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Local and global exact controllability in Hilbert space

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The controllability problem for the linear system with generator of strongly continuous group is investigated. The criteria of exact and local exact controllability are obtained. The principal results of the paper rely essentially on some deep facts of the operator theory. The application to the problem of control of the wave process with the braking force is considered.

Introduction

Consider the system

$$\dot{x} = Ax + Bu, \quad x \in X, \quad u \in \Omega \subset U = \overline{\mathrm{Lin}\Omega}\,, \tag{1}$$

where X, U are Hilbert spaces, A is generator of the strongly continuous group $\{e^{At}\}$, $-\infty < t < +\infty$, B is a bounded operator, $B \in [U, X]$, Ω is a convex set such that $0 \in \Omega$.

In this work, we shall investigate the problem of the exact local and global controllability for the system (1).

We shall say that a control $u(t)$, $t \in [0,T]$, is admissible on $[0,T]$ if $u(\cdot) \in L_2([0,T],\Omega)$. As it is well known, the set of controllability S_T, $T > 0$, for the system (1) has the following form:

$$S_T = \left\{ x_0 \in X : x_0 = -\int_0^T e^{-At} Bu(t)\, dt, u(\cdot) \in L_2([0,T],\Omega) \right\}.$$

Definition 1. *The system (1) is called globally (locally) controllable for the time $T > 0$, if $S_T = X$ ($0 \in \mathrm{int} S_T$).*

Definition 2. *The system (1) is called globally (locally) controllable for the free time or simply – controllable if*

$$S \stackrel{\text{def}}{=} U_{T>0} S_T = X \quad (0 \in \text{int} S)\,.$$

The main goal of our work is to get necessary and sufficient condition of the global and local controllability of the system (1) under a following general assumption:

$$\exists u : u \in \text{int}\Omega\,. \tag{2}$$

This work continues the previous investigations of its authors on the controllability problem [1–4].

In Section 1, we consider the exact controllability problem for the equation (1), that is the controllability problem in the case where $\Omega = U$. Necessary and sufficient conditions are obtained in the form which generalize criterion from [2].

Next we investigate the problem of the local controllability for the system (1), (2) (Section 2). The full solution of this problem is obtained.

In Section 3, another form of the criterion of the controllability is given. We investigate also relationships between the obtained results and the results from [3]. Examples are also given.

In the concluding section, the application of our results to the problem of control of the wave process is considered.

1. The exact controllability problem

As it is known, if the operator A is bounded, then the concepts of the exact controllability for the free time and for the arbitrary fixed time $T > 0$ are equivalent [2]. It essentially follows from the independence of the criterion of the exact controllability [2] from the time. In case of the unbounded operator A these concepts are different.

E x a m p l e 1. Let $X = L_2[0,1]$, $U = L_2[0.5,1]$, $A = d/ds$, and the domain of definition $D(A)$ of A is the class of absolutely continuous functions $x(\cdot)$ on $[0,1]$ such that

$$x'(\cdot) \in L_2[0,1], \quad x(0) = x(1)\,,$$

the operator B is defined by formula

$$Bu(\cdot) = \begin{cases} u(s), & s \in [0.5,1], \\ 0, & s \in [0,0.5]. \end{cases}$$

It is known [5] that the operator A generates the strongly continuous unitary group $\{e^{At}\}$, $-\infty < t < +\infty$: $e^{At}x(s) = \tilde{x}(s+t)$, $s \in [0,1]$, where $\tilde{x}(\cdot)$ is the 1-periodic continuation of the function $x(\cdot)$ on the real axis. Now we establish the exact controllability of the equation (1) for the time $T > 0.5$. Let us make use of the criterion from [6]. Let operator N_T be defined by the formula

$$N_T x = \int_0^T e^{-At} BB^* e^{-A^* t} x \, dt \,. \tag{3}$$

Then

$$(N_T x, x) = \int_0^T ||B^* e^{-A^* t} x||^2 \, dt = \int_0^T dt \int_{t+1/2}^{t+1} |\tilde{x}(s)|^2 \, ds$$

$$= \int_{1/2}^1 (s - 0.5)|\tilde{x}(s)|^2 \, ds + 0.5 \int_1^{T+1/2} |\tilde{x}(s)|^2 \, ds + \int_{T+1/2}^{T+1} (T + 1 - s)|\tilde{x}(s)|^2 \, ds$$

$$\geq \delta \int_{T/2+1/4}^1 (s - 0.5)|\tilde{x}(s)|^2 \, ds 0.5 \int_1^{T+1/2} |\tilde{x}(s)|^2 \, ds +$$

$$+\delta \int_{T+1/2}^{T/2+5/4} (T + 1 - s)|\tilde{x}(s)|^2 \, ds \geq (T/2 - q/4)||x||^2 \,;$$

parameters δ and q are defined here in the following way: if $0.5 \leq T \leq 1.5$, then $q = 1$, $\delta = 1$; if $T > 1.5$, then $q = 2(T - 1)$, $\delta = 0$. Therefore, if $T > 0.5$, then N_T is the strongly positive operator and the equation (1) is exactly controllable.

