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# On two models of interacting Bose gas: Bogolyubov's model of superfluidity and Huang-Yang-Luttinger model

D.Ya. Petrina

*Institute of Mathematics National Academy of Sciences of Ukraine*
*3, Tereshchenkovskaj Ave., 252601, Kiev-4, Ukraine*

V.I. Gredzhuk

*Institute of Cibernetics National Academy of Sciences of Ukraine*
*4, Academician Glushkov Ave., 252207, Kiev, Ukraine*

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The equations for Green's functions are investigated and thermodynamically equivalent approximating Hamiltonians are derived for the model Hamiltonian of Bogolyubov's theory of superfluidity and the Huang–Yang–Luttinger (HYL) model Hamiltonian. The approximating Hamiltonians contain terms linear and quadratic in operators of creation and annihilation with momentum zero. On this basis, we prove that Bogolyubov's model describes superfluidity for certain values of the chemical potential, high densities, and low temperatures. It is shown that the expression for pressure derived in this paper for the HYL model coincides with that obtained earlier.

## Introduction

The problem of theoretical description of Bose condensation and superfluidity is a permanent challenge for mathematicians and physicists. It is generally accepted that the theory explaining these phenomena should be based on the Hamiltonian for bosons interacting via a pair potential

$$H_\Lambda = \sum_k a_k^* a_k (\varepsilon_k - \mu) + \frac{1}{2V(\Lambda)} \sum_{k_1, k_2, k_3, k_4} \delta_{k_1 + k_2, k_3 + k_4} \Phi(k_1 - k_3) a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} =$$

$$= H_{0,\Lambda} + H_{I,\Lambda}. \tag{1}$$

Here $\Lambda$ is a cube centered at the origin, $L$ is the length of its edge, $V(\Lambda) = L^3$ is its volume, the summation is carried out over quasimomenta $k$ that take the values

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$k = 2\pi n/L$, $n = (n_1, n_2, n_3)$ is a vector with integer components, $\varepsilon_k = k^2/2m$, $m$ is the mass of a boson, $\mu$ is a chemical potential, $a_k$ and $a_k^*$ are the operators of annihilation and creation of bosons satisfying canonical commutation relations, $\delta_{k_1, k_2}$ is the Kronecker symbol, and $\Phi(k)$ is the Fourier transform of a potential, $\Phi(k) = \overline{\Phi}(-k)$. Furthermore, we assume that periodic boundary conditions are given at the boundary of the cube $\Lambda$.

The general Hamiltonian (1) contains all information about boson systems but, except for quite general properties, hardly anything can be acquired from it. Therefore a very important problem is to extract from (1) a model Hamiltonian which would describe superfluidity being exactly solvable. Certainly, it is desirable to clarify in what sense a model Hamiltonian approximates the general one.

For this purpose, Bogolyubov [1] suggested the following model Hamiltonian:

$$H_{M,\Lambda} = \sum_k a_k^* a_k (\varepsilon_k - \mu) + \frac{1}{2V(\Lambda)} \sum_{k_1, k_2 \neq 0} \delta_{k_1 + k_2, 0} \Phi(k_1) a_{k_1}^* a_{k_2}^* a_0 a_0 +$$

$$+ \frac{1}{2V(\Lambda)} \sum_{k_1, k_3 \neq 0} \delta_{0, k_3 + k_4} \Phi(k_3) a_0^* a_0^* a_{k_3} a_{k_4} + \frac{1}{2V(\Lambda)} \sum_{k_1, k_3 \neq 0} \delta_{k_1, k_3} \Phi(0) a_{k_1}^* a_0^* a_{k_3} a_0 +$$

$$+ \frac{1}{2V(\Lambda)} \sum_{k_1, k_4 \neq 0} \delta_{k_1, k_4} \Phi(k_1) a_{k_1}^* a_0^* a_0 a_{k_4} + \frac{1}{2V(\Lambda)} \sum_{k_2, k_4 \neq 0} \delta_{k_2, k_4} \Phi(0) a_0^* a_{k_2}^* a_0 a_{k_4} +$$

$$+ \frac{1}{2V(\Lambda)} \sum_{k_2, k_3 \neq 0} \delta_{k_2, k_3} \Phi(k_3) a_0^* a_{k_2}^* a_{k_3} a_0 + \frac{1}{2V(\Lambda)} \Phi(0) a_0^* a_0^* a_0 a_0 . \tag{2}$$

The model Hamiltonian (2) is obtained from the general Hamiltonian (1) by neglecting the terms that contain more than two operators of annihilation and creation with nonzero momentum in the interaction Hamiltonian $H_{I,\Lambda}$. It was unknown then in what sense the model Hamiltonian (2) approximates the general Hamiltonian (1); this was the first postulate of the Bogolyubov theory of superfluidity.

The second Bogolyubov's postulate was that the operators $a_0/\sqrt{V(\Lambda)}$ and $a_0^*/\sqrt{V(\Lambda)}$ can be replaced by $c$-numbers, because, in the thermodynamic limit, they commute with all operators $a_k$ and $a_k^*$ and are thus multiples of the identity operator ($c$-numbers), provided that the representation of the canonical commutation relations is irreducible.

By replacing all the operators $a_0/\sqrt{V(\Lambda)}$ and $a_0^*/\sqrt{V(\Lambda)}$ in (2) by $c$-numbers, namely, $a_0/\sqrt{V(\Lambda)} = c$ and $a_0^*/\sqrt{V(\Lambda)} = c$, we obtain the approximating Hamiltonian

$$H_{\text{appr},\Lambda} = \sum_{k \neq 0} (\varepsilon_k - \mu + \Phi(0)c^2) a_k^* a_k + \frac{c^2}{2} \sum_{k \neq 0} \Phi(k) a_k^* a_{-k}^* + \frac{c^2}{2} \sum_{k \neq 0} \Phi(k) a_k a_{-k} +$$

$$+ c^2 \sum_{k \neq 0} \Phi(k) a_k^* a_k + \left( -\mu c^2 + \frac{1}{2}\Phi(0)c^4 \right) V(\Lambda). \tag{3}$$

Hamiltonian (3) can be diagonalized by using Bogolyubov's canonical $u$-$v$ transformation, its spectrum can be exactly determined and coincides with that postulated by Landau in his phenomenological theory of superfluidity.

At first sight, the replacement of the operators $a_0/\sqrt{V(\Lambda)}$ and $a_0^*/\sqrt{V(\Lambda)}$ by $c$-numbers seems to be reasonable and well justified. However a more detailed analysis leads to a serious doubt about the validity of this postulate. First, for every $k_0 \neq 0$, the operators $a_{k_0}/\sqrt{V(\Lambda)}$ and $a_{k_0}^*/\sqrt{V(\Lambda)}$ (and certain expressions containing these) also commute with the operators $a_k$ and $a_k^*$ in the thermodynamic limit and are thus $c$-numbers, provided that the representation of the canonical commutation relations is irreducible. In this connection, it is not clear why the operators $a_0/\sqrt{V(\Lambda)}$ and $a_0^*/\sqrt{V(\Lambda)}$ play such a distinguished role.

To clarify the last remark, we write Hamiltonian (2) for an infinite cube $\Lambda \nearrow \mathbf{R}^3$, $V(\Lambda) \to \infty$, using the well-known rules

$$\frac{(2\pi)^3}{V(\Lambda)} \sum_k \ldots \to \int \ldots dk, \quad \frac{V(\Lambda)}{(2\pi)^3} \delta_{k_1, k_2} \to \delta(k_1 - k_2),$$

$$\frac{\sqrt{V(\Lambda)}}{(2\pi)^{3/2}} a_k \to a(k), \quad \frac{\sqrt{V(\Lambda)}}{(2\pi)^{3/2}} a_k^* \to a^*(k). \tag{4}$$

Before passing to the case of an infinite cube, we rewrite the Hamiltonian $H_{\mathrm{M},\Lambda}$ (2) in the form, where the summation is carried out over all momenta including $k = 0$ (this step is necessary for passing to integrals of $a(k)$ and $a^*(k)$). We have

$$H_{\mathrm{M},\Lambda} = \sum_k a_k^* a_k (\varepsilon_k - \mu) + \frac{1}{2V(\Lambda)} \sum_{k_1, k_2} \delta_{k_1 + k_2, 0} \Phi(k_1) a_{k_1}^* a_{k_2}^* a_0 a_0 +$$

$$+\frac{1}{2V(\Lambda)}\sum_{k_1,k_3}\delta_{0,k_3+k_4}\Phi(k_3)a_0^*a_0^*a_{k_3}a_{k_4}+\frac{1}{2V(\Lambda)}\sum_{k_1,k_3}\delta_{k_1,k_3}\Phi(0)a_{k_1}^*a_0^*a_{k_3}a_0+$$

$$+\frac{1}{2V(\Lambda)}\sum_{k_1,k_4}\delta_{k_1,k_4}\Phi(k_1)a_{k_1}^*a_0^*a_0a_{k_4}+\frac{1}{2V(\Lambda)}\sum_{k_2,k_4}\delta_{k_2,k_4}\Phi(0)a_0^*a_{k_2}^*a_0a_{k_4}+$$

$$+\frac{1}{2V(\Lambda)}\sum_{k_2,k_3}\delta_{k_2,k_3}\Phi(k_3)a_0^*a_{k_2}^*a_{k_3}a_0-\frac{5}{2V(\Lambda)}\Phi(0)a_0^*a_0^*a_0a_0. \tag{5}$$

For $\Lambda = \mathbf{R}^3$ ($L = \infty$), in view of (4), we get ($V = V(R^3)$)

$$H_M = \int a^*(k)a(k)(\varepsilon(k)-\mu)dk+$$

$$+\frac{(2\pi)^3}{2V^2}\int\delta(k_1+k_2)\Phi(k_1)a^*(k_1)a^*(k_2)a(0)a(0)dk_1dk_2+$$

$$+\frac{(2\pi)^3}{2V^2}\int\delta(k_3+k_4)\Phi(k_3)a^*(0)a^*(0)a(k_3)a(k_4)dk_3dk_4+$$

$$+\frac{(2\pi)^3}{2V^2}\int\delta(k_1-k_3)\Phi(0)a^*(k_1)a^*(0)a(k_3)a(0)dk_1dk_3+$$

$$+\frac{(2\pi)^3}{2V^2}\int\delta(k_1-k_4)\Phi(k_1)a^*(k_1)a^*(k_1)a^*(0)a(0)a(k_4)dk_1dk_4+$$

$$+\frac{(2\pi)^3}{2V^2}\int\delta(k_2-k_3)\Phi(k_3)a^*(0)a^*(k_2)a(k_3)a(0)dk_2dk_3+$$

$$+\frac{(2\pi)^3}{2V^2}\int\delta(k_2-k_4)\Phi(0)a^*(0)a^*(k_2)a(0)a(k_4)dk_2dk_4-$$

$$-5\frac{(2\pi)^6}{2V^3}\Phi(0)a^*(0)a^*(0)a(0)a(0). \tag{6}$$

Recall that we consider an infinite cube and thus all the integrals divided by the volume in (6) are understood as the limits

$$\frac{1}{V}\int\dots dk=\lim_{V(\Lambda)\to\infty}\frac{1}{V(\Lambda)}\int_\Lambda\dots dk. \tag{7}$$

Let us examine a typical term, e.g., the second one, on the right-hand side of (6) and show how certain operator expressions turn into c-numbers in the thermodynamic limit and what problems arise in realizing this procedure. We have

$$\frac{(2\pi)^3}{2V^2}\int \delta(k_1 + k_2)\Phi(k_1)a^*(k_1)a^*(k_2)a(0)a(0)dk_1dk_2. \tag{8}$$

Obviously, we can consider the factor $1/V^2$ together with the operator $a(0)a(0)$ and replace the expression $(2\pi)^3 \frac{a(0)}{V}\frac{a(0)}{V}$ by a c-number. In this case, we obtain

$$\frac{c^2}{2}\int \delta(k_1 + k_2)\Phi(k_1)a^*(k_1)a^*(k_2)dk_1dk_2.$$

At the same time, there exists another possibility, namely, to join the factors $1/V$ to one of the operators $a(0)$ and the operator expression

$$\int \delta(k_1 + k_2)\Phi(k_1)a^*(k_1)a^*(k_2)dk_1dk_2,$$

respectively. The operator

$$\frac{1}{V}\int \delta(k_1 + k_2)\Phi(k_1)a^*(k_1)a^*(k_2)dk_1dk_2$$

commutes with all $a_k$ and $a_k^*$ and hence it is also a c-number; denote it by $c_1$. Thus interpreted, operator (8) takes the form $(2\pi)^{3/2}cc_1a(0)$.

