}$$

The constant $\alpha(a,\lambda,\lambda')$ is given by (8) and (16) with $t=5a$:

$$\alpha(a,\lambda,\lambda')=e^{-5a(\lambda+\lambda^2/\lambda')/2-(1-\lambda^2/\lambda'^2)\lambda' a\chi_1(1)}\le e^{-5a\lambda/2}. \tag{18}$$

Estimates for the derivatives. We compute now the size of the derivatives of v and B_{ij} constructed above. For D_i only the first order derivative is needed in the proof of Theorem 1, and we give a bound for it.

Let k, l and m be three natural numbers, $k+l+m>0$. Then during the second step, $B_{11}=1+\tilde{B}$, where $\tilde{B}$ is given by (11) and we have

$$\partial_t^k\partial_x^l\partial_y^m B_{11}=\partial_t^k\partial_x^l\partial_y^m\tilde{B}=$$

$$=\partial_t^k\tilde{\chi}(t)\,\partial_x^l\partial_y^m\frac{\cos\lambda' y\cos\lambda x}{\lambda^2}-\partial_t^k\tilde{\chi}(t)\,\tilde{c}\,e^{t(\lambda-\lambda'\rho)}\chi\left(\frac{t-a}{a}\right)\partial_x^l\partial_y^m\frac{2\sin^2\lambda' y}{\lambda^2}. \tag{19}$$

The k-th derivative of $\tilde{\chi}$ is (see (9))

$$\partial_t^k\tilde{\chi}(t)=\tilde{c}\sum_{j=0}^{k}\binom{k}{j}\partial_t^j e^{(\lambda-\rho\lambda')t}\partial_t^{k-j}\left(\frac{1}{a^2}\chi''\left(\frac{t-a}{a}\right)-\frac{2\rho\lambda'}{a}\chi'\left(\frac{t-a}{a}\right)\right),$$

and its absolute value is bounded by

$$\tilde{c}e^{(\lambda-\rho\lambda')t}\sum_{j=0}^{k}\binom{k}{j}(\lambda-\rho\lambda')^j\times$$

$$\times\left(\frac{1}{a^{k-j+2}}\left|\chi^{(k-j+2)}\left(\frac{t-a}{a}\right)\right|+\frac{2\rho\lambda'}{a^{k-j+1}}\left|\chi^{(k-j+1)}\left(\frac{t-a}{a}\right)\right|\right).$$

Let us set $C_{\chi,k}=\sup_{\substack{i\le k\\ t\in\mathbf{R}}}|\chi^{(i)}(t)|$. Using $\lambda>\rho\lambda'$ and recalling that $\tilde{c}=e^{-5a(\lambda-\rho\lambda')/2}$, we infer that

$$\tilde{c}\, e^{t(\lambda - \rho\lambda')} \le e^{-a(\lambda - \rho\lambda')/2} \text{ for any } f \in [a, 2a].$$

Now we use $\lambda > 1/a$ and obtain:

$$|\,\partial_t^k \tilde{\chi}\,| \le \tilde{c}\, e^{(\lambda - \rho\lambda')t} \sum_{j=0}^{k} \binom{k}{j} (\lambda - \rho\lambda')^j (1/a)^{k-j} \left(\frac{1}{a^2} C_{\chi, k+2} + \frac{2\rho\lambda'}{a} C_{\chi, k+1} \right) \le$$

$$\le \tilde{c}\, e^{(\lambda - \rho\lambda')t} \left(\lambda + \frac{1}{a}\right)^k 3\lambda^2 C_{\chi, k+2} \le e^{-a(\lambda - \rho\lambda')/2} \cdot 3 \cdot 2^k C_{\chi, k+2} \lambda^{k+2}. \quad (20)$$

The same kind of computation will give

$$\left|\, \partial_t^k \tilde{\chi}(t)\, \tilde{c}\, e^{(\lambda - \lambda'\rho)t} \chi\left(\frac{t-a}{a}\right) \right| =$$

$$= \left| \tilde{c}^2 \sum_{i+j+h=k} \binom{k}{i\,j\,h} \partial_t^i e^{2(\lambda - \rho\lambda')t} \partial_t^j \left(\frac{1}{a^2} \chi''\left(\frac{t-a}{a}\right) - \frac{2\rho\lambda'}{a} \chi'\left(\frac{t-a}{a}\right) \right) \partial_t^h \chi\left(\frac{t-a}{a}\right) \right| \le$$

$$\le \tilde{c}^2 e^{2(\lambda - \rho\lambda')t} \sum_{i+j+h=k} \binom{k}{i\,j\,h} (2\lambda - 2\rho\lambda')^i (1/a)^j \times$$

$$\times \left(\frac{1}{a^2} C_{\chi, k+2} + \frac{2\rho\lambda'}{a} C_{\chi, k+1} \right) (1/a)^h C_{\chi, k} \le$$

$$\le \tilde{c}^2 e^{2(\lambda - \rho\lambda')t} \left(2\lambda + \frac{2}{a}\right)^k \cdot 3 \cdot \lambda^2 C_{\chi, k+2} C_{\chi, k} \le$$

$$\le e^{-a(\lambda - \rho\lambda')} C^2_{\chi, k+2} \cdot 3 \cdot 4^k \lambda^{k+2}. \quad (21)$$