On the other hand, if $T = 0.5$, then

$$\begin{aligned} (N_{\frac{1}{2}} x, x) &= \int_{1/2}^1 (s - 0.5)|\tilde{x}(s)|^2 \, ds + \int_0^{1/2} (0.5 - s)|\tilde{x}(s)|^2 \, ds \\ &= \int_0^1 |s - 0.5||\tilde{x}(s)|^2 \, ds \,. \end{aligned}$$

Let $\{x_n\}_{n=2}^{\infty} \subset L_2[0, 1]$, $||x_n|| = 1$, $n = 2, 3, \ldots$, is a sequence of the form

$$x_n(s) = \begin{cases} \sqrt{n}, & s \in [0.5, 0.5 + 1/n], \\ 0, & s \in [0.1] \backslash (0.5, 0.5 + 1/n]. \end{cases}$$

We have then

$$(N_{\frac{1}{2}} x_n, x_n) = n \int_{1/2}^{1/2+1/n} (s - 0.5) \, ds = \frac{1}{2n} \to 0, \quad n \to \infty.$$

Therefore the equation (1) is not controllable for the time $T = 0.5$ and, of course, for the time $T < 0.5$.

Thus the equation (1) is exactly controllable for the free time but is not exactly controllable for the arbitrary fixed time.

Introduce into consideration the bounded operator $N(\lambda)$, $\lambda > 2\omega_0(-A) = 2\lim_{t\to\infty} \frac{\ln ||e^{-At}||}{t}$, which is defined by the equality

$$N(\lambda)x = \int_0^\infty e^{-\lambda t} e^{-At} BB^* e^{-A^*t} x\, dt\,.$$

Proposition 1. *The equation (1) is exactly controllable for the free time iff $N(\lambda)$ is strongly positive operator.*

P r o o f. Assume that the equation (1) is exactly controllable for the free time. Consider a bounded operator $F_\lambda = F_\lambda(A)$: $L_2([0,\infty], U) \to X$ defined by the formula

$$F_\lambda u(\cdot) = \int_0^\infty e^{-\lambda t/2} e^{-At} Bu(t)\, dt, \quad \lambda > 2\omega_0(-A)\,.$$

Then this operator will be surjective. Therefore, due to [7], we get

$$(N(\lambda)x, x) = (F_\lambda F_\lambda^* x, x) = ||F_\lambda^* x||^2 \geq \gamma_\lambda ||x||^2, \quad \gamma_\lambda > 0.$$

Contrary, let $N(\lambda)$ be a strongly positive operator, that is,

$$(N(\lambda)x, x) = \gamma_\lambda ||x||^2, \quad \gamma_\lambda > 0.$$

From the estimation

$$\int_T^\infty e^{-\lambda t} ||B^* e^{-A^*t} x||^2\, dt \leq \frac{M_\varepsilon}{\lambda - 2\omega_0(-A) - \varepsilon}\, e^{-(\lambda - 2\omega_0(-A) - \varepsilon)T} ||x||^2,$$

$M_\varepsilon > 0$, $\varepsilon < \lambda - 2\omega(-A)$, we get that

$$\int_0^T ||B^* e^{-A^*t} x||^2\, dt \geq \min\{1, e^{\lambda T}\} \left((N(\lambda)x, x) - \int_T^\infty e^{-\lambda t} ||B^* e^{-A^*t} x||^2\, dt \right)$$

$$\geq \min\{1, e^{\lambda T}\} \left(\gamma - \frac{M_\varepsilon}{\lambda - 2\omega_0(-A) - \varepsilon}\, e^{-(\lambda - 2\omega_0(-A) - \varepsilon)T} \right) ||x||^2\,.$$

Consequently, the operator N_T defined by (3) is strongly positive. Then the system (1) is exactly controllable for the time T due to the criterion [6].