We have encountered an ambiguity, since we have two factors $1/V$ and three operator expressions in (8) to which they may be joined. Note that, following the first scheme (joining these factors only to the operators $a(0)$ and $a^*(0)$), we obtain the approximating Hamiltonian (3). The detailed analysis carried out in [2, 3] showed that both the possibilities must be taken into account; in this case, the following thermodynamically equivalent operator corresponds to operator (8):

$$\frac{c^2}{2}\int \delta(k_1 + k_2)\Phi(k_1)a^*(k_1)a^*(k_2)dk_1dk_2 + (2\pi)^{3/2}cc_1a(0).$$

The thermodynamically equivalent approximating Hamiltonian which corresponds to Hamiltonian (6) has the form

$$H_{\text{appr}} = \int a^*(k)a(k)(\varepsilon(k) - \mu)dk + \frac{c^2}{2}\int \delta(k_1 + k_2)\Phi(k_1)a^*(k_1)a(k_2)dk_1dk_2 +$$

$$+\frac{c^2}{2}\int \delta(k_1+k_2)\Phi(k_1)a(k_1)a(k_2)dk_1dk_2 + c^2\Phi(0)\int a^*(k)a(k)dk +$$

$$+ c^2\int \Phi(k)\, a^*(k)a(k)dk + A(2\pi)^{3/2}a^*(0) + A(2\pi)^{3/2}a(0) + B, \tag{9}$$

where $A$ and $B$ are certain constants determined by averages of the operators $a(k)$ and $a^*(k)$.

The approximating Hamiltonian (9) was obtained by analysis of equations of a state (more precisely, the equations for temperature Green's functions, see Section 2). It follows from these equations that both possibilities of joining the two factors $1/V$ to the three operator expressions in term (8) are realized. This means that an argument based on the assumption that only the operators $a_0/\sqrt{V(\Lambda)}$ and $a_0^*/\sqrt{V(\Lambda)}$ commute with all operators $a_k$ and $a_k^*$ in the thermodynamic limit and are thus $c$-numbers, provided that the representation of the canonical commutation relations is irreducible, fails because the Hamiltonian $H_{\mathrm{M}}$ also contains other terms that possess this property and can be replaced by $c$-numbers. This problem is completely solved by using the equations for Green's functions, the analysis of which shows that, as has been mentioned above, all possible replacements of operator expressions by $c$-numbers are realized.

According to (4), Hamiltonian (9) is associated with the following Hamiltonian in the cube $\Lambda$:

$$H_{\mathrm{appr},\Lambda} = \sum_k a_k^* a_k(\varepsilon_k - \mu) + \frac{c^2}{2}\sum_{k_1,k_2} a_{k_1}^* a_{k_2}^* \Phi(k_1)\delta_{k_1+k_2,0} +$$

$$+\frac{c^2}{2}\sum_{k_1,k_2} a_{k_1}a_{k_2}\Phi(k_1)\delta_{k_1+k_2,0} + c^2\Phi(0)\sum_k a_k^*a_k +$$

$$+ c^2\sum_k a_k^*a_k\,\Phi(k) + AV(\Lambda)^{1/2}a_0^* + A(V(\Lambda))^{1/2}a_0 + B, \tag{10}$$

where the constants $A$ and $B$ are determined via the same averages (taken over the cube $\Lambda$) as those in (9).

The approximating Hamiltonian $H_{\mathrm{appr}}$ (9), (10) is thermodynamically equivalent to the model Hamiltonian $H_{\mathrm{M}}$ (2), (5) in the sense that the Green's functions of both these systems coincide in the thermodynamic limit.

Comparing (3) and (10) show that Hamiltonian (10) contains terms with the product $a_0^*a_0$ of operators with momentum zero as well as terms linear in $a_0$ and $a_0^*$. Note that, in Hamiltonians (9) and (10), there are no operators $a(0)/V$, $a^*(0)/V$ or

$a_0/\sqrt{V(\Lambda)}$, $a_0^*/\sqrt{V(\Lambda)}$, which turn into $c$-numbers in the thermodynamic limit. Therefore, for $H_{\text{appr}}$ and $H_{\text{appr},\Lambda}$, these expressions are final (we have no intention of considering the possibility of constructing artificial expressions such as $V \cdot a(0)/V$, $V \cdot a^*(0)/V$ or $\sqrt{V(\Lambda)} \cdot a_0/\sqrt{V(\Lambda)}$, $\sqrt{V(\Lambda)} \cdot a_0^*/\sqrt{V(\Lambda)}$ and replacing then the corresponding operators by $c$-numbers).

By using canonical transformations, one can eliminate the terms linear in $a_0$ and $a_0^*$ from operator (10). As a result, we obtain the following Hamiltonian:

$$H_{\text{appr},\Lambda} = \sum_k a_k^* a_k(\varepsilon_k - \mu) + \frac{c^2}{2}\sum_{k_1,k_2} a_{k_1}^* a_{k_2}^* \Phi(k_1)\delta_{k_1+k_2,0} + \\ + \frac{c^2}{2}\sum_{k_1,k_2} a_{k_1} a_{k_2} \Phi(k_1)\delta_{k_1+k_2,0} + c^2\Phi(0)\sum_k a_k^* a_k + \\ + c^2\sum_k a_k^* a_k \Phi(k) + \left(-\mu c^2 + \frac{1}{2}\Phi(0)c^4\right)V(\Lambda), \tag{11}$$

where the constant $c$ is determined by the condition $c = \langle a_0\rangle/\sqrt{V(\Lambda)} = \langle a_0^*\rangle/\sqrt{V(\Lambda)}$ and $\langle A\rangle$ denotes the statistical average of an operator $A$. (We preserve the notation $a_0$ and $a_0^*$ for the new operators, though 'heir averages are equal to zero.)

It follows from (10) that the difference between Hamiltonian (3), which was generally used before, and the correctly defined thermodynamically equivalent Hamiltonian (10) is that latter contains operators $a_0$ and $a_0^*$. This fact is very important because Bose condensation is possible only if a Hamiltonian contains operators of creation and annihilation with momentum zero (under periodic boundary conditions). Certainly, one must prove that the condition $c = \langle a_0\rangle/\sqrt{V(\Lambda)} = \langle a_0^*\rangle/\sqrt{V(\Lambda)}$ has a non-trivial solution.

For certain model Hamiltonians, one cannot use the procedure of replacing some operators by $c$-numbers because it is quite difficult (or even impossible) to select operators that commute with all $a_k$ and $a_k^*$ in the thermodynamic limit. Nevertheless, by analyzing the equations for Green's functions, one can obtain a thermodynamically equivalent approximating Hamiltonian . For example, such a situation takes place in the case of the Huang–Yang–Luttinger (HYL) Hamiltonian, which has the form

$$H_{\text{M},\Lambda} = H_{0,\Lambda} + H_{\text{I},\Lambda} =$$

$$= \sum_k a_k^* a_k(\varepsilon_k - \mu) + \frac{a}{2V(\Lambda)}\left[2\left(\sum_k a_k^* a_k\right)^2 - \sum_k (a_k^* a_k)^2\right], \tag{12}$$

where $a$ is a coupling constant. The system of bosons is again considered in a cube $\Lambda$ under periodic boundary conditions.

If we pass to the thermodynamic limit in (12), we obtain the following Hamiltonian for $\Lambda = \mathbf{R}^3$:

$$H_{\mathrm{M}} = \int a^*(k)a(k)(\varepsilon(k) - \mu)dk +$$

$$+ \frac{a}{2V}\left\{2\left[\int a^*(k)a(k)dk\right]^2 - \frac{(2\pi)^3}{V}\int \left[a^*(k)a(k)\right]^2 dk\right\} = H_0 + H_{\mathrm{I}}. \tag{13}$$

Let us select in $H_{\mathrm{I}}$ operators that commute with all $a(k)$ and $a^*(k)$ and thus can be replaced by $c$-numbers. It is obvious that the operator $\frac{1}{V}\int a^*(k)a(k)dk$ in the first term of $H_{\mathrm{I}}$ possesses this property and can be replaced by a $c$-number. At first sight, the entire second term can be replaced by a $c$-number. However the analysis of the equations for Green's functions shows that this is not true and the thermodynamically equivalent approximating Hamiltonian has the form [4]

$$H_{\mathrm{appr}} = \int a^*(k)a(k)(\varepsilon(k) - \mu + 2ac_1)dk - ac^3(2\pi)^{3/2}[a^*(0) + a(0)] -$$

$$- ac_1^2 V + \frac{3}{2}ac^4 V, \tag{14}$$

where

$$c = (2\pi)^{3/2}\frac{\langle a(0)\rangle}{V} = (2\pi)^{3/2}\frac{\langle a^*(0)\rangle}{V},\ c_1 = \frac{1}{V}\int \langle a^*(k)a(k)\rangle dk. \tag{15}$$

For a system in a cube $\Lambda$, Hamiltonian (14) has the form

$$H_{\mathrm{appr},\Lambda} = \sum_k a_k^* a_k(\varepsilon_k - \mu + 2ac_1) - ac^3 V(\Lambda)^{1/2}[a_0^* + a_0] -$$

$$- ac_1^2 V(\Lambda) + \frac{3}{2}ac^4 V(\Lambda). \tag{16}$$

It is easy to see that the approximating Hamiltonians (14) and (16) for the HYL model have the same features as the corresponding approximating Hamiltonians (10) and (11) for the Bogolyubov model, namely, they contain quadratic and linear terms with respect to the operators of creation and annihilation with momentum zero and there is no reason to replace them by $c$-numbers. By using canonical transformations, one can eliminate the terms linear in $a_0^*$ and $a_0$ from (16). As a result, the Hamiltonian takes the following final form:

$$H_{\text{appr},\Lambda} = \sum_k a_k^* a_k(\varepsilon_k - \mu + 2ac_1) - ac_1^2 V(\Lambda) + \frac{3}{2} ac^4 V(\Lambda) - \frac{a^2 c^6 V(\Lambda)}{-\mu + 2ac_1}. \quad (17)$$

Here the constants $c$ and $c_1$ are determined by (15) or, which is the same, from the condition of the minimum of free energy.