Now we can estimate the derivatives of B_{11} (see (19)), using (20) and (21):

$$|\,\partial_t^k \partial_x^l \partial_y^m B_{11}(t, x, y)\,| \le$$

$$\le e^{-a(\lambda - \rho\lambda')/2} \cdot 3 \cdot 2^k C_{\chi, k+2} \lambda^{k+2} \frac{\lambda^l \lambda'^m}{\lambda^2} +$$

$$+ e^{-a(\lambda - \rho\lambda')} C^2_{\chi, k+2} \cdot 3 \cdot 4^k \lambda^{k+2} \frac{2^{m+1} \lambda'^m}{\lambda^2} \le$$

$$\le e^{-a(\lambda - \rho\lambda')/2} C'_{\chi, k, m} \lambda^{k+l} \lambda'^m. \quad (22)$$

Here the constant $C'_{\chi,k,m}$ depends only on χ, k and m. For the coefficient B_{12} the computation is simpler and we obtain in view of (10) and using the estimate (20) and $\lambda' > \lambda$:

$$| \partial_t^k \partial_x^l \partial_y^m B_{12}(t,x,y) | \le e^{-a(\lambda-\rho\lambda')/2} \cdot 3 \cdot 2^k C_{\chi,k+2} \lambda^{k+2} 2 \frac{\lambda^l \lambda'^m}{\lambda\lambda'} \le$$

$$\le e^{-a(\lambda-\rho\lambda')/2} C''_{\chi,k,m} \lambda^{k+l} \lambda'^m. \tag{23}$$

For the fourth step the estimate is similar. We use that

$$\frac{e^{t(\rho\lambda'-\lambda)}}{\widetilde{c}} \le e^{-a(\lambda-\rho\lambda')/2} \text{ for any } t \in [3a, 4a],$$

and obtain in view of (12) that

$$| \partial_t^k \tilde{\tilde{\chi}}(t) | \le e^{-a(\lambda-\rho\lambda')/2} \cdot 3 \cdot 2^k C_{\chi,k+2} \lambda^{k+2}.$$

The computation goes in the same way, and we obtain that B_{22} satisfies (22) (replace B_{11} by B_{22} and $[a, 2a]$ by [3a, 4a]) and B_{12} satisfies (23) for any t in [3a, 4a] (and any $x, y \in \mathbf{R}$). Since $B_{ij} = \delta_{ij}$ during the first, third and fifth step, we conclude from the relations (22) and (23), that we have for any $t \in [0, 5a]$:

$$| \partial_t^k \partial_x^l \partial_y^m B_{ij}(t,x,y) | \le e^{-a(\lambda-\rho\lambda')/2} C'''_{\chi,k,l,m} \lambda^{k+l} \lambda'^m. \tag{24}$$

Now we turn to the derivatives of v. We have from (16):

$$\partial_t^k \partial_x^l \partial_y^m v = \partial_t^k \chi\left(\frac{4a-t}{a}\right) e^{-t\lambda} \partial_x^l \partial_y^m \cos\lambda x +$$

$$+ \partial_t^k \chi\left(\frac{t-a}{a}\right) e^{-\rho\lambda' t - (1-\rho)\lambda' a \chi_1(t/a-4)} \partial_x^l \partial_y^m \cos\lambda' y. \tag{25}$$

We take care first of the t derivatives. Using that $t \ge 0$:

$$| \partial_t^k \chi(4 - t/a) e^{-t\lambda} | \le$$

$$\le e^{-t\lambda} \sum_{j=0}^{k} \binom{k}{j} (1/a)^j C_{\chi,j} \lambda^{k-j} \le e^{-t\lambda} (2\lambda)^k C_{\chi,k} \le (2\lambda)^k C_{\chi,k}. \tag{26}$$

By induction we prove the existence of a constant $\widetilde{C}_{\chi,k}$, depending only on χ and k, such that

$$| \partial_t^k e^{-\rho\lambda' t - (1-\rho)\lambda' a \chi_1(t/a-4)} | \le \widetilde{C}_{\chi,k} \lambda'^k. \tag{27}$$

This is true for $k = 0$ since the exponent is negative. We prove that if (27) holds for $k = 0, 1, \dots, m$, then it holds also, for a certain $\widetilde{C}_{\chi, m+1}$, for $k = m + 1$. Indeed,

$$| \partial_t^{m+1} e^{-\rho\lambda' t - (1-\rho)\lambda' a\chi_1(t/a - 4)} | =$$

$$= | \partial_t^m (-\rho\lambda' - (1-\rho)\lambda' \chi(t/a - 4)) e^{-\rho\lambda' t - (1-\rho)\lambda' a\chi_1(t/a - 4)} | \le$$

$$\le \lambda' \sum_{j=0}^{m} \binom{m}{j} \left| \left(\partial_t^j (-\rho - (1-\rho)\chi(t/a - 4))\right) \partial_t^{m-j} e^{-\rho\lambda' t - (1-\rho)\lambda' a\chi_1(t/a - 4)} \right| \le$$

$$\le \lambda' \sum_{j=0}^{m} \binom{m}{j} | (1/a)^j C_{\chi, j} \widetilde{C}_{\chi, m-j} \lambda'^{m-j} | \le \widetilde{C}_{\chi, m+1} \lambda'^{m+1}.$$

We used $\lambda' > 1/a$. Applying (27), we obtain

$$| \partial_t^k \chi(t/a - 1) e^{-\rho\lambda' t - (1-\rho)\lambda' a\chi_1(t/a - 4)} | \le$$

$$\le \lambda' \sum_{j=0}^{k} \binom{k}{j} | (1/a)^j C_{\chi, j} \widetilde{C}_{\chi, k-j} \lambda'^{k-j} | \le \widetilde{\widetilde{C}}_{\chi, k} \lambda'^k. \tag{28}$$