R e m a r k. From the proof of the Proposition 1 it follows that if the equation (1) is exactly controllable then there exist $T > 0$ such that operator N_T is positive defined. In this case, for arbitrary $x \in X$ we have for $u_x(\cdot) \in L_\infty\{[0,T], U\}$

$$u_x(t) = \begin{cases} -B^* e^{\lambda t/2} e^{-A^*t} N_T^{-1} x, & t \in [0,T], \\ 0, & t > T, \end{cases}$$

$F_\lambda u_x(\cdot) = x$.

This fact we will write in the form

$$F_\lambda L_\infty\{[0,T],U\} = X\,.$$

We now establish some properties of the operator $N(\lambda)$.

Proposition 2. *The operator $N(\lambda)$ maps the domain of definition $D(A^*)$ of the operator A^* into the domain of definition $D(A)$ of the operator A. Moreover, if $x \in D(A^*)$, then the following relation holds:*

$$AN(\lambda)x + N(\lambda)A^*x + \lambda N(\lambda)x = BB^*x\,. \tag{4}$$

P r o o f. Let $x \in D(A^*)$. Since

$$\begin{aligned} e^{A\Delta}N(\lambda)x &= \int_0^\infty e^{-A(t-\Delta)}BB^*e^{-A^*t}e^{-\lambda t}\,xdt \\ &= \int_{-\Delta}^\infty e^{-At}BB^*e^{-(A^*+\lambda I)t}e^{-(A^*+\lambda I)\Delta}x\,dt = N(\lambda)e^{-(A^*+\lambda I)\Delta}x \\ &\quad + \int_\Delta^0 e^{-At}BB^*e^{-(A^*+\lambda I)t}e^{-(A^*+\lambda I)\Delta}x\,dt\,, \end{aligned}$$

then we have

$$\begin{aligned} \lim_{\Delta\to 0}\frac{1}{\Delta}(e^{A\Delta}-I)N(\lambda)x &= N(\lambda)\lim_{\Delta\to 0}\frac{1}{\Delta}\left(e^{-(A^*+\lambda I)\Delta}-I\right)x \\ &+ \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{-\Delta}^0 e^{-At}BB^*e^{-(A^*+\lambda I)t}e^{-(A^*+\lambda I)\Delta}x\,dt \\ &= -N(\lambda)(A^*+\lambda I)x + BB^*x\,. \end{aligned} \tag{5}$$

Therefore $\lim_{\Delta\to 0}\frac{1}{\Delta}(e^{A\Delta}-I)N(\lambda)x$ exists. This implies [5] at first that $N(\lambda)x \in D(A)$ and at second that

$$\lim_{\Delta\to 0}\frac{1}{\Delta}(e^{A\Delta}-I)N(\lambda)x = AN(\lambda)x\,. \tag{6}$$

From (5), (6) we have relation (4). The proof is complete.

Denote $R_\lambda = R_\lambda(A) = (\lambda I + A)^{-1}$, $T_\lambda = T_\lambda(A) = AR_\lambda = I - \lambda R_\lambda$, $\lambda > \omega_0(-A)$. Then the equality (4) can be written in the form

$$((A+\lambda I)N(\lambda) + N(\lambda)(A^*+\lambda I) - \lambda N(\lambda))x = BB^*x, \quad x \in D(A^*)\,.$$

Since every $x \in D(A^*)$ can be uniquely represented in the form $x = R_\lambda^* y$, then we have for $\lambda > \max\{2\omega_0(-A), \omega_0(-A)\}$,

$$(A + \lambda I)N(\lambda)R_\lambda^* y + N(\lambda)y - \lambda N(\lambda)R_\lambda^* y = BB^* R_\lambda^* y, y \in X\,.$$

From this we obtain the operator equality

$$\lambda N(\lambda)R_\lambda^* + \lambda R_\lambda N(\lambda) - \lambda^2 R_\lambda N(\lambda)R_\lambda^* = \lambda R_\lambda BB^* R_\lambda^*\,,$$

which can be written in the following form:

$$N(\lambda) = T_\lambda N(\lambda)T_\lambda^* + \lambda R_\lambda BB^* R_\lambda^*\,. \tag{7}$$

From the formula (7) we have at last that

$$N(\lambda) = T_\lambda^{n+1} N(\lambda) T_\lambda^{*^{n+1}} + \lambda R_\lambda \sum_{k=0}^{n} T_\lambda^k BB^* T_\lambda^{*^k} R_\lambda^*\,, \tag{8}$$

$\lambda > \max\{2\omega_0(-A), \omega_0(-A)\}$, $k = 1, 2, \ldots$.