It should be noted that the expression for pressure obtained by Van den Berg, Lewis, and Pulé [5] for the HYL model coincides with the corresponding expression calculated for a system with Hamiltonian (16). This fact corroborates that $H_{\text{appr}}$ (16) is a correct thermodynamically equivalent Hamiltonian.

Let us formulate the main results of this paper.

In Section 1, we study the model Hamiltonian of Bogolyubov's theory of superfluidity and the HYL model Hamiltonian.

In Section 2, we investigate the equations for Green's functions in the theory of superfluidity and derive the approximating Hamiltonian (10), (11). It is shown that if $-\mu + c^2\Phi(0) < 0$, the pressure is infinite, i.e., the system is unstable; if $-\mu + c^2\Phi(0) > 0$, there is no condensate in the system, i.e., $c = 0$ and $H_{\text{appr}}$ is reduced to a free Hamiltonian with an energy of excitation $\varepsilon_k$. Only for $\mu = c^2\Phi(0)$ and sufficiently high densities and low temperatures, there is a condensate in the system and Hamiltonian (2) (or $H_{\text{appr}}$ thermodynamically equivalent to it) describes the phenomenon of superfluidity. Thus the doubts whether Bogolyubov's model Hamiltonian (2) describes superfluidity expressed in [6] is reasonless.

In Section 3, we determine $H_{\text{appr}}$ (16), (17) for the HYL model. It is shown that the expression for pressure obtained here coincides with the corresponding expression derived in [5].

Some of results presented in this paper were published earlier in the brief notes [2, 3] and the preprint [4].

## 1. Description of Hamiltonians of model systems

In this section, we describe two models of interacting bosons, namely, the Bogolyubov model of superfluidity and the Huang–Yang–Luttinger (HYL) model.

**1.1. Bogolyubov Hamiltonian in the theory of superfluidity [1].** Denote by $a_k^*$ and $a_k$ the creation and annihilation operators for a boson with momentum $k$. These operators satisfy the well-known canonical commutation relations. Assume that they interact via a pair potential $\Phi(x)$. For this system of particles contained in the region (cube) $\Lambda$ and subjected to periodic boundary conditions, the Hamiltonian has the form

$$H_\Lambda = \sum_k a_k^* a_k \left(\frac{k^2}{2m} - \mu\right) +$$

$$+ \frac{1}{2V(\Lambda)} \sum_{k_1, k_2, k_3, k_4} \delta_{k_1 + k_2, k_3 + k_4} \Phi(k_1 - k_3) a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4}. \tag{1.1}$$

Here, as usual, summation is carried out over all quasimomenta; $\Phi(k)$ are the coefficients of the Fourier series for the potential $\Phi(x) = \Phi(|x|)$

$$\Phi(x_1 - x_2) = \frac{1}{V} \sum_k e^{-ik(x_1 - x_2)} \Phi(k), \quad \overline{\Phi}(x) = \Phi(x),$$

$$\overline{\Phi}(k) = \Phi(-k) = \Phi(k), \quad \int |\Phi(x)|dx < \infty. \tag{1.2}$$

We assume that $\Phi(0) > 0$ and $\Phi(k)$ is a real continuous function with a bounded support. The model Bogolyubov Hamiltonian describing superfluidity can be obtained from the general Hamiltonian (1.1) if we retain in the Hamiltonian of interaction all terms that have at most two operators with non-zero momentum and neglect all the others. One can easily show that this Hamiltonian takes the form

$$H_{\mathrm{M}, \Lambda} = \sum_k a_k^* a_k \left(\frac{k^2}{2m} - \mu\right) + \frac{1}{2V(\Lambda)} \sum_{k_1, k_2} \delta_{k_1 + k_2, 0} \Phi(k_1) a_{k_1}^* a_{k_2}^* a_0 a_0 +$$

$$+ \frac{1}{2V(\Lambda)} \sum_{k_3, k_4} \delta_{0, k_3 + k_4} \Phi(k_3) a_0^* a_0^* a_{k_3} a_{k_4} + \frac{1}{2V(\Lambda)} \sum_{k_1, k_3} \delta_{k_1, k_3} \Phi(0) a_{k_1}^* a_0^* a_{k_3} a_0 +$$

$$+ \frac{1}{2V(\Lambda)} \sum_{k_1, k_4} \delta_{k_1, k_4} \Phi(k_1) a_{k_1}^* a_0^* a_0 a_{k_4} + \frac{1}{2V(\Lambda)} \sum_{k_2, k_4} \delta_{k_2, k_4} \Phi(0) a_0^* a_{k_2}^* a_0 a_{k_4} +$$

$$+ \frac{1}{2V(\Lambda)} \sum_{k_2, k_3} \delta_{k_2, k_3} \Phi(k_3) a_0^* a_{k_2}^* a_{k_3} a_0 - \frac{5}{2V(\Lambda)} \Phi(0) a_0^* a_0^* a_0 a_0. \tag{1.3}$$

Note that we can also consider the case, where summation in the terms corresponding to $H_1$ in (1.3) is carried out over non-zero momenta. This results in the Hamiltonian that differs from (1.3) only by the coefficient in the last term ( + 1 instead of − 5). Therefore we consider Hamiltonian (1.3).

By passing to the thermodynamic limit in (1.3) according to (4), we obtain the model Hamiltonian of the infinite system

$$H_{\mathrm{M}} = \int a^*(k)a(k)\left(\frac{k^2}{2m} - \mu\right) dk +$$

$$+ \frac{(2\pi)^3}{2V^2} \int \delta(k_1 + k_2)\, \Phi(k_1)a^*(k_1)a^*(k_2)a(0)a(0)dk_1dk_2 +$$

$$+ \frac{(2\pi)^3}{2V^2} \int \delta(k_3 + k_4)\, \Phi(k_3)a^*(0)a^*(0)a(k_3)a(k_4)dk_3dk_4 +$$

$$+ \frac{(2\pi)^3}{2V^2} \int \delta(k_1 - k_3)\, \Phi(0)a^*(k_1)a^*(0)a(k_3)a(0)dk_1dk_3 +$$

$$+ \frac{(2\pi)^3}{2V^2} \int \delta(k_1 - k_4)\, \Phi(k_1)a^*(k_1)a^*(0)a^*(0)a(k_4)dk_1dk_4 +$$

$$+ \frac{(2\pi)^3}{2V^2} \int \delta(k_2 - k_3)\, \Phi(k_3)a^*(0)a^*(k_2)a(k_3)a(0)dk_2dk_3 +$$

$$+ \frac{(2\pi)^3}{2V^2} \int \delta(k_2 - k_4)\, \Phi(0)a^*(0)a^*(k_2)a(0)a(k_4)dk_2dk_4 -$$

$$- 5\frac{(2\pi)^6}{2V^3}\Phi(0)a^*(0)a^*(0)a(0)a(0). \tag{1.4}$$

By using the operators of creation and annihilation of bosons in the configuration space

$$a^*(k) = \frac{1}{(2\pi)^{3/2}}\int e^{-ikx}a^*(x)dx, \quad a(k) = \frac{1}{(2\pi)^{3/2}}\int e^{ikx}a(x)dx,$$

$$\Phi(x) = \frac{1}{(2\pi)^3}\int e^{-ikx}\,\Phi(k)dk, \tag{1.5}$$

we arrive at the model Hamiltonian of the infinite system in the configuration space

$$H_{\mathrm{M}} = \int a^*(x)\left(-\frac{\Delta}{2m} - \mu\right)a(x)dx +$$

$$+ \frac{1}{2V^2}\int\int a^*(x_1)a^*(x_2)\Phi(x_1 - x_2)dx_1dx_2\left(\int a(x_3)dx_3\right)\left(\int a(x_4)dx_4\right) +$$

$$+\frac{1}{2V^2}\left(\int a^*(x_1)dx_1\right)\left(\int a^*(x_2)dx_2\right)\int\int a(x_3)a(x_4)\Phi(x_3-x_4)dx_3dx_4+$$

$$+\frac{1}{2V^2}\tilde{\Phi}(0)\int\int\delta(x_1-x_3)a^*(x_1)\left(\int a^*(x_2)dx_2\right)a(x_3)dx_1dx_3\left(\int a(x_4)dx_4\right)+$$

$$+\frac{1}{2V^2}\int\int\Phi(x_1-x_4)a^*(x_1)\left(\int a^*(x_2)dx_2\right)\left(\int a(x_3)dx_3\right)a(x_4)dx_1dx_4+$$

$$+\frac{1}{2V^2}\int\int\Phi(x_2-x_3)\left(\int a^*(x_1)dx_1\right)a^*(x_2)a(x_3)\left(\int a(x_4)dx_4\right)dx_2dx_3+$$

$$+\frac{1}{2V^2}\tilde{\Phi}(0)\int\int\delta(x_2-x_4)\left(\int a^*(x_1)dx_1\right)a^*(x_2)\left(\int a(x_3)dx_3\right)a(x_4)dx_2dx_4-$$

$$-\frac{5}{2V^3}\tilde{\Phi}(0)\left(\int a^*(x_1)dx_1\right)\left(\int a^*(x_2)dx_2\right)\left(\int a(x_3)dx_3\right)\left(\int a(x_4)dx_4\right),$$

$$\tilde{\Phi}(0)=\int\Phi(x)dx. \tag{1.6}$$

Note that here $V=V(\mathbf{R}^3)$ and $\Lambda=\mathbf{R}^3$; therefore all integrals in (1.6) should be regarded as limits (7). For example, the second term in (1.6) should be understood as the limit

$$\lim_{\substack{V(\Lambda)\to\infty\\(\Lambda\nearrow\mathbf{R}^3)}}\frac{1}{2V^2(\Lambda)}\int_\Lambda\int_\Lambda a^*(x_1)a^*(x_2)\Phi(x_1-x_2)dx_1dx_2\times \tag{1.7}$$

$$\times\left(\int_\Lambda a(x_3)dx_3\right)\left(\int_\Lambda a(x_4)dx_4\right).$$

The other terms in (I.6) have similar meaning.

**1.2. Huang–Yang–Luttinger (HYL) model [5].** The Hamiltonian of this model has the form

$$H_{M,\Lambda}=H_{0,\Lambda}+H_{I,\Lambda}=\sum_k a_k^*a_k\left(\frac{k^2}{2m}-\mu\right)+\frac{a}{2V(\Lambda)}\left[2\left(\sum_k a_k^*a_k\right)^2-\sum_k(a_k^*a_k)^2\right], \tag{1.8}$$

where $a_k^*$ and $a_k$ are creation and annihilation operators for bosons, $a$ is a coupling constant of the model, and parameters $m$ and $\mu$ denote, as usual, mass and chemical potential, respectively.