Using (25), (26) and (28), we conclude that

$$| \partial_t^k \partial_x^l \partial_y^m v | \le \mathring{C}_{\chi, k} \lambda'^{k+m} \lambda^l. \tag{29}$$

There is left to estimate the derivative of D_i. The function D_1 is identically 0, and D_2 is constant during the second, the third and the fourth step (i.e., on $[a, 4a]$). We have, in view of (6) and (15):

$$| \partial_t D_2 | \le C_{\chi, 1} (1 - \rho^2)/a \le 2C_{\chi, 1} (1 - \rho)/a$$

for any $t \in [0, a]$,

$$| \partial_t D_2 | =$$

$$= \left| \frac{2\rho(1-\rho)}{a} \chi'\left(\frac{t - 4a}{a}\right) + \frac{2(1-\rho)^2}{a} \chi'\left(\frac{t-4a}{a}\right) \chi\left(\frac{t-4a}{a}\right) - \frac{(1-\rho)}{a^2\lambda'} \chi''\left(\frac{t-4a}{a}\right) \right| \le$$

$$\le (1-\rho)(2C_{\chi, 1}/a + 2C_{\chi, 1} C_{\chi, 0}/a + C_{\chi, 2}/a) \le 5C_{\chi, 2}(1-\rho)/a$$

for any $t \in [4a, 5a]$, and we can conclude that

$$| \partial_t D_i | \le 5C_{\chi, 2}(1 - (\lambda/\lambda')^2)/a \tag{30}$$

for any $t \in [0,5a]$.

Boundary conditions. The function u satisfies the Neumann boundary condition for the equation (1) in the open set $\Omega \subset \mathbf{R}^n$ if and only if

$$\sum_{i,j=1}^{n} n_i a_{ij} \partial_j u(x) = 0,$$

for any $x \in \partial\Omega$, where $(n_1, \dots, n_n)$ is the normal vector to $\partial\Omega$.

We want our function v to satisfy this condition for the equation (5), seen in the variables x and y, in the open set $(0, 2\pi)\times(0, 2\pi)$. In this case, the above relation reads:

$$(B_{11} + D_1)\,\partial_x v + B_{12}\,\partial_y v = 0 \text{ on } \{0, 2\pi\}\times[0, 2\pi], \tag{31}$$

$$B_{12}\,\partial_y v + (B_{22} + D_2)\,\partial_x v = 0 \text{ on } [0, 2\pi]\times\{0, 2\pi\}. \tag{32}$$

We have

$$\partial_x v = \chi(4 - t/a)e^{-t\lambda}(-\lambda \sin \lambda x),$$

$$\partial_y v = \tilde{c}\chi(t/a - 1)e^{-t\lambda' - (1-\rho)\lambda' a \chi(t/a - 4)}(-\lambda' \sin \lambda' y).$$

Since B_{12} is a multiple of $\sin \lambda x \sin \lambda' y$ (see (10) and (13)), the conditions

$$\lambda \in \mathrm{N},\ \lambda' \in \mathrm{N} \tag{33}$$

are sufficient in order to ensure the boundary condition (31) and (32). These relations will imply that u, b_{ij} and d_i constructed below fulfill condition (iv) of Theorem 1. They satisfy also the periodicity condition (iii).

P r o o f o f T h e o r e m 1. Let $\{a_k\}_{k\geq 1}$ and $\{\lambda_k\}_{k\geq 1}$ be two sequences of positive numbers. We will suppose

$$\sum_{j=1}^{\infty} a_j < \infty,\ 1/a_k < \lambda_k < \lambda_{k+1}. \tag{34}$$

We denote $T_k = \sum_{j=1}^{k-1} a_j$ for $k \geq 1$ and $T = \sum_{j=1}^{\infty} a_j$. The sequence $\{\rho_k\}_{k\geq 1}$ is defined by $\rho_k = \lambda_k^2/\lambda_{k+1}^2$. We postpone as much as we can the choice of these sequences, in order to derive first all the conditions they have to fulfill. We shall use the indices a, λ, λ' for the functions B_{ij} and D_i, with $i, j = 1, 2$ (similarly to (16)), since we will use them for different values of these parameters. Let $k_0 > 0$ be an even natural number, to be chosen later. We are ready for the definition of the functions u, b_{ij} and d_i.

$$u(t,x,y) = \begin{cases} e^{-(t-T_{k_0})\lambda_{k_0}} \cos \lambda_{k_0} x & \text{for all } t \in (-\infty, T_{k_0}], \\ c_k v_{a_k, \lambda_k, \lambda_{k+1}}(t - T_k, x, y) & \text{for } k \text{ even} \\ c_k v_{a_k, \lambda_k, \lambda_{k+1}}(t - T_k, y, x) & \text{for } k \text{ odd} \end{cases} \Bigg\} \forall t \in [T_k, T_{k+1}], \forall k \geq k_0, \\ 0 & \text{for all } t \in [T, \infty). \end{cases} \tag{35}$$