Lemma 1. *For arbitrary* $\lambda > \max\{2\omega_0(-A), 0\}$ *and for arbitrary* $x \in X$ *the following equalities hold:*

$$\lim_{k\to+\infty} T_\lambda^k x = 0. \qquad \lim_{k\to+\infty} T_\lambda^{*^k} x = 0\,.$$

P r o o f. Let the operator $N_*(\lambda)$ be defined by formula

$$N_*(\lambda)x = \int_0^\infty e^{-\lambda t} e^{-At} e^{-A^* t} x\, dt, \quad x \in X\,.$$

Since

$$||e^{-A^* t}x||^2 \geq m e^{-\nu t}||x||^2, \quad t > 0, \quad m, \nu > 0\,,$$

then

$$(N_*(\lambda)x, x) = \int_0^\infty e^{-\lambda t}||e^{-A^* t}x||^2\, dt \geq m \int_0^\infty e^{-(\lambda+\nu)t}\, dt||x||^2 = \frac{m}{m+\nu}||x||^2\,.$$

Consequently, $N_*(\lambda)$ is a bounded strongly positive operator. Then the Hilbert norma $||x||_* = \|\sqrt{N_*(\lambda)}x\|$ is equivalent to the former norm, that is,

$$c_1||x||_* \leq ||x|| \leq c_2||x||_*, \quad c_1, c_2 > 0\,. \tag{9}$$

Let us now make use of formula (6), replacing the operators $N(\lambda)$ and B by $N_*(\lambda)$ and I, respectively:

$$||x||_*^2 = (N_*(\lambda)x, x) = (N_*(\lambda)T_\lambda^* x, T_\lambda^* x) + \lambda||R_\lambda^* x||^2 \geq ||T_\lambda^* x||_*^2\,. \tag{10}$$

Therefore T_λ^* is a contracting operator in the norm $||\cdot||_*$. The spectrum $\sigma(T_\lambda^*)$ of the operator T_λ^* is defined by the equality [8]

$$\sigma(T_\lambda^*) = \sigma(A)/(\lambda + \sigma(A))^*,$$

if the operator A is unbounded, we take that $\infty \in \sigma(A)$. Since for every finite $\mu \in \sigma(A)$ the unequality $\mathrm{Re}\,\mu \geq -\omega_0(-A)$ holds, then for $\lambda > \max\{2\omega_0(-A), 0\}$ we get $\mathrm{Im}\,\mu = \mathrm{Im}\,(\lambda + \mu)$, $\mathrm{Re}\,\mu < \mathrm{Re}\,(\lambda + \mu)$. Therefore $|\mu/(\lambda + \mu)| < 1$. Thus the intersection $\sigma(T_\lambda^*) \cap \{z : |z| = 1\}$ may contain at most one point, namely, the point 1 (in the case of nonbounded operator A). Moreover, number 1 is not eigenvalue of T_λ^*. Therefore [9], the powers of the operator T_λ^* strongly converge to zero in the norm $||\cdot||_*$. From this and (9) the first statement of lemma follows. The proof of the second equality may be obtained by the formal substitution $A = A^*$.

Theorem 1. *For every* $\lambda > \max\{2\omega_0(-A), 0\}$ *and for every* $x \in X$, *the following decomposition holds:*

$$N(\lambda)x = \lambda \sum_{k=0}^{\infty} R_\lambda T_\lambda^k B B^* T_\lambda^{*^k} R_\lambda^* x\,.$$

P r o o f. From (9), (10) we have for every $k = 0, 1, 2, \ldots$

$$\| T_\lambda^{n+1} x \| \leq c_2 \| T_\lambda^{n+1} x \|_* \leq c_2 \| T_\lambda \|_*^{n+1} ||x||_* \leq c_2 ||x||_* \leq c_2/c_1 ||x||\,.$$

Therefore $\| T_\lambda^{n+1} \| \leq c_2/c_1$. Then due to (8) the following relation holds:

$$\| N(\lambda)x - \lambda \sum_{k=0}^{n} R_\lambda T_\lambda^k B B^* T_\lambda^{*^k} R_\lambda^* x \| \leq \| T_\lambda^{n+1} \| \| N(\lambda) \| \| T_\lambda^{*^{n+1}} x \|$$

$$\leq c_2/c_1 \| N(\lambda) \| \| T_\lambda^{*^{n+1}} x \| \to 0, \quad n \to \infty\,.$$

Thus the theorem is proved.