We now pass to the thermodynamic limit by using (4). The Hamiltonian of the infinite system has the form

$$H_M = \int a^*(k)a(k)\left(\frac{k^2}{2m} - \mu\right)dk +$$

$$+ \frac{a}{2V}\left\{2\left[\int a^*(k)a(k)dk\right]^2 - \frac{(2\pi)^3}{V}\int \left[a^*(k)a(k)\right]^2 dk\right\} = H_0 + H_I. \quad (1.9)$$

By using (1.5), we can pass to the configuration space in (1.9). As a result, we obtain the model Hamiltonian of the infinite system in configuration space

$$H_M = H_0 + H_I = \int a^*(x)\left(-\frac{\Delta}{2m} - \mu\right)a(x)dx + \frac{a}{2V}\left\{\left[\int a^*(x)a(x)dx\right]^2 -\right.$$

$$\left. - \frac{1}{V}\int \left[\int a^*(x+y)a(y)dy\right]\left[\int a^*(-x+z)a(z)dz\right]\right\}dx. \quad (1.10)$$

## 2. Equations for Green's functions in the model of superfluidity

**2.1.** In this model, Green's functions are defined in a standard way as statistical averages of $T$-products of the Heisenberg operators $a(t, x)$ and $a^*(t, x)$,

$$G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$

$$= \lim_{V(\Lambda)\to\infty} (\operatorname{Tr} e^{-\beta H_\Lambda})^{-1} \operatorname{Tr}\left[T(a(t_1, x_1) \dots a(t_m, x_m) \times\right.$$

$$\times a^*(t_{m+1}, x_{m+1}) \dots a^*(t_{m+n}, x_{m+n}) e^{-\beta H_\Lambda})] =$$

$$= \langle T(a(t_1, x_1) \dots a(t_m, x_m) a^*(t_{m+1}, x_{m+1}) \dots a^*(t_{m+n}, x_{m+n}))\rangle, \quad (2.1)$$

$$m, n = 0, 1, 2, \dots, m + n \geq 1.$$

Here $H_\Lambda$ is a Hamiltonian (1.3) of the model of superfluidity and $a^*(t, x)$ and $a(t, x)$ are the operators of creation and annihilation in the Heisenberg representation. We assume that $G_{mn}$ exist.

To derive equations for Green's functions, we need the Heisenberg equation for $a(t, x)$ and $a^*(t, x)$. Let us write the equation for $a(t, x)$. We have

$$i\frac{\partial a(t,x)}{\partial t} = [a(t,x), H\,] = -\left(-\frac{\Delta}{2m} - \mu\right) a(t,x) +$$

$$+\frac{1}{V^2}\int \Phi(x-y_1)a^*(t,y_1)dy_1 \int a(t,y_2)dy_2 \int a(t,y_3)dy_3 +$$

$$+\frac{1}{V^2}\int a^*(t,y_1)dy_1 \int\int a(t,y_2)\,\Phi(y_2-y_3)a(t,y_3)dy_2dy_3 +$$

$$+\frac{1}{2V^2}\tilde{\Phi}(0)\left[\int a^*(t,y_1)dy_1 a(t,x)\int a(t,y_2)dy_2 +\right.$$

$$\left.+\int a^*(t,y_1)a(t,y_1)dy_1 \int a(t,y_2)dy_2\right] +$$

$$+\frac{1}{2V^2}\int \Phi(x-y_3)\left(\int a^*(t,y_1)dy_1\right)\left(\int a(t,y_2)dy_2\right)a(t,y_3)dy_3 +$$

$$+\frac{1}{2V^2}\int\int \Phi(y_1-y_3)a^*(t,y_1)\left(\int a(t,y_2)dy_2\right)a(t,y_3)dy_1dy_3 +$$

$$+\frac{1}{2V^2}\int\int \Phi(y_1-y_2)a^*(t,y_1)a(t,y_2)dy_1dy_2\left(\int a(t,y_3)dy_3\right) +$$

$$+\frac{1}{2V^2}\int \Phi(x-y_2)\left(\int a^*(t,y_1)dy_1\right)a(t,y_2)dy_2\left(\int a(t,y_3)dy_3\right) +$$

$$+\frac{1}{2V^2}\tilde{\Phi}(0)\int a^*(t,y_1)\left(\int a(t,y_2)dy_2\right)a(t,y_1)dy_1 +$$

$$+\frac{1}{2V^2}\tilde{\Phi}(0)\left(\int a^*(t,y_1)dy_1\right)\left(\int a(t,y_2)dy_2\right)a(t,x) -$$

$$-\frac{5}{V^3}\tilde{\Phi}(0)\left(\int a^*(t,y_1)dy_1\right)\left(\int a(t,y_2)dy_2\right)\left(\int a(t,y_3)dy_3\right). \qquad (2.2)$$

The equation for $a^*(t,x)$ can be obtained from (2.2) by the operation of Hermitian conjugation.

By using the Heisenberg equations and one-time commutation relations, we can now deduce equations for Green's functions in the standard way. We obtain

$$i\frac{\partial}{\partial t_1}G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$

$$= \left(-\frac{\Delta_1}{2m} - \mu\right) G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) +$$

$$+ \frac{1}{V^2} \int\int\int \Phi(x_1 - y_1) G_{m+1\, n+1}(t_1, y_2, t_1, y_3, t_2, x_2, \dots, t_m, x_m;$$
$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 dy_3 +$$

$$+ \frac{1}{V^2} \int\int\int \Phi(y_2 - y_3) G_{m+1\, n+1}(t_1, y_2, t_1, y_3, t_2, x_2, \dots, t_m, x_m;$$
$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 dy_3 +$$

$$+ \frac{\tilde{\Phi}(0)}{2V^2} \int\int G_{m+1\, n+1}(t_1, x_1, t_1, y_2, t_2, x_2, \dots, t_m, x_m;$$
$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 +$$

$$+ \frac{1}{2V^2} \int\int\int \Phi(x_1 - y_3) G_{m+1\, n+1}(t_1, y_1, t_1, y_2, t_2, x_2, \dots, t_m, x_m;$$
$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 dy_3 +$$

$$+ \frac{\tilde{\Phi}(0)}{2V^2} \int\int G_{m+1\, n+1}(t_1, y_1, t_1, y_2, t_2, x_2, \dots, t_m, x_m;$$
$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 +$$

$$+ \frac{1}{2V^2} \int\int\int \Phi(y_1 - y_3) G_{m+1\, n+1}(t_1, y_2, t_1, y_3, t_2, x_2, \dots, t_m, x_m;$$
$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 dy_3 +$$

$$+ \frac{1}{2V^2} \int\int\int \Phi(y_1 - y_2) G_{m+1\, n+1}(t_1, y_2, t_1, y_3, t_2, x_2, \dots, t_m, x_m;$$
$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 dy_3 +$$

$$+ \frac{1}{2V^2} \int\int\int \Phi(x_1 - y_2) G_{m+1\, n+1}(t_1, y_2, t_1, y_3, t_2, x_2, \dots, t_m, x_m;$$
$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 dy_3 +$$

$$+ \frac{\tilde{\Phi}(0)}{2V^2} \int \int G_{m+1\, n+1}(t_1, y_1, t_1, y_2, t_2, x_2, \dots, t_m, x_m;$$

$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dy_1 dy_2 +$$

$$+ \frac{\tilde{\Phi}(0)}{2V^2} \int \int G_{m+1\, n+1}(t_1, x_1, t_1, y_2, t_2, x_2, \dots, t_m, x_m;$$

$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dy_1 dy_2 -$$

$$- \frac{5\tilde{\Phi}(0)}{V^3} \int \int \int G_{m+1\, n+1}(t_1, y_2, t_1, y_3, t_2, x_2, \dots, t_m, x_m;$$

$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dy_1 dy_2 dy_3 +$$

$$+ i \sum_{j=m+1}^{m+n} \delta(t_1 - t_j)\, \delta(x_1 - x_j) G_{m-1\, n-1}(t_2, x_2, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \overset{j}{\check{\dots}}, t_{m+n}, x_{m+n}),$$

$$m, n = 0, 1, 2, \dots, \; m + n \geq 1. \tag{2.3}$$

**2.2.** It is now necessary to give rigorous meaning to the terms on the right-hand side of (2.3) that contain $V = V(\mathbf{R}^3) = \infty$ in the denominators. We understand this term in the sense of limit (1.7). To define these terms, we first make some assumptions concerning the structure of the functions $G_{mn}$.

Let us split a set of $m + n$ points $(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})$ into $l$ subsets $\sigma_1, \dots, \sigma_l$ consisting at least of the point

$$(t_{i_1}, x_{i_1}, \dots, t_{i_{m_1}}, x_{i_{m_1}}; t_{i_{m_1+1}}, x_{i_{m_1+1}}, \dots, t_{i_{m_1+n_1}}, x_{i_{m_1+n_1}}), \dots,$$

$$(t_{j_1}, x_{j_1}, \dots, t_{j_{m_l}}, x_{j_{m_l}}; t_{j_{m_l+1}}, x_{j_{m_l+1}}, \dots, t_{j_{m_l+n_l}}, x_{j_{m_l+n_l}}). \tag{2.4}$$

These subsets $\sigma_1, \dots, \sigma_l$ are associated with the functions

$$g_{m_1 n_1}(t_{i_1}, x_{i_1}, \dots, t_{i_{m_1}}, x_{i_{m_1}}; t_{i_{m_1+1}}, x_{i_{m_1+1}}, \dots, t_{i_{m_1+n_1}}, x_{i_{m_1+n_1}}) =$$

$$= g_{m_1 n_1}((t, x)_{m_1}; (t, x)_{n_1}), \dots,$$

$$g_{m_l n_l}(t_{j_1}, x_{j_1}, \dots, t_{j_{m_l}}, x_{j_{m_l}}; t_{j_{m_l+1}}, x_{j_{m_l+1}}, \dots, t_{j_{m_l+n_l}}, x_{j_{m_l+n_l}}) =$$

$$= g_{m_l n_l}((t, x)_{m_l}; (t, x)_{n_l}) \tag{2.5}$$

which are assumed to be translation invariant with respect to all variables, i.e., they remain unchanged under the replacements of all four-dimensional points $(t, x)$ by the points $(t + a_0, x + a)$, where $a_0$ and $a$ are an arbitrary number and an arbitrary threevector, respectively. This means that the functions

$$g_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})$$

depend on $m + n - 1$ independent variables that can be chosen in the form of differences

$$(t_2 - t_1, x_2 - x_1, \dots, t_m - t_1, x_m - x_1;$$
$$t_{m+1} - t_1, x_{m+1} - x_1, \dots, t_{m+n} - t_1, x_{m+n} - x_1),$$

namely,

$$g_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$
$$= g_{mn}(0, 0, t_2 - t_1, x_2 - x_1, \dots, t_m - t_1, x_m - x_1;$$
$$t_{m+1} - t_1, x_{m+1} - x_1, \dots, t_{m+n} - t_1, x_{m+n} - x_1). \tag{2.6}$$

Assume that Green's functions can be represented in the form

$$G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$
$$= \sum_{\sigma} g_{m_1 n_1}((t, x)_{m_1}; (t, x)_{n_1}), \dots, g_{m_l n_l}((t, x)_{m_l}; (t, x)_{n_l}). \tag{2.7}$$

Here summation is carried out over all possible decompositions $\sigma$†. In view of the fact that all subsets of the decomposition $\sigma$ contain at least one point, we have

$$G_{10} = g_{10}, G_{01} = g_{01}. \tag{2.8}$$

It follows from (2.7) and (2.8) that the representation

$$G_{m+1\, n+1}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$
$$= G_{10}(t_1, x_1)G_{10}(t_2, x_2)G_{m-2\, n}(t_3, x_3, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) +$$
$$+ M_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+1+n+1}, x_{m+1+n+1}) \tag{2.9}$$

† We assume that the functions $Gmn$ and $gmn$ are symmetric with respect to permutations.

holds, where $M_{mn}$ is given by (2.7) but summation is carried out over all decompositions $\sigma$ which do not contain two subsets consisting of one point and containing just the points $(t_1, x_1)$ and $(t_2, x_2)$.