Here c_k are constants which ensure the continuity (and therefore, the smoothness) of u. They are defined by the relations

$$c_{k_0} = 1,$$

$$\frac{c_{k+1}}{c_k} = \alpha\,(a_k, \lambda_k, \lambda_{k+1}),$$

where $\alpha(a, \lambda, \lambda')$ is defined by the relation (18). We have therefore (see (18)):

$$c_k \leq \exp\left(-\frac{5}{2} \sum_{j=k_0}^{k-1} a_j \lambda_j \right). \tag{36}$$

The coefficients are

$$b_{ij}(t,x,y) = \begin{cases} \delta_{ij} & \text{for any } t \notin [T_{k_0}, T), \\ B_{ij\, a_k, \lambda_k, \lambda_{k+1}}(t - T_k, x, y) & \text{for } t \in [T_k, T_{k+1}] \text{ with } k \text{ even}, \\ B_{\underline{j}\,\underline{i}\, a_k, \lambda_k, \lambda_{k+1}}(t - T_k, y, x) & \text{for } t \in [T_k, T_{k+1}] \text{ with } k \text{ odd}, \end{cases}$$

for all $i, j = 1, 2$ with $i \leq j$, where $\underline{i} = 3 - i$ and $\underline{j} = 3 - j$. This inversion is necessary, since the derivatives with respect to x and y — and therefore the coefficients — switch their roles in the odd intervals. The singular coefficients are defined in a similar manner:

$$d_i(t,x,y) = \begin{cases} 0 & \text{for any } t \notin [T_{k_0}, T), \\ D_{i\, a_k, \lambda_k, \lambda_{k+1}}(t - T_k, x, y) & \text{for } t \in [T_k, T_{k+1}] \text{ with } k \text{ even}, \\ D_{\underline{i}\, a_k, \lambda_k, \lambda_{k+1}}(t - T_k, y, x) & \text{for } t \in [T_k, T_{k+1}] \text{ with } k \text{ odd}. \end{cases}$$

The above u, b_{ij} and d_i fulfill the equation (3): indeed, they are obtained by simple changes of variables (the translation $t \to T_k + t$ and the symmetry which reverts the roles of x and y) from the functions satisfying (5).

Notice that $B_{ija,\lambda,\lambda'} = \delta_{ij}$ for t in a neighborhood of 0 or in a neighborhood of $5a$, and therefore b_{ij} are smooth in $\mathbf{R} \setminus \{T\} \times \mathbf{R}^2$. In order to obtain that b_{ij} are smooth at $t = T$ too, it is enough that all their derivatives are continuous and have the limit 0 as $t \uparrow T$. In view of (24), we have for any $i, j = 1, 2$:

$$\sup_{\substack{t \in [T_k, T_{k+1}] \\ x, y \in \mathbf{R}}} | \partial_t^p \partial_x^l \partial_y^m b_{ij}(t, x, y) | \le C'''_{\chi, p, l, m}\, e^{-a(\lambda_k - \lambda_k^2/\lambda_{k+1})/2} \lambda_k^{p+l} \lambda_{k+1}^m,$$

and due to the monotony of $\{\lambda_k\}$ the following condition ensures that b_{ij} are smooth on $\mathbf{R}^3$:

$$\lim_{k \to \infty} e^{-a_k(\lambda_k - \lambda_k^2/\lambda_{k+1})/2} \lambda_{k+1}^m = 0 \text{ for any } m \in \mathbf{N}. \tag{37}$$

Remark that if we suppose d_i continuous then $\lim_{k \to \infty} (1 - \rho_k^2) = 0$, since d_i takes the value $(1 - \rho_k^2)$ on a subset of $[T_k, T_{k+1}]$, for $i = 2$ for even k and $i = 1$ for odd k, and $d_i = 0$ for $t \ge T$ for $i = 1,2$. This implies that $\rho_k \to 1$ and since $\rho_k = \lambda_k^2/\lambda_{k+1}^2$, we have

$$\lim_{k \to \infty} \lambda_k/\lambda_{k+1} = 1. \tag{38}$$

For the smoothness of u we use the relation (29), and obtain

$$| \partial_t^p \partial_x^l \partial_y^m u(t, x, y) | \le c_k \mathring{C}_{\chi, p} \lambda_k^{p+l} \lambda_{k+1}^m \quad \forall k \ge k_0, \forall t \in [T_k, T_{k+1}], \forall x, y \in \mathbf{R},$$

and in view of (36) a sufficient condition for the smoothness of u is

$$\lim_{k \to \infty} \exp\left(-\frac{5}{2} \sum_{j=k_0}^{k-1} a_j \lambda_j \right) \lambda_{k+1}^m = 0 \quad \text{for any } m \in \mathbf{N}. \tag{39}$$

Due to the relation (38), we can replace in the limit above λ_{k+1}^m by λ_k^m or, equivalently, take the sum under exponential from k_0 to k. We have

$$-\frac{5}{2} \sum_{j=k_0}^{k} a_j \lambda_j \le -a_k \lambda_k/2 \le -a_k(\lambda_k - \lambda_k^2/\lambda_{k+1})/2,$$

and therefore (39) is a consequence of (37). Since we will put conditions on λ_k and a_k that ensure the continuity of d_1 and d_2 (hence (38) hold) we will omit the condition (39).