Now we can obtain the criterion of the exact controllability. Let U^∞ be Hilbert space with elements

$$u^\infty = (u_0, u_1, \ldots, u_k, \ldots), u_k \in U, k = 0, 1, \ldots\,,$$

and with the norm

$$||u^\infty|| = \left(\sum_{k=0}^{\infty} ||u_k||^2\right)^{1/2}.$$

Denote for $\lambda > \max\{2\omega_0(-A), 0\}$ by $F_{\lambda,A}$ operator

$$F_{\lambda,A} : X \to U^\infty, F_{\lambda,A}x = (B^* R^*_\lambda x, B^* R^*_\lambda T^*_\lambda x, \ldots, B^* R^*_\lambda T^{*^k}_\lambda x, \ldots) .$$

From Theorem 1 we have

$$\| F_{\lambda,A}x \|^2 = \sum_{k=0}^{\infty} (R_\lambda T^k_\lambda BB^* T^{*^k}_\lambda R^*_\lambda x, x) = \frac{1}{\lambda}(N(\lambda)x, x) \le \frac{1}{\lambda}||N(\lambda)||||x||^2 . \quad (11)$$

Hence the operator $F_{\lambda,A}$ is bounded. Then the conjugate operator $G_{\lambda,A} = F^*_{\lambda,A}$, $G_{\lambda,A}$: $U^\infty \to X$ is also bounded. This operator will play an important role for us further. It is defined by formula

$$G_{\lambda,A}u^\infty = \sum_{k=0}^{\infty} R_\lambda T^k_\lambda B u_k ,$$

which also may be written in the form

$$G_{\lambda,A} = (R_\lambda B, R_\lambda T_\lambda B, \ldots, R_\lambda T^k_\lambda B, \ldots) .$$

Theorem 2 (criterion of the exact controllability). *The equation (1) is exactly controllable iff the operator $G_{\lambda,A}$ for $\lambda > \max\{2\omega_0(-A), 0\}$ is surjective:*

$$G_{\lambda,A}U^\infty = X ,$$

that is,

$$(R_\lambda B, R_\lambda T_\lambda B, \ldots, R_\lambda T^k_\lambda B, \ldots)U^\infty = X. \quad (12)$$

P r o o f. Let the equation (1) be exactly controllable. Then, by Proposition 1, the operator $N(\lambda)$ is strongly positive and thus $N(\lambda)$ is surjective. But we have from Theorem 1

$$N(\lambda) = \lambda G_{\lambda,A} G^*_{\lambda,A} .$$

Consequently, the operator $G_{\lambda,A}$ is surjective. Conversely, let the operator $G_{\lambda,A}$ be surjective. Then [7] the following unequality is true:

$$||G(\lambda)x|| \ge \gamma||x||, \quad \gamma > 0 .$$

Hence from (11) we have

$$(N(\lambda)x, x) \ge \lambda\gamma^2||x||^2 .$$

Therefore $N(\lambda)$ is a strongly positive operator and the equation (1) is exact controllable due to the Proposition 1. Thus the theorem is proved.

In the concluding part of this section we shall establish the direct connection between the obtained criterion and the criterion from [2] in the case, when the operator A is bounded. We choose $\lambda_0 > \max\{2\omega_0(-A), 0\}$ such large that for $\lambda > \lambda_0$ unequality $\| R_\lambda \| \leq q||A||^{-1}$, $q < 1$, is valid. Let the operator $P_n(\lambda) \in [U^\infty, X]$, $n = 0, 1, 2, \dots$ be defined by equality

$$P_n(\lambda)u^\infty = \sum_{k=0}^{n} R_\lambda T_\lambda^k B u_k \,.$$

Then

$$\| (G_{\lambda,A} - P_n(\lambda))u^\infty \| = \| T_\lambda^{n+1} \sum_{k=0}^{\infty} R_\lambda T_\lambda^k B u_{k+n+1} \| \leq q^{n+1} \| G_{\lambda,A} \| \, ||u^\infty|| \,,$$

that is, $P_n(\lambda) \to G_{\lambda,A}$, $n \to \infty$ in the operator norm. Hence, if the operator A is surjective, then, [7], there exist a number m such that the operator $P_m(\lambda)$ is also surjective, that is,

$$\mathrm{Lin}\{R_\lambda BU, R_\lambda T_\lambda BU, \dots, R_\lambda T_\lambda^m BU\} = X \,. \tag{13}$$

Since the operator $(A + \lambda I)$ is invertible, then from (13) we have

$$\mathrm{Lin}\{(A + \lambda I)^m BU, (A + \lambda I)^{m-1} BU, \dots, A^m BU\} = X \,.$$

The last equality and the equality

$$\mathrm{Lin}\{BU, ABU, \dots, A^m BU\} = X \tag{14}$$

(from [2]) are evidently equivalent. Thus the criterion of the exact controllability [2] arises from the (12).