We require that the functions $g_{mn}$ must be summable with respect to the difference variables $x_i - x_1$ together with the potential $\Phi$ (all other variables are fixed). This means, for example, that

$$\int |\Phi(x_2 - x_1)| \, |g_{mn}(0, 0, t_2 - t_1, x_2 - x_1, \dots, t_m - t_1, x_m - x_1;$$

$$t_{m+1} - t_1, x_{m+1} - x_1, \dots, t_{m+n} - t_1, x_{m+n} - x_1)| \, d(x_2 - x_1) < \infty. \qquad (2.10)$$

In the models investigated in physical literature, potentials $\Phi$ are, as usual, bounded and summable; therefore it suffices to require that $g_{mn}$ must summable with respect to difference variables.

2.3. Let us proceed to the determination of the right-hang side of equations (2.3), which contains uncertainties connected with the factors $1/V^2$ and $1/V^3$. To eliminate these uncertainties, we assumed that Green's functions can be represented in the form (2.7) and decompositions $\sigma$ also contain subsets consisting of a single point. The functions $g_{10}(t, x)$ and $g_{01}(t, x)$ are translation invariant, i.e., they are constants. The other functions $g_{mn}$, $m + n \geq 2$, are also translation invariant and integrable with respect to difference variables; furthermore, they are bounded.

We now determine the second term on the right-hand side of (2.3). By using representation (2.7), (2.9) for $G_{m+1\, n+1}$ and inequality (2.10), we get

$$\frac{1}{V^2}\int\int\int \Phi(x_1 - y_1) G_{m+1\, n+1}(t_1, y_2, t_1, y_3, t_2, x_2, \dots, t_m, x_m;$$

$$t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 dy_2 dy_3 =$$

$$= c^2 \int \Phi(x_1 - y_1) G_{m-1\, n+1}(t_2, x_2, \dots, t_m, x_m; t_1, y_1, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) dy_1 , \qquad (2.11)$$

where

$$c = G_{10}(0, 0) = \lim_{V(\Lambda) \to \infty} \frac{1}{V(\Lambda)} \int_\Lambda G_{10}(t_1, y_2) dy_2 = \langle a(0, 0) \rangle. \qquad (2.12)$$

The proof consists in establishing the equalities

$$\lim_{V(\Lambda)\to\infty}\frac{1}{V(\Lambda)^2}\int_\Lambda\int_\Lambda G_{10}(t_1,y_2)G_{10}(t_1,y_3)dy_2dy_3=c^2,$$

$$\lim_{V(\Lambda)\to\infty}\frac{1}{V(\Lambda)^2}\int_\Lambda\int_\Lambda\int_\Lambda \Phi(x_1-y_1)M_{m+1\,n+1}(t_1,y_2,t_1,y_3,t_2,x_2,\dots,t_m,x_m;$$
$$t_1,y_1,t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})dy_1dy_2dy_3=0,$$

which follow from integrability of $g_{mn}$ and the fact that $M_{m+1\,n+1}$ depends on $y_2$ or $y_3$. The other terms can be determined analogously.

By using the argument similar to that used for the term considered above, we conclude that, after passing to the thermodynamic limit and eliminating the uncertainties connected with the factors $1/V^2$ and $1/V^3$, equations (2.3) for Green's functions take the following form:

$$i\frac{\partial}{\partial t_1}G_{mn}(t_1,x_1,\dots,t_m,x_m;t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})=$$
$$=\left(-\frac{\Delta_1}{2m}-\mu\right)G_{mn}(t_1,x_1,\dots,t_m,x_m;t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})+$$
$$+c^2\int\Phi(x_1-y_1)G_{m-1\,n+1}(t_2,x_2,\dots,t_m,x_m;t_1,y_1,t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})dy_1+$$
$$+c^*c_1G_{m-1\,n}(t_2,x_2,\dots,t_m,x_m;t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})+$$
$$+c^*c\,\tilde\Phi(0)G_{m-1\,n}(t_2,x_2,\dots,t_m,x_m;t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})+$$
$$+cc_2\tilde\Phi(0)G_{m-1\,n}(t_2,x_2,\dots,t_m,x_m;t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})+$$
$$+c^*c\int\Phi(x_1-y_1)G_{mn}(t_1,y_1,t_2,x_2,\dots,t_m,x_m;t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})dy_1+$$
$$+cc_3G_{m-1\,n}(t_2,x_2,\dots,t_m,x_m;t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})-$$
$$-5c^2c^*\tilde\Phi(0)G_{m-1\,n}(t_2,x_2,\dots,t_m,x_m;t_{m+1},x_{m+1},\dots,t_{m+n},x_{m+n})+$$
$$+i\sum_{j=m+1}^{m+n}\delta(t_1-t_j)\,\delta(x_1-x_j)G_{m-1\,n-1}(t_2,x_2,\dots,t_m,x_m;t_{m+1},x_{m+1},\overset{j}{\check{\dots}},t_{m+n},x_{m+n}),$$
$$m,n=0,1,2,\dots,\ m+n\ge1. \tag{2.13}$$

Here the constants $c$, $c^*$, $c_1$, $c_2$, and $c_3$ are defined as follows:

$$c = G_{10}(0, 0) = g_{10}(0, 0) = \langle a(0, 0)\rangle = \langle a(t, x)\rangle = \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda \langle a(t, x)\rangle \, dx,$$

$$c^* = G_{01}(0, 0) = g_{01}(0, 0) = \langle a^*(0, 0)\rangle = \langle a^*(t, x)\rangle = \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda \langle a^*(t, x)\rangle \, dx,$$

$$c_1 = \int \Phi(x-y) G_{20}(0, x-y, 0, 0)\, d(x-y) =$$

$$= \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda \int_\Lambda \Phi(x-y)\langle T(a(t, x)a(t,y))\rangle \, dxdy,$$

$$c_2 = G_{11}(0, 0; 0, 0) = G_{11}(t, x; t, x) = \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda \langle T(a(t, x)a^*(t, x)\rangle \, dx,$$

$$c_3 = \int \Phi(x-y) G_{11}(0, x-y; 0, 0)\, d(x-y) =$$

$$= \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda \int_\Lambda \Phi(x-y) G_{11}(t, x; t, y)\, dxdy =$$

$$= \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda \int_\Lambda \Phi(x-y)\langle T(a(t, x)a^*(t,y))\rangle \, dxdy. \tag{2.14}$$

By using the canonical transformation

$$a(x) \to e^{i\varphi} a(x), \quad a^*(x) \to e^{-i\varphi} a^*(x),$$

we can always guarantee that $c = c^*$ and $c_1 = c_1^*$. This enables us to assume in what follows that the constants $c$ and $c_1$ are real.

Let us introduce the approximating Hamiltonian

$$H_{\text{appr}} = \int a^*(x)\left(-\frac{\Delta}{2m} - \mu\right) a(x)dx + \frac{c^2}{2} \int\int a^*(x_1)\Phi(x_1 - x_2)a^*(x_2)dx_1 dx_2 +$$

$$+ \frac{c^2}{2} \int\int a(x_1)\Phi(x_1 - x_2)a(x_2)dx_1 dx_2 + c^2 \tilde{\Phi}(0) \int a^*(x)a(x)\, dx +$$

$$+ c^2 \int\int a^*(x_1)\Phi(x_1 - x_2)a(x_2)dx_1 dx_2 + A\int a^*(x)\, dx + A\int a(x)\, dx + BV, \tag{2.15}$$

where

$$A = cc_1 + cc_2\,\widetilde{\Phi}(0) + cc_3 - 5c^3\,\widetilde{\Phi}(0),$$

$$B = -2c^2c_1 - 2c^2c_2\,\Phi(0) - 2c^2c_3 + \frac{15}{2}c^4\,\Phi(0).$$

The constant $B$ was determined from the equation $\langle H_M\rangle = \langle H_{\text{appr}}\rangle$.

By direct calculation, we find that the equations for Green's functions in the system with approximating Hamiltonian coincide with equations (2.13) for the system with model Hamiltonian provided that the constants $c$, $c_1$, $c_2$, and $c_3$ in (2.13) are defined according to (2.14). This means that Green's functions in the system with approximating Hamiltonian coincide with Green's functions in the system with model Hamiltonian, i.e., that these systems are equivalent in the thermodynamic sense.

In the momentum space, $H_{\text{appr}}$ has the form

$$\begin{aligned} H_{\text{appr}} = &\int a^*(k)\left(\frac{k^2}{2m} - \mu\right)a(k)dk + \\ &+ \frac{c^2}{2}\int a^*(k_1)a^*(k_2)\Phi(k_1)\,\delta(k_1 + k_2)\,dk_1dk_2 + \\ &+ \frac{c^2}{2}\int a(k_1)a(k_2)\Phi(k_1)\,\delta(k_1 + k_2)\,dk_1dk_2 + \\ &+ c^2\,\Phi(0)\int a^*(k_1)a(k_2)\,\delta(k_1 - k_2)\,dk_1dk_2 + \\ &+ c^2\int a^*(k_1)a(k_2)\Phi(k_1)\,\delta(k_1 - k_2)\,dk_1dk_2 + \\ &+ A(2\pi)^{3/2}a^*(0) + A(2\pi)^{3/2}a^*(0) + BV; \end{aligned} \tag{2.16}$$

for a bounded region $\Lambda$, we have

$$\begin{aligned} H_{\text{appr},\Lambda} = &\sum_k a_k^*\left(\frac{k^2}{2m} - \mu\right)a_k + \frac{c^2}{2}\sum_{k_1,k_2} a_{k_1}^*a_{k_2}^*\Phi(k_1)\,\delta_{k_1+k_2,0} + \\ &+ \frac{c^2}{2}\sum_{k_1,k_2} a_{k_1}a_{k_2}\Phi(k_1)\,\delta_{k_1+k_2,0} + c^2\,\Phi(0)\sum_{k_1,k_2} a_{k_1}^*a_{k_2}\,\delta_{k_1,k_2} + \\ &+ c^2\sum_{k_1,k_2} a_{k_1}^*a_{k_2}\,\Phi(k_1)\,\delta_{k_1,k_2} + A(V(\Lambda))^{1/2}a_0^* + \\ &+ A(V(\Lambda))^{1/2}a_0 + BV(\Lambda). \end{aligned} \tag{2.17}$$

**2.4.** Let us make the canonical transformation in (2.17)

$$a_0 = \tilde{a}_0 + d, \quad a_0^* = \tilde{a}_0^* + d. \tag{2.18}$$

Then the term in $H_{\mathrm{appr}}$ (2.17) containing $a_0$ and $a_0^*$ takes the form

$$(-\mu + 2c^2\Phi(0))\tilde{a}_0^*\tilde{a}_0 + \frac{c^2}{2}\Phi(0)\,\tilde{a}_0^*\tilde{a}_0^* + \frac{c^2}{2}\Phi(0)\,\tilde{a}_0\tilde{a}_0 +$$

$$+ (-\mu + 3c^2\Phi(0))(\tilde{a}_0^* d + \tilde{a}_0 d) + A(V(\Lambda))^{1/2}\tilde{a}_0^* +$$

$$+ A(V(\Lambda))^{1/2}\tilde{a}_0 + (-\mu + 3c^2\Phi(0))d^2 + 2A(V(\Lambda))^{1/2}d. \tag{2.19}$$