Continuity of d_i. We will prove that for $i \in \{1, 2\}$ we have

$$| d_i(t_1) - d_i(t_2) | \leq$$
$$\leq 10C_{\chi,2} \sup_{k \geq k_0} ((1 - \lambda_k^2/\lambda_{k+1}^2) \min(5, |t_1 - t_2|/a_k)), \forall t_1, t_2 \in \mathbf{R}. \tag{40}$$

In order to do so, we show that for any t_1 and t_2 there is a $k \geq k_0$ such that

$$| d_i(t_1) - d_i(t_2) | \leq 10C_{\chi,2}(1 - \lambda_k^2/\lambda_{k+1}^2) \min(5, |t_1 - t_2|/a_k). \tag{41}$$

Since $\bigcup_{k=k_0}^{\infty} [T_k, T_{k+1}] = [T_{k_0}, T)$, there are three cases to treat:

(a) There is a $k \geq k_0$ such that $t_1, t_2 \in [T_k, T_{k+1}]$.

(b) One of the t_i belongs to $\mathbf{R} \setminus [T_{k_0}, T)$.

(c) $t_1 \in [T_{k_1}, T_{k_1+1}]$ and $t_2 \in [T_{k_2}, T_{k_2+1}]$, with $k_1 \neq k_2$.

Case (a). Using the theorem of Cauchy, and the upper bound of the derivative of d_i given by (30) we obtain:

$$| d_i(t_1) - d_i(t_2) | \leq |t_1 - t_2| 5C_{\chi,2}(1 - \lambda_k^2/\lambda_{k+1}^2)/a_k.$$

Using further that

$$t_1, t_2 \in [T_k, T_{k+1}] \Rightarrow |t_1 - t_2| \leq T_{k+1} - T_k = 5a_k,$$

we obtain

$$| d_i(t_1) - d_i(t_2) | \leq 5C_{\chi,2}(1 - \lambda_k^2/\lambda_{k+1}^2) \min(5, |t_1 - t_2|/a_k).$$

Case (b). Suppose that $t_1 \notin [T_{k_0}, T)$. Then $d_i(t_1) = 0$. If t_2 is also outside this interval, then $d_i(t_2) = d_i(t_1) = 0$ and there is nothing to prove. So, we may suppose that $t_2 \in [T_k, T_{k+1}]$, with $k \geq k_0$. Then one of T_k and T_{k+1} (let us denote it by t_1') must lie between t_1 and t_2 (or equal t_2). Then $|t_1 - t_2| \geq |t_1' - t_2|$ and since $d_i(T_k) = d_i(T_{k+1}) = 0$, we have $d_i(t_1') = 0 = d_i(t_1)$. Applying the case (a) to t_1' and t_2, we obtain:

$$|d_i(t_1) - d_i(t_2)| = |d_i(t_1') - d_i(t_2)| \leq 5C_{\chi,2}(1 - \lambda_k^2/\lambda_{k+1}^2) \min(5, |t_1' - t_2|/a_k) \leq$$

$$\leq 5C_{\chi,2}(1 - \lambda_k^2/\lambda_{k+1}^2) \min(5, |t_1 - t_2|/a_k).$$

Case (c). The method is similar to the one used in case (b). Suppose $t_1 \in [T_{k_1}, T_{k_1+1}]$ and $t_2 \in [T_{k_2}, T_{k_2+1}]$ with $k_1 \neq k_2$. By symmetry we may suppose that $t_1 < t_2$, hence $k_1 < k_2$. Let $t_1' = T_{k_1+1}$ and $t_2' = T_{k_2}$. Then we have $d_i(t_j') = 0$ for $j = 1,2$ and

$$|d_i(t_1) - d_i(t_2)| \leq |d_i(t_1)| + |d_i(t_2)| = |d_i(t_1) - d_i(t_1')| + |d_i(t_2) - d_i(t_2')| \leq$$

$$\leq 5C_{\chi,2}(1 - \lambda_{k_1}^2/\lambda_{k_1+1}^2) \min(5, |t_1 - t_1'|/a_{k_1}) +$$

$$+ 5C_{\chi,2}(1 - \lambda_{k_2}^2/\lambda_{k_2+1}^2) \min(5, |t_2 - t_2'|/a_{k_2}) \leq$$

$$\leq 10C_{\chi,2} \max_{j=1,2} \left((1 - \lambda_{k_j}^2/\lambda_{k_j+1}^2) \min(5, |t_j - t_j'|/a_{k_j})\right).$$

The proof of (41) is complete.

We turn back to Theorem 1, condition (v). In order to obtain Hölder continuous coefficients of any order $0 < \alpha < 1$ our sequences $\{a_k\}$ and $\{\lambda_k\}$ must satisfy (in view of (41)):

$$\forall \alpha \in (0,1)\ \exists C > 0 \text{ s.t. } (1 - \lambda_k^2/\lambda_{k+1}^2) \min(5, |t|/a_k) \leq Ct^\alpha, \forall t \geq 0, \forall k \geq k_0.$$

Since the r.h.s. is concave and increasing, while the l.h.s. is linear on $[0, 5a_k]$ and constant on $[5a_k, \infty]$ and is continuous, it is enough to check the inequality at $t = 5a_k$. We obtain in this way the condition:

$$\forall \alpha < 1\ \exists C > 0 \text{ s.t. } (1 - \lambda_k^2/\lambda_{k+1}^2) \leq Ca_k^\alpha, \forall k \geq k_0.$$

Sumarising, we need two sequences $\{a_k\}_{k \geq 1}$ and $\{\lambda_k\}_{k \geq 1}$ which must satisfy:

(α) $\sum_1^\infty a_k < \infty$ (the constuction is to be achieved in finite time),

(β) $1/a_k < \lambda_k < \lambda_{k+1}$ (technical condition),

(γ) $\lambda_k \in \mathbb{N}$ (in order to ensure the 2π-periodicity and the boundary conditions),

(δ) $\lim_{k\to\infty} e^{-a_k(\lambda_k - \lambda_k^2/\lambda_{k+1})/2}\lambda_{k+1}^m = 0$ for any $m \in \mathbb{N}$ (to ensure the smoothness of b_{ij} and implicitly that of u).