Conversely, the equation (14) for some $m = 0, 1, \dots$ implies that the operator $P_m(\lambda)$ is invertible, $\lambda > \omega_0(-A)$. On the other hand, $\mathrm{Im} G_{\lambda,A} \supset \mathrm{Im}\, P_m(\lambda)$. Thus the condition (12) follows from the criterion [2] in case of bounded operator A.

2. The main result

The necessary and sufficient conditions of the local controllability of the system (1), (2) with a bounded operator A is [3] $\exists m \in \{0, 1, 2, \dots\}$, that is,

$$0 \in \mathrm{int}\,\mathrm{co}\{B\Omega, AB\Omega, \dots, A^m B\Omega\} \,,$$

$$0 \in \mathrm{int}\,\mathrm{co}\{B\Omega, -AB\Omega, \dots, (-1)^m A^m B\Omega\} \,.$$

The main goal of this part is to generalize this criterion on the case of unbounded operator A. Let $Q \subset U$. We denote further by $Q^\infty \subset U^\infty$ the set

$$Q^\infty = \{u^\infty \subset U^\infty : u^\infty = (u_0, u_1, \ldots, u_k, \ldots), \quad u_k \in Q, k = 0, 1, \ldots\}.$$

Then the following criterion of local controllability holds.

Theorem 3. *The system (1), (2) is locally controllable for the free time iff*

$$\frac{1}{\lambda} B\bar{u} \in \operatorname{int} G_{\lambda,A}(\bar{u} - \Omega)^\infty, \quad \frac{1}{\lambda} B\bar{u} \in \operatorname{int} G_{\lambda,-A}(\bar{u} - \Omega)^\infty,$$
$$\lambda > \max\{2\omega_0(A), 2\omega_0(-A), 0\}.$$

P r o o f. N e c e s s i t y. Let the system (1), (2) be locally controllable for the free time, that is $0 \in \operatorname{int} S$. Then the equation (1) is exactly controllable and due to the theorem 2

$$G_{\lambda,A} U^\infty = X, \quad \lambda > \max\{2\omega_0(-A), 0\}.$$

Besides, the equation

$$\dot{x} = -Ax + Bu, \quad u \in \Omega, \tag{15}$$

is also exactly controllable (this fact can be proved completely similarly as in the case of the bounded operator A [3]). Therefore (theorem 2)

$$G_{\lambda,-A} U^\infty = X, \quad \lambda > \max\{2\omega_0(A)\}.$$

Introduce into consideration a Banach space $\tilde{U}^\infty = U^\infty \times L$, where $L = \{\mu\bar{u}^\infty, \mu \in R\}$, $\bar{u}^\infty = \{\bar{u}, \bar{u}, \ldots, \bar{u}, \ldots\}$ with elements $\tilde{u}^\infty = (u^\infty, \mu\bar{u}^\infty)$ and with the norm $||\tilde{u}^\infty|| = ||u^\infty|| + |\mu|$. Now we extend the operators $G_{\lambda,\mp A}$ to the space $\tilde{U}^\infty$. For this purpose we note that

$$\sum_{k=0}^{\infty} R_\lambda T_\lambda^k B\bar{u} = \frac{1}{\lambda} B\bar{u}. \tag{16}$$

In fact,

$$\sum_{k=0}^{n} R_\lambda T_\lambda^k B\bar{u} = (I - T_\lambda^{n+1})(I - T_\lambda)^{-1} R_\lambda B\bar{u} = \frac{1}{\lambda} B\bar{u} - \frac{1}{\lambda} T_\lambda^{n+1} B\bar{u}.$$

Since, by the Lemma 1, the relation $T_\lambda^{n+1} B\bar{u} \to 0$ holds, then the equality (16) holds too. For this reason, we define extensions $\tilde{G}_{\lambda,\mp A}$ of operators $G_{\lambda,\mp A}$ to the space $\tilde{U}^\infty$ by the equality

$$\tilde{G}_{\lambda,\mp A} \tilde{u}^\infty = G_{\lambda,\mp A} u^\infty + \frac{\mu}{\lambda} B\bar{u}.$$