We choose the number $d$ so that the coefficients at $\tilde{a}_0$ and $\tilde{a}_0^*$ are equal to zero. We obtain the equation

$$(-\mu + 3c^2\Phi(0))d + A(V(\Lambda))^{1/2} = 0. \tag{2.20}$$

It follows from (2.20) that

$$d = \frac{-A(V(\Lambda))^{1/2}}{3c^2\Phi(0) - \mu}. \tag{2.21}$$

The constant $A$ can be defined by $c$. To show this, we use the equations

$$i\frac{\partial a_0(t)}{\partial t} = \frac{\delta H_{\mathrm{M}}}{\delta a_0^*(t)}, \quad -i\frac{\partial a_0^*(t)}{\partial t} = \frac{\delta H_{\mathrm{M}}}{\delta a_0(t)}$$

or the equations

$$i\frac{\partial a_0(t)}{\partial t} = \frac{\delta H_{\mathrm{appr}}}{\delta a_0^*(t)}, \quad -i\frac{\partial a_0^*(t)}{\partial t} = \frac{\delta H_{\mathrm{appr}}}{\delta a_0(t)}$$

and average them, by using the fact that

$$\frac{\langle a_0(t)\rangle}{(V(\Lambda))^{1/2}} = \frac{\langle a_0(0)\rangle}{(V(\Lambda))^{1/2}} = c, \quad \frac{\langle a_0^*(t)\rangle}{(V(\Lambda))^{1/2}} = \frac{\langle a_0^*(0)\rangle}{(V(\Lambda))^{1/2}} = c.$$

Both the equations result in the same equations

$$-\mu c - 2c^3\Phi(0) + cc_1 + cc_2\Phi(0) + cc_3 = 0, \tag{2.22}$$

or, equivalently,

$$A = \mu c - 3c^3\Phi(0).$$

If we take (2.22) into account, then we find $d$ from (2.21), namely,

$$d = c(V(\Lambda))^{1/2}. \tag{2.23}$$

Thus the operators linear in $a_0$, $\widetilde{a}_0^*$ vanish and operator (2.19) takes the form

$$(-\mu + 2c^2\Phi(0))\widetilde{a}_0^*\widetilde{a}_0 + \frac{c^2}{2}\Phi(0)\,\widetilde{a}_0^*\widetilde{a}_0^* + \frac{c^2}{2}\Phi(0)\,\widetilde{a}_0\widetilde{a}_0 +$$

$$+ \left[\mu c^2 - 3c^4\Phi(0)\right]V(\Lambda). \tag{2.24}$$

Taking (2.24) into account, we obtain from (2.17) the final expression for $H_{\mathrm{appr},\Lambda}$

$$H_{\mathrm{appr},\Lambda} = \sum_k a_k^* a_k\left(\frac{k^2}{2m} - \mu\right) + \frac{c^2}{2}\sum_k a_k^* a_{-k}^*\Phi(k) +$$

$$+ \frac{c^2}{2}\sum_k a_k a_{-k}\Phi(k) + c^2\Phi(0)\sum_k a_k^* a_k +$$

$$+ c^2\sum_k a_k^* a_k\Phi(k) + \left[-\mu c^2 + \frac{1}{2}c^4\Phi(0)\right]V(\Lambda). \tag{2.25}$$

In (2.25), we use the fact that, by virtue of (2.22) and $A = \mu c^2 - 3c^3\Phi(0)$, the expression for $B$ takes the form

$$B = \left(-2\mu c^2 + \frac{7}{2}c^4\widetilde{\Phi}(0)\right)V(\Lambda). \tag{2.26}$$

In addition, in (2.25), we use the notation $a_0$, $a_0^*$ instead of $\widetilde{a}_0$, $\widetilde{a}_0^*$; however $\langle a_0\rangle = \langle a_0^*\rangle = 0$ according to (2.18) and (2.23).

Let us show that, in the thermodynamic limit, the extremum condition for the specific free energy for $H_{\mathrm{appr},\Lambda}$ (2.25) coincides with equation (2.22). Indeed, the extremum condition gives

$$\lim_{V(\Lambda)\to\infty}(c_1' + \Phi(0)c_2' + c_3' - \mu + c^2\Phi(0)) = 0. \tag{2.27}$$

Here

$$c_1' = \frac{1}{V(\Lambda)}\sum_k{}'\langle a_k^* a_{-k}^*\rangle\Phi(k) = \frac{1}{V(\Lambda)}\sum_k{}'\langle a_k a_{-k}\rangle\Phi(k),$$

$$c_2' = \frac{1}{V(\Lambda)} \sum_k{}' \langle a_k^* a_k \rangle , \quad c_3' = \frac{1}{V(\Lambda)} \sum_k{}' \langle a_k^* a_k \rangle \Phi(k), \tag{2.28}$$

and we have taken into account that

$$\lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \langle \tilde{a}_0 \tilde{a}_0 \rangle = \lim_{V(\Lambda)\to\infty} \left\langle \frac{1}{V(\Lambda)} \int_\Lambda \tilde{a}(x)dx \frac{1}{V(\Lambda)} \int_\Lambda \tilde{a}(x)dx \right\rangle = 0,$$

$$\lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \langle \tilde{a}_0^* \tilde{a}_0^* \rangle = \lim_{V(\Lambda)\to\infty} \left\langle \frac{1}{V(\Lambda)} \int_\Lambda \tilde{a}^*(x)dx \frac{1}{V(\Lambda)} \int_\Lambda \tilde{a}^*(x)dx \right\rangle = 0,$$

$$\lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \langle \tilde{a}_0^* \tilde{a}_0 \rangle = \lim_{V(\Lambda)\to\infty} \left\langle \frac{1}{V(\Lambda)} \int_\Lambda \tilde{a}^*(x)dx \frac{1}{V(\Lambda)} \int_\Lambda \tilde{a}(x)dx \right\rangle = 0.$$

If we make an identical transformation in (2.27), adding and subtracting $3c^3\Phi(0)$, we arrive at the equation

$$-\mu - 2c^2\Phi(0) + c_1 + \Phi(0)c_2 + c_3 = 0, \tag{2.29}$$

which coincides with (2.22) with the constants $c$, $c_1$, $c_2$, $c_3$ given by (2.14) in terms of the initial fields $a(x)$ and $a^*(x)$. This equation was taken into account in $H_{\rm appr}$ by setting $A = \mu c - 3c^3\Phi(0)$.

The approximating Hamiltonian (2.17) (or (2.25)) thus obtained is thermodynamically equivalent to Bogolyubov's model Hamiltonian (1). It contains the creation and annihilation operators with momentum zero, and the constant $c$ will be determined below.

**2.5. The determination of the constant $c$ and the spectrum of $H_{\rm appr}$.** Thus, after taking into account the condition of minimum of a free energy and eliminating terms linear in $a_0^*$ and $a_0$, we have obtained the final expression (2.25) for $H_{\rm appr}$, which contains only one constant $c$. To complete the investigation of the Hamiltonians $H_{\rm M}$ and $H_{\rm appr}$, it remains to determine the constant $c$ and the spectrum of $H_{\rm appr}$ and show that this Hamiltonian describes superfluidity.

First we determine the constant $c$. For this purpose, we use the following relation:

$$c^2 = \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda \langle a^*(x) \rangle dx \frac{1}{V(\Lambda)} \int_\Lambda \langle a(y) \rangle dy =$$

$$= \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)^2} \int_{\Lambda^2} \langle a^*(x)a(y)\rangle dxdy = \lim_{V(\Lambda)\to\infty} \left\langle \frac{a_0^*}{\sqrt{V(\Lambda)}} \frac{a_0}{\sqrt{V(\Lambda)}} \right\rangle, \qquad (2.30)$$

where the average is taken with respect to $H_{\text{appr}}$ [7].

To determine the averages $\langle a_k^* a_k\rangle$, we reduce the Hamiltonian $H_{\text{appr}}$ to a diagonal form by using Bogolyubov's canonical $u - v$ transformation

$$a_k = u_k b_k + v_k b_{-k}^*. \qquad (2.31)$$

If we set

$$u_k^2 = \frac{1}{2}\left(\frac{f_k}{E_k} + 1\right), \quad v_k^2 = \frac{1}{2}\left(\frac{f_k}{E_k} - 1\right), \quad E_k = (f_k^2 - h_k^2)^{1/2},$$

$$f_k = \varepsilon_k - \mu + c^2(\Phi(0) + \Phi(k)), \quad h_k = c^2\Phi(k),$$

$$E_k = \sqrt{(\varepsilon_k - \mu + c^2\Phi(0))^2 + 2(\varepsilon_k - \mu + c^2\Phi(0))c^2\Phi^2(k)}, \qquad (2.32)$$

then $H_{\text{appr},\Lambda}$ takes the form

$$H_{\text{appr},\Lambda} = \sum_k E_k b_k^* b_k + \frac{1}{2}\sum_k (E_k - f_k) + \left(-\mu c^2 + \frac{1}{2}\Phi(0)c^4\right)V. \qquad (2.33)$$

In this subsection, we assume that $\Phi(k) > 0$. Clearly, $E_k$ must be non-negative, $E_k \geq 0$, and therefore $f_k^2 \geq h_k^2$. The latter inequality holds if $-\mu + c^2\Phi(0) \geq 0$; in what follows, we assume that this condition is satisfied.

In terms of $\langle b_k^* b_k\rangle$, the averages $\langle a_k^* a_k\rangle$ are expressed as follows:

$$\langle a_k^* a_k\rangle = v_k^2 + (2u_k^2 - 1)\langle b_k^* b_k\rangle =$$

$$= \frac{1}{2}\left(\frac{f_k}{E_k} - 1\right) + \frac{f_k}{E_k}\frac{1}{e^{\beta E_k} - 1} = \frac{1}{2}\left(\frac{f_k}{E_k}\coth\frac{\beta E_k}{2} - 1\right). \qquad (2.34)$$

It is obvious that $\langle a_k^* a_k\rangle$ has singularities for $-\mu + c^2\Phi(0) \geq 0$ only if

$$E_0 = \sqrt{(-\mu + 2c^2\Phi(0))^2 - c^4\Phi^2(0)} = 0,$$

i.e., if

$$\mu = c^2\Phi(0). \tag{2.35}$$

Below we assume that this relation is satisfied.

Hamiltonian (2.33) is a free one with respect to the new operators $b_k^*$ and $b_k$ and its energy of excitation is $E_k$. It is well known that the Bose condensation of a free Bose system is possible only if $E_k = 0$ for $k = 0$, i.e., for $\mu = c^2\Phi(0)$. The density of condensed particles is equal to $c^2$, therefore, for Bose condensation, it is necessary to have $c \neq 0$. We proceed to determining this constant. For this purpose, we employ the well-known idea from the theory of Bose condensation and consider the formula for the number of particles in a system

$$N = \langle a_0^* a_0 \rangle + \sum_{k \neq 0} \langle a_k^* a_k \rangle = n_0 + \sum_{k \neq 0} n_k, \tag{2.36}$$

where $n_0$ is the average number of particles with momentum zero and $n_k$ is the average number of particles with momentum $k$. Dividing relation (2.36) by $V(\Lambda)$, we get

$$\frac{1}{v} = \frac{1}{v_0} + \frac{1}{V(\Lambda)} \sum_{k \neq 0} n_k,$$

where $1/v = N/V(\Lambda)$ is the density of all particles and $1/v_0 = n_0/V(\Lambda)$ is the density of particles with momentum zero.