(ε) $\forall\, \alpha < 1\ \exists C > 0$ s.t. $(1 - \lambda_k^2/\lambda_{k+1}^2) \le C a_k^\alpha$ for any $k \ge k_0$ (the Hölder continuity of d_1, d_2, of any order $\alpha < 1$).

The following sequences satisfy all these conditions:

$$\lambda_k = (k+1)^3,$$

$$a_k = (k \ln^2(k+1))^{-1}.$$

The condition (α) is easy to prove, and also (β), and (γ). We have for (δ):

$$e^{-a_k(\lambda_k - \lambda_k^2/\lambda_{k+1})/2}\lambda_{k+1}^m = e^{-\frac{(k+1)^3 - (k+1)^6/(k+2)^3}{k\ln^2(k+1)}}(k+2)^{3m} =$$

$$= e^{-(k+1)^3\frac{3k^2+9k+7}{(k+2)^3 k\ln^2(k+1)}}(k+2)^{3m}.$$

The exponent is assimptotically

$$-(k+1)^3\frac{3k^2+9k+7}{(k+2)^3 k\ln^2(k+1)} = -(1 + O(1/k))3k\ln^{-2}(k+1),$$

and therefore the whole above expression has limit zero as $k \to \infty$. For the condition (ε) we have

$$(1 - \lambda_k^2/\lambda_{k+1}^2) = (1 - (k+1)^6/(k+2)^6) = \frac{6k^5 + 45k^4 + \dots + 63}{(k+2)^6} \le Ck^{-1}, C > 0,$$

and since $\lim_{k\to\infty} k^{-1+\alpha}\ln^{-2\alpha}(k+1) = 0$ for any $\alpha < 1$, we have

$$\forall\,\alpha < 1\ \exists C_\alpha > 0 \text{ such that } (1 - \lambda_k^2/\lambda_{k+1}^2) \le C_\alpha k^{-\alpha}\ln^{-2\alpha}(k+1).$$

It remains to choose k_0. We must ensure the uniform ellipticity of the equation (3), as required in the point (vi) of the theorem. This is possible since the coefficients that we have constructed are uniformly continuous: d_i and $b_{ij} - \delta_{ij}$ have compact support in the t variable and are periodic in x and y. Now, passing from a k_0 to a bigger $\tilde{k}_0$ has the only effect that these function become zero for $t \in [T_{k_0}, T_{\tilde{k}_0}]$ and remain as they were for

$t \in [T_{\tilde{k}_0}, T]$. Since they tend uniformly to zero as $t \uparrow T$, we can choose k_0 such that $| d_i | \leq 1/18$ and $| b_{ij} - \delta_{ij} | \leq 1/18$ and then

$$\left\| \begin{pmatrix} b_{11} - 1 + d_1 & b_{12} \\ b_{12} & b_{22} - 1 + d_2 \end{pmatrix} \right\| \leq 6 \times 1/18 = 1/3,$$

and we obtain

$$1 - 1/3 \leq \begin{pmatrix} b_{11} + d_1 & b_{12} \\ b_{12} & b_{22} + d_2 \end{pmatrix} \leq 1 + 1/3.$$

The proof is complete.

The construction for the parabolic problem (4) is similar to the one for the elliptic equation and will be not done here.

R e m a r k. From the condition $\lambda_k \to \infty$ we infer that

$$\lim_{k \to \infty} \lambda_k^{-4} = \lambda_{k_0}^{-4} \prod_{k = k_0}^{\infty} \frac{\lambda_k^4}{\lambda_{k+1}^4} = \lambda_{k_0}^{-4} \prod_{k = k_0}^{\infty} \rho_k^2 = 0,$$

and since $\rho_k^2 \in (0, 1)$ for any k, we can pass to the infinite sum associated to the infinite product, and obtain from the above relation:

$$\sum_{k = k_0}^{\infty} (1 - \rho_k^2) = \infty.$$

Since in each of the intervals $[T_k, T_{k+1}]$ one of the functions d_1, d_2 takes the value $-(1 - \rho_k^2)$ and gets back to the value 0 at the end of the interval, the above relation implies that either d_1 or d_2 have unbounded variation. Thus we cannot obtain $W^{1,1}$ coefficients in the construction above.

Professor N. Lerner raised the problem of the refinement of the above result, considering the continuity moduli of the coefficients. He asked in particular whether the below result hold. The following corollary is actually a corollary of the *proof* of Theorem 1.