Let $K \subset \tilde{U}^\infty$ be the cone generated by the set $((\Omega - \bar{u})^\infty, \bar{u}^\infty)$, that is,

$$K = \{\tilde{u}^\infty \in \tilde{U}^\infty : \ \tilde{u}^\infty = \gamma((\Omega - \bar{u})^\infty, \bar{u}^\infty), \ \gamma > 0\}.$$

Then the element $(\bar{u}, 0^\infty)$ belongs to $\operatorname{int} K$. We denote by K^+, K^- convex cones defined by the equalities $K^+ = \tilde{G}_{\lambda,A} K$, $K^- = \tilde{G}_{\lambda,-A} K$. Since $\operatorname{Im} \tilde{G}_{\lambda,A} = \operatorname{Im} \tilde{G}_{\lambda,-A} = X$, then by the theorem on an open mapping we have that $\operatorname{int} K^+ \neq 0$, $\operatorname{int} K^- \neq 0$.

Next we shall show that the cone K^+ is invariant with respect to the operator $T_\lambda(A)$. In fact, let $x \in K^+$. Hence, from (16)

$$x = \gamma \left(\frac{1}{\lambda} B\bar{u} + \sum_{k=0}^{\infty} R_\lambda T_\lambda^k B u_k \right) = \gamma \sum_{k=0}^{\infty} R_\lambda T_\lambda^k B(u_k + \bar{u}),$$

$$\gamma > 0, \quad (u_1, u_2, \ldots) \in (\Omega - \bar{u})^\infty.$$

Therefore

$$\begin{aligned} T_\lambda x &= \gamma \sum_{k=0}^{\infty} R_\lambda T_\lambda^{k+1} B(u_k + \bar{u}) = \gamma \left(\sum_{k=0}^{\infty} R_\lambda T_\lambda^k B\bar{u} - R_\lambda B\bar{u} \right. \\ &\left. + \sum_{k=1}^{\infty} R_\lambda T_\lambda^k B u_{k-1} \right) = \gamma \left(\frac{1}{\lambda} B\bar{u} + \sum_{k=0}^{\infty} R_\lambda T_\lambda^k B {u'}_k \right), \end{aligned}$$

where ${u'}_0 = -\bar{u}$, ${u'}_k = u_{k-1}$, $k = 1, 2, \ldots$, and element $(u'_0, u'_1, \ldots) \in (\Omega - \bar{u})^\infty$. Thus $T_\lambda x \in K^+$. In the same way we can establish the invariance of the cone K^- with respect to the operator $T_\lambda(-A)$.

We shall now prove that $K^+ = X$, $K^- = X$. In fact, suppose that $K^+ \neq X$. Then due to the theorem on invariant cone [10] there exists the eigenvector $f \in X$ of the operator $T_\lambda^*(A)$: $T_\lambda^* f = \mu f$, $\mu \geq 0$ such that $(f, x) \geq 0$, $x \in K^+$. Since $\mu = 1$ is not an eigenvalue of the operator T_λ^*, then $f \in D(A^*)$ and $A^* f = \nu f$, where $\nu = \mu\lambda/(1 - \mu)$.

Next we shall establish that $(f, Bu) \geq 0$ for every $u \in \Omega$. Denote

$$x^n = \sum_{k=0}^{n} R_\lambda T_\lambda^k B u + \sum_{k=n+1}^{\infty} R_\lambda T_\lambda^k B\bar{u}.$$

Therefore, by (16),

$$x^n = \frac{1}{\lambda} B\bar{u} + \sum_{k=0}^{\infty} R_\lambda T_\lambda^k B u_k^n,$$

where $u_k^n = u - \bar{u}$, $k = 1, \ldots, n$; $u_k^n = 0$, $k = n+1, n+2, \ldots$. Thus

$$x^n = G_{\lambda,A}(u_n^\infty, \bar{u}),$$

where $u_n^\infty = (u_1^n, u_2^n, \ldots) \in ((\Omega - \bar{u})^\infty, \bar{u})$ and $x^n \in K^+$. Besides, due to Lemma 1,

$$x^n \to \sum_{k=0}^{\infty} R_\lambda T_\lambda^k Bu = \frac{1}{\lambda} Bu, \ n \to \infty .$$

Hence we have

$$(f, Bu) = \lambda \lim_{n \to +\infty} (f, x^n) \geq 0 .$$

From this relation it follows

$$(f, -\int_0^T e^{-At} Bu(t)\, dt) = -\int_0^T e^{-\nu t} (f, Bu(t))\, dt \leq 0$$

for every $T \geq 0$ and for every function $u(\cdot) \in L_2([0,T), \Omega)$. This unequality contradicts the assumption of the local controllability (1), (2). Thus $K^+ = X$.