By employing the well-known procedure† and passing to the thermodynamic limit $V(\Lambda) \to \infty$ in view of (2.30), we obtain

$$\frac{1}{v} = c^2 + \frac{1}{(2\pi)^3} \int \frac{1}{2} \left( \frac{f(k)}{E(k)} \coth \frac{\beta E(k)}{2} - 1 \right) dk. \tag{2.37}$$

Let us show that this equation has a non-trivial solution with respect to $c^2$ for

$$\frac{1}{v} > c^2 + \frac{1}{v_c(\infty)}, \quad \frac{1}{v_c(\beta)} = \frac{1}{(2\pi)^3} \int \frac{1}{2} \left( \frac{f(k)}{E(k)} \coth \frac{\beta E(k)}{2} - 1 \right) dk$$

and some $\beta$. For this purpose, we fix $c^2$ arbitrarily and consider the value $1/v_c(\beta)$ as a function of $\beta$. For a finite potential $\Phi(k)$, the integral in the definition of $1/v_c(\beta)$ is absolutely convergent; moreover, it is a decreasing function of $\beta$ and

$$\frac{1}{v_c(\infty)} = \frac{1}{(2\pi)^3} \int \frac{1}{2} \left( \frac{f(k)}{E(k)} - 1 \right) dk,$$

† For detailed presentation, see [8-10].

$$\frac{1}{v_c(0)} = \infty, \quad \frac{1}{v_c(0)} > \frac{1}{v_c(\beta)} > \frac{1}{v_c(\infty)}.$$

Therefore, if

$$\infty > \frac{1}{v} > \frac{1}{v_c(\infty)} + c^2,$$

equation (2.37) has a unique solution with respect to $c^2$ for a certain fixed $\beta_c$. This means that, for certain high densities and low temperatures, Bose condensation takes place in the system.

Note that if $\mu$ satisfies condition (2.35), the behavior of the energy of elementary excitations

$$E_k = \sqrt{\varepsilon_k^2 + 2\varepsilon_k c^2 \Phi(k)} \tag{2.38}$$

at zero is as follows:

$$E_k \sim |k| \left( \frac{c^2 \Phi(0)}{m} \right)^{1/2}$$

Note that if $-\mu + c^2\Phi(0) > 0$, then $E_k \neq 0$ for all $k$ and Bose condensation is absent, i.e., $c = 0$ and $H_{\text{appr}}$ is reduced to the free Hamiltonian with respect to $a_k^*$ and $a_k$ and with an energy of excitation $\varepsilon_k$. If $-\mu + c^2\Phi(0) < 0$, then $E_k = 0$ for some $k$, a pressure is equal to infinity, and, in this sense, the system is unstable. Only for $-\mu + c^2\Phi(0) = 0$, Bose condensation is possible, $c \neq 0$, and an energy of excitation has a desired behavior for $k \sim 0$.

Thus Bogolyubov's model Hamiltonian (1.3) completely describes superfluidity for a non-zero density of a condensate $c^2 = 1/v_0$ with the corresponding energy of elementary excitations $E_k$.

Note that this result was obtained due to the fact that $H_{\text{appr},\Lambda}$ contains operators of creation and annihilation $a_0^*$ and $a_0$ with momentum zero.

Equation (2.37) has a solution $c^2 = 0$ for $\mu < 0$, wich corresponds to a free Bose system. Note that equation (2.37) also has a non-zero solution for fixed $\mu$ and $c \neq 0$ such that $\mu - c^2\Phi(0) < 0$. Indeed, the function $1/v_c(\beta)$ is a decreasing function of $\beta$ and $1/v_c(0) = \infty$. Therefore, if $\infty > 1/v > 1/v_c(\infty) + c^2$, equation (2.37) has a unique solution with respect to $c^2 \neq 0$ for a certain fixed $\beta_c$. Physical meaning of this solution is not clear yet.

### 3. Equations for Green's functions in the HYL model

Green's functions in the HYL model are defined according to the formula

$$G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) = \lim_{V(\Lambda)\to\infty} (\mathrm{Tr}\, e^{-\beta H_\Lambda})^{-1} \times$$

$$\times \mathrm{Tr}\left[T(a(t_1, x_1) \dots a(t_m, x_m) a^*(t_{m+1}, x_{m+1}) \dots a^*(t_{m+n}, x_{m+n}) e^{-\beta H_\Lambda})\right],$$

$$m, n = 0, 1, 2, \dots, \quad m + n \geq 1. \tag{3.1}$$

Here we assume that the thermodynamic limit of (3.1) exists.

When deriving equations for Green's functions, we shall need the Heisenberg equations for $a^*(t, x)$ and $a(t, x)$. The equation for $a(t, x)$ has the form

$$i\frac{\partial a(t_1, x_1)}{\partial t} = [a(t_1, x_1), H\,] =$$

$$= \left(-\frac{\Delta_1}{2m} - \mu\right) a(t_1, x_1) + \frac{a}{2V}\left\{2a(t_1, x_1)\left(\int a^*(t_1, x)a(t_1, x)dx\right) + \right.$$

$$+ 2\left(\int a^*(t_1, x)a(t_1, x)dx\right) a(t_1, x_1) -$$

$$- \frac{1}{V}\int a(t_1, x_1 - x)\left(\int a^*(t_1, -x + y)a(t, y)dy\right)dx -$$

$$\left. - \frac{1}{V}\int\left(\int a^*(t_1, x + y)a(t, y)dy\right)a(t_1, x_1 + x)dx\right\}. \tag{3.2}$$

The equation for $a^*(t_1, x_1)$ can be obtained from (3.2) by the operation of Hermitian conjugation.

By using the Heisenberg equations and the canonical commutation relations, we get the following equations for Green's functions:

$$i\frac{\partial}{\partial t_1} G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$

$$= \left(-\frac{\Delta_1}{2m} - \mu\right) G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) +$$

$$+ \frac{a}{2V}\left\{4\int G_{m+1\, n+1}(t_1, x, t_1, x_1, t_2, x_2, \dots, t_m, x_m;\right.$$

$$t_1, x, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dx -$$

$$- 2\frac{1}{V}\int G_{m+1\,n+1}(t_1, x_1 + x, t_1, y, t_2, x_2, \dots, t_m, x_m;$$

$$t_1, x + y, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dxdy\Bigg\} +$$

$$+ i\sum_{j=m+1}^{m+n} \delta(t_1 - t_j)\,\delta(x_1 - x_j)G_{m-1\,n-1}(t_2, x_2, \dots, t_m, x_m; t_{m+1},$$

$$x_{m+1}, \overset{j}{\check{\dots}}, t_{m+n}, x_{m+n}),$$

$$m, n = 0, 1, 2, \dots, \ m + n \geq 1. \tag{3.3}$$

On the right-hand side of equation (3.3) the terms in braces contain the factors $1/V$ and $1/V^2$, where $V$ is the volume of the entire three-dimensional space. To make these expressions meaningful, we assume that the functions $G_{mn}$ can be represented in the form (2.7), where the set of decomposition $\sigma$ also contain decompositions with one-point subsets; moreover, we assume that the functions $G_{mn}$ and $g_{mn}$ are symmetric. The functions $g_{10}(t, x)$ and $g_{01}(t, x)$ are translation invariant; therefore they are constants. The other functions $g_{mn}$ , $m + n \geq 2$ are summable with respect to difference spatial variables as in Bogolyubov's model (they are also bounded).

By using relation (2.7) for the functions $G_{mn}$ , we can represent them in the form

$$G_{m+1\,n+1}(t_1, x, t_1, x_1, \dots, t_m, x_m; t_1, x, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$

$$= G_{11}(t_1, x; t_1, x)G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) +$$

$$+ M^{\mathrm{I}}_{m+1\,n+1}(t_1, x, t_1, x_1, \dots, t_m, x_m; t_1, x, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}),$$

where $M^{\mathrm{I}}_{m+1\,n+1}(t_1, x, t_1, x_1, \dots, t_m, x_m; t_1, x, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})$ is representable in the form (2.7) but summation is carried out over all decompositions $\sigma$ which do not contain subsets that include exactly two points $t_1, x$ and $t_1, x$ corresponding to the operators $a(t_1, x)$ and $a^*(t_1, x)$.

We have

$$\lim_{V(\Lambda)\to\infty} \frac{4a}{2V(\Lambda)} \int_\Lambda G_{m+1\, n+1}(t_1, x, t_1, x_1, \dots, t_m, x_m;$$

$$t_1, x, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dx =$$

$$= 2ac_1 G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}), \tag{3.4}$$

where

$$c_1 = \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda G_{11}(t_1, x; t_1, x)dx = G_{11}(0, 0; 0, 0). \tag{3.5}$$

This can be proved by using the relation

$$\lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda M^{\mathrm{I}}_{m+1\, n+1}(t_1, x, t_1, x_1, \dots, t_m, x_m;$$

$$t_1, x, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dx = 0,$$

which follows from the integrability of $g_{mn}$ with respect to difference spatial variables. We also have

$$\lim_{V(\Lambda)\to\infty} \frac{a}{V^2(\Lambda)} \int_{\Lambda^2} G_{m+1\, n+1}(t_1, x_1 + x, t_1, y, t_2, x_2, \dots, t_m, x_m;$$

$$t_1, x + y, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dxdy =$$

$$= ac^3 G_{m-1\, n}(t_2, x_2, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}), \tag{3.6}$$

where

$$c = \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda G_{10}(t, x)dx = G_{10}(0, 0) =$$

$$= \lim_{V(\Lambda)\to\infty} \frac{1}{V(\Lambda)} \int_\Lambda G_{01}(t, x)dx = G_{01}(0, 0), \quad G_{10} = g_{10}, \quad G_{01} = g_{01}.$$

This relation can be proved with the help of the representation

$$G_{m+1\, n+1}(t_1, x_1 + x, t_1, y, t_2, x_2, \dots, t_m, x_m; t_1, x + y, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$

$$= G_{10}(t, x_1 + x)G_{10}(t_1, y)G_{01}(t, x + y)G_{m-1\,n}(t_2, x_2, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) + M^{\mathrm{II}}_{m+1\,n+1}(t_1, x_1 + x, t_1, y, t_2, x_2, \dots, t_m, x_m; t_1, x + y, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}),$$

where $M^{\mathrm{II}}_{m+1\,n+1}$ is representable in the form (2.7) but summation is carried out over all decompositions $\sigma$ that do not contain subsets just with the points $t_1, x_1 + x, t_1, y$, and $t_1, x + y$ (one point in each subset). Here we take into account that the integrability of $g_{mn}$ with respect to difference spatial variables implies the equality

$$\lim_{V(\Lambda)\to\infty} \frac{a}{V^2(\Lambda)} \int_{\Lambda^2} M^{\mathrm{II}}(t_1, x_1 + x, t_1, y, t_2, x_2, \dots, t_m, x_m; t_1, x + y, t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n})dxdy = 0.$$

In view of (3.4)-(3.6), we can rewrite equations (3.3) in the form

$$i\frac{\partial}{\partial t_1} G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) =$$
$$= \left(-\frac{\Delta_1}{2m} - \mu\right) G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) +$$
$$+ 2ac_1 G_{mn}(t_1, x_1, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) -$$
$$- ac^3 G_{m-1\,n}(t_2, x_2, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \dots, t_{m+n}, x_{m+n}) +$$
$$+ i \sum_{j=m+1}^{m+n} \delta(t_1 - t_j)\,\delta(x_1 - x_j) G_{m-1\,n-1}(t_2, x_2, \dots, t_m, x_m; t_{m+1}, x_{m+1}, \overset{j}{\check{\dots}}, t_{m+n}, x_{m+n}). \tag{3.7}$$

One can easily show that equations (3.7) coincide with the equations for Green's functions in the model with the following approximating Hamiltonian:

$$H_{\mathrm{appr}} = \int a^*(x)\left(-\frac{\Delta_1}{2m} - \mu\right) a(x)dx + 2ac_1 \int a^*(x)a(x)dx -$$

$$- ac^3\left[\int a^*(x)dx + \int a(x)dx\right] - ac_1^2V + \frac{3}{2}ac^4V. \tag{3.8}$$

Note that the second term in the square brackets in $H_{\text{appr}}$ appears if we differentiate Green's functions with respect to the variables $t_j$, $m + 1 \le j \le m + n$, and the constants $BV = - ac_1^2V + \frac{3}{2}c^4V$ are chosen to guarantee the coincidence of average specific energies of the model and approximating Hamiltonians.