Corollary 1. *Let* $\omega : [0, \infty) \to [0, \infty)$ *be a continuous, non-decreasing and concave function such that* $\omega(0) = 0$ *and* $\omega(1) > 0$. *Suppose that*

$$\int_0^1 \frac{dt}{\omega(t)} < \infty. \qquad (42)$$

Then there exist u, b_{ij} and d_i, where $i, j = 1,2$, satisfying all the conditions of Theorem 1, and such that

$$| d_i(t_1) - d_i(t_2) | \le \omega(|t_1 - t_2|), \ \forall t_1, t_2 \in \mathbf{R}, \ i = 1,2. \qquad (43)$$

R e m a r k. If $f : \mathbf{R}^n \to \mathbf{R}$ then the modulus of continuity of f is by definition the function

$$\omega_f : [0, \infty) \to [0, \infty), \omega_f(t) = \sup_{|x_1 - x_2| \le t} | f(x_1) - f(x_2) |.$$

It is easy to prove that ω_f is non-decreasing and satisfy the relation

$$\omega_f'(\alpha t_1 + (1 - \alpha)t_2) \ge 1/2(\alpha\omega_f(t_1) + (1 - \alpha)\omega_f(t_2)), \forall t_1, t_2 \ge 0, \forall \alpha \in [0, 1]. \qquad (44)$$

This shows that there is a concave function $\tilde{\omega}_f$, more precisely,

$$\tilde{\omega}_f(t) = \sup_{0 \le t_1 < t < t_2} \frac{(t_2 - t)\,\omega(t_1) + (t - t_1)\,\omega(t_2)}{t_2 - t_1}$$

such that $\frac{1}{2}\tilde{\omega}_f \le \omega_f \le \tilde{\omega}_f$. It follows that the restriction to concave funstions ω in the above corollary does not affect the generality.

P r o o f o f C o r o l l a r y 1. We may suppose that

$$\omega(t) \le \sqrt{t}\,. \qquad (45)$$

Indeed, replacing ω by the function

$$\tilde{\omega}(t) = \min(\omega(t), \sqrt{t}),$$

the hypotheses of the corollary remains true: $\tilde{\omega}$ is continuous, non-decreasing, concave and

$$\int_0^1 \frac{dt}{\tilde{\omega}(t)} = \int_0^1 \max\left(\frac{1}{\omega(t)}, t^{-1/2}\right) dt \le \int_0^1 \frac{dt}{\omega(t)} + \int_0^1 t^{-1/2}\,dt < \infty.$$

We will make another choice of the constant k_0, of the sequences $\{a_k\}_{k\geq 1}$ and $\{\lambda_k\}_{k\geq 1}$ in the proof of Theorem 1, such that the conditions (α)-(δ) and the relation (43) are satisfied. We choose

$$\lambda_k = k^4.$$

Let

$$\delta_k \overset{\text{def}}{=} 1 - \lambda_k^2/\lambda_{k+1}^2 = \frac{(k+1)^8 - k^8}{(k+1)^8}.$$

We have to make some preparations in view of the construction of the sequence $\{a_k\}$. Let $a = \sup\{x \in [0,1] \,|\, \omega(x) < \omega(1)\}$. Since $\omega(0) = 0$ and $\omega(1) > 0$, we have $a \in (0,1]$. Then by continuity we have $\omega(a) = \omega(1)$ and the function $\omega : [0, a] \to [0, \omega(1)]$ is bijective. Indeed, suppose $0 \leq x < y \leq a$. Since ω is non-decreasing, $\omega(x) < \omega(a)$ by the definition of a. Using that ω is concave,

$$\omega(y) \geq \frac{(a-y)\,\omega(x) + (y-x)\,\omega(a)}{a-x} > \frac{(a-y)\,\omega(x) + (y-x)\,\omega(x)}{a-x} = \omega(x),$$

which proves that ω is strictly increasing on $[0, a]$. We put

$$a_k = 1/5\omega^{-1}(50C_{\chi,2}\delta_k), \text{ for any } k \geq k_0.$$

This requires that the argument of ω^{-1} lies in $[0, \omega(1)]$. To this end, we impose

$$50C_{\chi,2}\delta_{k_0} \leq \omega(1),$$

relation satisfied for k_0 big enough since $\delta_k \to 0$. Since $0 < a_{k_1} \leq a_{k_0}$ for any $k_1 \geq k_0$, we obtain from the concavity of ω:

$$50C_{\chi,2}\delta_{k_1} = \omega(5a_{k_1}) \geq \frac{(5a_{k_0} - 5a_{k_1})\omega(0) + 5a_{k_1}\,\omega(5a_{k_0})}{5a_{k_0}} = \frac{a_{k_1}}{a_{k_0}} 50C_{\chi,2}\delta_{k_0},$$

and we infer

$$\frac{a_{k_1}}{\delta_{k_1}} \leq \frac{a_{k_0}}{\delta_{k_0}} \quad \text{for all } k_1 \geq k_0. \tag{46}$$

Now we will check in order the conditions (α), (β), (γ) and (δ) stated at the end of the proof of Theorem 1.

We prove first that $\sum a_i < \infty$. Using the monotony of ω and then the relation (42):

$$\sum_{k=k_0}^{k_1} \frac{1}{\delta_k}(a_k - a_{k+1}) = 50C_{\chi,2} \sum_{k=k_0}^{k_1} \frac{1}{50C_{\chi,2}\delta_k}(a_k - a_{k+1}) =$$

$$= 10C_{\chi,2} \sum_{k=k_0}^{k_1} \frac{1}{\omega(5a_k)}(5a_k - 5a_{k+1}) \le 10C_{\chi,2} \sum_{k=k_0}^{k_1} \int_{5a_{k+1}}^{5a_k} \frac{dt}{\omega(t)} \le$$

$$\le 10C_{\chi,2} \int_0^{5a_{k_0}} \frac{dt}{\omega(t)} = M < \infty \quad \text{for any } k_1 \ge k_0 .$$

We will associate differently the terms in the first sum above, in order to obtain information about the series $\sum a_k$. We have

$$\sum_{k=k_0}^{k_1} \frac{1}{\delta_k}(a_k - a_{k+1}) = \sum_{k=k_0}^{k_1-1} \left(\frac{1}{\delta_{k+1}} - \frac{1}{\delta_k}\right) a_{k+1} + \frac{a_{k_0}}{\delta_{k_0}} - \frac{a_{k_1+1}}{\delta_{k_1}}.$$

and we obtain using (46):

$$\sum_{k=k_0}^{k_1-1} \left(\frac{1}{\delta_{k+1}} - \frac{1}{\delta_k}\right) a_{k+1} \le M - \frac{a_{k_0}}{\delta_{k_0}} + \frac{a_{k_1}}{\delta_{k_1}} \le M \text{ for any } k_1 \ge k_0 .$$

Since $\{\delta_k\}$ is decreasing, $\left(\frac{1}{\delta_{k+1}} - \frac{1}{\delta_k}\right) a_{k+1} > 0$ for any $k \ge k_0$. We obtain that the series

$$\sum_{k=k_0}^{\infty} \left(\frac{1}{\delta_{k+1}} - \frac{1}{\delta_k}\right) a_{k+1}$$

is convergent. It remains now to use the fact that

$$\lim_{k \to \infty} \left(\frac{1}{\delta_{k+1}} - \frac{1}{\delta_k}\right) = 1/8 \tag{47}$$

and the positivity of a_k to conclude that

$$\sum_{k=k_0}^{\infty} a_k < \infty .$$

In order to show that the relation (47) holds, we compute

$$\frac{1}{\delta_k} = \frac{(k+1)^8}{8k^7 + 28k^6 + O(k^5)} = \frac{1}{8}\frac{k^8 + 8k^7 + O(k^6)}{k^7 + 7/2k^5 + O(k^5)} = \frac{1}{8}(k + 9/2 + O(1/k)).$$

The proof of the condition (α) is complete.

Due to the relation (45), we have $\omega^{-1}(t) \ge t^2$ for $t \in [0, \omega(1)]$ and in particular $5a_k = \omega^{-1}(50C_{\chi,2}\delta_k) \ge (50C_{\chi,2}\delta_k)^2$ for any $k \ge k_0$. Since $\delta_k = 8/k + O(1/k^2)$, we obtain the existence of a $C > 0$ such that

$$a_k \ge Ck^{-2}. \tag{48}$$

Choosing k_0 big enough, we obtain $1/a_k < k^4 = \lambda_k$ for any $k \ge k_0$ and the condition (β) is fulfilled.

The condition (γ) is obviously satisfied: $\lambda_k = k^4 \in \mathbf{N}$.

We have from (48):

$$e^{-a_k(\lambda_k - \lambda_k^2/\lambda_{k+1})/2}\lambda_{k+1}^m \le e^{-Ck^{-2}k^4(1 - k^4/(k+1)^4)/2}(k+1)^{4m} \le .$$

$$\le e^{-Ck^2(4k^3/(k+1)^4)/2}(k+1)^{4m}.$$

The limit of the above expression is 0 as $k \to \infty$, since the exponent is $-2Ck(1 + O(1/k))$, hence the condition (δ) is satisfied.

It remains to prove the inequality (43). In order to do so it is enough to prove that

$$10C_{\chi,2}\sup_{k \ge k_0}((1 - \lambda_k^2/\lambda_{k+1}^2)\min(5, t/a_k)) \le \omega(t)$$

for any $t \in [0, \infty)$, since (43) is then a consequence of (40). We will prove the inequality for each $k \ge k_0$:

$$\omega(t) \ge 10C_{\chi,2}\delta_k \min(5, t/a_k).$$

We use the concavity of ω and the fact that it is non-decreasing. This implies that it is enough to prove the above inequality at the point $t = 5a_k$, where the r.h.s. passes from a linear function to a constant one. Indeed, suppose the inequality proved at $t = 5a_k$. Then it results, on one hand, by the monotony of ω, that the inequality holds in the interval $[5a_k, \infty)$. On the other hand, it is obviously true for $t = 0$ and from the concavity of ω it is true in the interval $[0, 5a_k]$. We have to check that

$$\omega(5a_k) \geq 10C_{\chi, 2}\delta_k \cdot 5,$$

or by the definition of a_k we have equality. The proof is complete.

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О контрпримере, связанном с единственностью продолжения для эллиптических уравнений в дивергентной форме

Н. Мандаке

Предложен пример дивергентного эллиптического дифференциального оператора трех переменных, у которого существует ненулевое решение, обращающееся в нуль на полупространстве. Коэффициенты уравнения удовлетворяют условию Гельдера с любым показателем $\alpha < 1$. Пример улучшает известный результат Миллера. Показано также, что существует возможность сделать коэффициенты в построенном примере более гладкими.

Про контрприклад, пов'язаний з єдиністю продовження для еліптичних рівнянь у дивергентній формі

М. Мандаке

Запропоновано приклад дивергентного еліптичного оператора трьох змінних, у якого є ненульовий розв'язок, що обертається у нуль на напівпросторі. Коефіцієнти рівняння задовольняють умові Гельдера з любим показником $\alpha < 1$. Приклад покращує відомий результат Міллера. Показано теж, що існує можливість зробити коефіцієнти у побудованому прикладі більш гладкими.