Indeed, assume that the average energies of the model and approximating Hamiltonians coincide, i.e.,

$$\frac{1}{V}\langle H\rangle = \frac{1}{V}\langle H_{\text{appr}}\rangle;$$

this means that

$$\frac{1}{V}\langle H_0\rangle + ac_1^2 - \frac{ac^4}{2} = \frac{1}{V}\langle H_0\rangle + 2ac_1^2 - 2ac^4 + B$$

whence

$$B = - ac_1^2 + \frac{3}{2}ac^4.$$

It is easy to show that the conditions of extremum of the specific free energy (with respect to $c_1$ and $c$) in the system with approximating Hamiltonian (21.38) coincide with equations (3.5) and (3.7) for $c_1$ and $c$.

The approximating Hamiltonian (3.8) is not diagonal since its third term is not diagonal. Let us transform it to the diagonal form. It is convenient to do this in the momentum space. We have

$$H_{\text{appr}} = \int a^*(k)\left(\frac{k^2}{2m} - \mu + 2ac_1\right)a(k)dk -$$

$$- ac^3(2\pi)[a^*(0) + a(0)] - ac_1^2V + \frac{3}{2}ac^4V. \tag{3.9}$$

Let us diagonalize the Hamiltonian of a finite system

$$H_{\text{appr},\Lambda} = \sum_k a_k^* a_k\left(\frac{k^2}{2m} - \mu + 2ac_1\right) -$$

$$- ac^3(V(\Lambda))^{1/2}[a_0^* + a_0] - ac_1^2V(\Lambda) + \frac{3}{2}ac^4V(\Lambda). \tag{3.10}$$

We select in $H_{\text{appr},\Lambda}$ the terms with $a_0^*$ and $a_0$, namely,

$$a_0^* a_0(-\mu + 2ac_1) - ac^3(V(\Lambda))^{1/2}[a_0^* + a_0]$$

and introduce new operators $\tilde{a}_0^*$ and $\tilde{a}_0$ by using the canonical transformations

$$a_0^* = \tilde{a}_0^* + \frac{ac^3 V(\Lambda)^{1/2}}{-\mu + 2ac_1}, \quad a_0 = \tilde{a}_0 + \frac{ac^3 V(\Lambda)^{1/2}}{-\mu + 2ac_1}.$$

This enables us transform the terms indicated above to the diagonal form

$$\tilde{a}_0^* \tilde{a}_0(-\mu + 2ac_1) - \frac{a^2c^6 V(\Lambda)}{-\mu + 2ac_1}.$$

We can now write $H_{\text{appr},\Lambda}$ in the diagonal form

$$H_{\text{appr},\Lambda} = \sum_k a_k^* a_k \left(\frac{k^2}{2m} - \mu + 2ac_1\right) - ac_1^2 V(\Lambda) + \frac{3}{2} ac^4 V(\Lambda) - \frac{a^2c^6 V(\Lambda)}{-\mu + 2ac_1} \tag{3.11}$$

(here, for convenience, $\tilde{a}_0^*$ and $\tilde{a}_0$ are denoted by $a_0^*$ and $a_0$, respectively). The parameter $c$ appears in $H_{\text{appr},\Lambda}$ as a separate term independent of the operators $a_k^*$, $a_k$, $a_0^*$, and $a_0$; therefore the condition of extremum of the free energy with respect to $c$ coincides with the condition of extremum of the function

$$f(c_2) = \frac{3}{2} ac^4 - \frac{a^2c^6}{-\mu + 2ac_1}.$$

It is easy to show that, for $-\mu + 2ac_1 > 0$, the function $f(c)$ attains maximum value at the point $c^2 = a^{-1}(-\mu + 2ac_1)$ and its minimum value at $c^2 = 0$. In the fitst case, $H_{\text{appr},\Lambda}$ depends on the parameter $c_1$ as follows:

$$H_{\text{appr},\Lambda} = \sum_k a_k^* a_k \left(\frac{k^2}{2m} - \mu + 2ac_1\right) - ac_1^2 V(\Lambda) + \frac{(-\mu + 2ac_1)^2}{2a} V(\Lambda). \tag{3.12}$$

In the second case, one should omit the third term. In the first case, the grand partition function $\Xi(V(\Lambda), \beta, \mu)$ of the system with $H_{\text{appr},\Lambda}$ has the form

$$\Xi(V(\Lambda), \beta, \mu) =$$

$$= \mathrm{Tr}\exp\left(-\beta\left[\sum_k a_k^* a_k\left(\frac{k^2}{2m} - \mu + 2ac_1\right) - ac_1^2 V(\Lambda) + \frac{(-\mu + 2ac_1)^2}{2a} V(\Lambda)\right]\right) =$$

$$= \Xi_0(V(\Lambda), \beta, \mu - 2ac_1)\exp\left(-\beta\left[-ac_1^2 V(\Lambda) + \frac{(-\mu + 2ac_1)^2}{2a} V(\Lambda)\right]\right), \qquad (3.13)$$

where $\Xi_0(V(\Lambda), \beta, \mu - 2ac_1)$ is the grand partition function of the free Bose gas at an inverse temperature $\beta$ with a chemical potential equal to the difference between the chemical potential $\mu$ of the original system and $2ac_1$.

Let us define pressure in the system

$$P(\beta, \mu) = \lim_{V(\Lambda)\to\infty} (\beta V(\Lambda))^{-1} \log \Xi(V(\Lambda), \beta, \mu) =$$

$$= \lim_{V(\Lambda)\to\infty} (\beta V(\Lambda))^{-1} \log \Xi_0(V(\Lambda), \beta, \mu - 2ac_1) +$$

$$+ ac_1^2 - \frac{(\mu - 2ac_1)^2}{2a} = P_0(\beta, \mu - 2ac_1) + ac_1^2 - \frac{(\mu - 2ac_1)^2}{2a}, \qquad (3.14)$$

where $P_0$ is a pressure of the free Bose gas. Passing in (3.14) to a new variable $\mu - 2ac_1 = d < 0$, we obtain

$$P(\beta, \mu) = P_0(\beta, d) + \frac{(\mu - d)^2}{4a} - \frac{d^2}{2a}. \qquad (3.15)$$

In addition, one should take the maximum of (3.15) with respect to $d < 0$.

In the second case, by a similar calculation, we get

$$P(\beta, \mu) = P_0(\beta, \mu - 2ac_1) + ac_1^2$$

or, in terms of the variable $d$,

$$P(\beta, \mu) = P_0(\beta, d) + \frac{(\mu - d)^2}{4a}. \qquad (3.16)$$

Here one should take the minimum value with respect to $d < 0$.

Relations (3.15) and (3.16) coincide with the formulas obtained by Van den Berg, Lewis, and Pulé [5]. In this paper, the authors also established the existence of the desired maximum of (3.15) and minimum of (3.16) with respect to $d$.

This means that the system under consideration may be in two states corresponding to $c^2 = a^{-1}(-\mu - 2ac_1)$ and $c^2 = 0$. The values of pressure in these states are given by relations (3.15) and (3.16), respectively. It is known that the problem of determining the state in which the system is situated can be solved as follows: if a pressure has the same value in both states but chemical potentials are different, then the system is situated in the state with the lower chemical potential. It was proved by Van den Berg, Lewis, and Pulé in [ [5] that there is $\mu^*$ such that, for $\mu \geq \mu^*$, the system is situated in the first state while, for $\mu \leq \mu^*$, it is in the second state.

## Conclusions

Note a link between our article and important papers of Angelescu, Verbeure, and Zagrebnov [6] and Van den Berg, Lewis, and Pulé [9].

In [6], the model Hamiltonian of the theory of superfluidity (2) was investigated and it was established that, for some $\mu > 0$, pressure is equal to infinity and, for some $\mu < 0$, it coincides with that of the free system. From our point of view, it is not a shortage of the Bogolyubov model, because the free Bose gas analogous properties and the model Hamiltonian (2) is thermodynamically equivalent to the free Hamiltonian of quasiparticles (2.33).

It follows from our results that, for $\mu - c^2\Phi(0) = 0$, Hamiltonian (2) or Hamiltonian (2.33) thermodynamically equivalent to it describes superfluidity; for $\mu - c^2\Phi(0) > 0$, pressure of the system is equal to infinity; and, for $\mu - c^2\Phi(0) < 0$, pressure coincides, generally speaking, with that of the free system, i.e., $c = 0$.

In [9], by using the large deviation principle, the exact expression was rigorously calculated for pressure in the HYL model.

In the present paper, the state of the HYL model is exactly determined — this is the state of a free Boson system with a renormalized chemical potential. The expression for pressure obtained in this paper completely coincides with that obtained in [9].

Finally, note that the thermodynamically equivalent Hamiltonians (2.17), (3.10) contain quadratic and linear terms with respect to $a_0^*$ and $a_0$. This means that the commonly accepted method of replacement of all operators $a_0^*$ and $a_0$ by $c$-numbers fails. In order to emphasize this fact, we described this principal replacement procedure in detail in the introduction and Sections 1 and 2.

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## О двух моделях взаимодействующего Бозе газа: модель сверхтекучести Боголюбова и модель Хуанга–Янга–Латтинжера

Д. Я. Петрина, В. И. Греджук

Исследованы уравнения для функций Грина, выведены термодинамически эквивалентные аппроксимирующие гамильтонианы для модельного гамильтониана теории сверхтекучести Боголюбова и модельного гамильтониана Хуанга–Янга–Латтинжера (HYL). Аппроксимирующие гамильтонианы содержат члены, квадратичные и линейные по операторам рождения и уничтожения с нулевым импульсом. На этой основе доказано, что модель Боголюбова описывает явление сверхтекучести при определенных значениях химического потенциала, высоких плотностях и низких температурах. Давление, вычисленное в нашей работе для модели HYL, совпадает с полученным ранее в работе Люиса и др.

## Про дві моделі взаємодіючого Бозе газу: модель надплинності Боголюбова та модель Хуанга–Янга–Латтінжера

Д. Я. Петрина, В. І. Греджук

Вивчено рівняння для функцій Гріна, виведено термодинамічно еквівалентні апроксимуючі гамільтоніани для модельного гамільтоніана теорії надплинності Боголюбова та модельного гамільтоніана Хуанга–Янга–Латтінжера (HYL). Апроксимуючі гамільтоніани містять члени, квадратичні та лінійні відносно операторів народження та знищення з нульовим імпульсом. На цій основі доведено, що модель Боголюбова описує явище надплинності при певних значеннях хімічного потенціалу, високих густинах і низьких температурах. Тиск, який підрахований в нашій роботі для моделі HYL, співпадає з отриманим раніше в роботі Льюіса та інших.