The supposition that $K^- \neq X$ contradicts the local controllability of the system (15), (2). Hence $K^- = X$.

Now note that, due to (2), $0^\infty \in \operatorname{int}(\Omega - \bar{u})^\infty \subset U^\infty$. Then, according to the theorem on open mapping, each of the sets

$$M^\mp = \tilde{G}_{\lambda, \mp A}(\bar{u}^\infty, (\Omega - \bar{u})^\infty) = \frac{1}{\lambda} B\bar{u} + G_{\lambda, \mp A}(\Omega - \bar{u})^\infty \subset X$$

has non-empty interior. We shall prove that

$$0 \in \operatorname{int} \left(\frac{1}{\lambda} B\bar{u} + G_{\lambda, \mp A}(\Omega - \bar{u})^\infty \right) . \qquad (17)$$

In fact, if $0 \notin \operatorname{int} M^+$ or $0 \notin \operatorname{int} M^-$, then there exists a functional $f \in X^* = X$ such that $(f, x) \geq 0$, $x \in M^+$ or $x \in M^-$. This contradicts the established relations $K^+ = K^- = X$. Finally, from (17) evidently it follows

$$\frac{1}{\lambda} B\bar{u} \in \operatorname{int} G_{\lambda, \mp A}(\bar{u} - \Omega)^\infty .$$

The necessity is proved.

S u f f i c i e n c y. Since by the assumption of the theorem the sets $G_{\lambda, \mp A}(\bar{u} - \Omega)^\infty$ have a non-emply interior, then $\operatorname{int} \operatorname{Im} G_{\lambda, \mp A} \neq 0$ and $G_{\lambda, \mp A} U^\infty = X$. Hence the equation (1) is the exact controllable. Therefore, by remark to Proposition 1, we have

$$F_\lambda L_\infty\{[0,T], U\} = X . \qquad (18)$$

Next we argue by contradiction. Let the system (1), (2) be not locally controllable. The set $L_\infty\{[0,T], \Omega\} \subset L_\infty\{[0,T], U\}$ has an inner point $\bar{u}(t) = \bar{u}$, $t \in [0,T]$. Then, by (18) due to the theorem on an open mapping, the set $F_\lambda L_\infty\{[0,T], \Omega\}$ has non-empty interior in X. Since $F_\lambda L_\infty\{[0,T], \Omega\} \subset$

$F_\lambda L_2\{[0,\infty],\Omega\}$, then $P = F_\lambda L_2\{[0,\infty],\Omega\} \subset X$ is the set with non-empty interior. Besides, this set is convex and the point 0 belongs to its boundary.

Denote by Q the cone generated by P, that is $Q = \{q : q = \gamma x, x \in P, \gamma > 0\}$. Then Q is a convex cone with non-empty interior, which is invariant with respect to the commutative semi-group of operators $\{e^{-At}\}$, $t \geq 0$. Due to the theorem on invariant cone [10], there exist a vector $h \in X$ such that

$$e^{-A^*t}h = \gamma(t)h, \quad \gamma(t) \geq 0\,, \tag{19}$$

$$(h,x) \geq 0, \quad x \in Q\,. \tag{20}$$

Since the space Lin $\{h\}$ is invariant with respect to the semi-group $\{e^{-A^*t}\}$, $t \geq 0$, then $h \in D(A^*)$. Hence, by (19), $A^*h = \mu h$, $\mu = \gamma'(0)$. Therefore h is an eigenvector of the operators $T^*_\lambda(A)$ and $T^*_\lambda(-A)$:

$$T^*_\lambda(A)h = \frac{\mu}{\lambda+\mu}h, \quad T^*_\lambda(-A)h = \frac{-\mu}{\lambda-\mu}h\,. \tag{21}$$

Next let for some $T > 0$ and arbitrary $u \in \Omega$

$$u(t) = \begin{cases} u \in \Omega, & 0 \leq t \leq T\,, \\ 0, & T < t\,. \end{cases}$$

Then, by (20),

$$(h, F_\lambda u(\cdot)) = (h, R_\lambda Bu) - (h, e^{-(A+\lambda I)T}R_\lambda Bu) \geq 0\,.$$

From this we have when $T \to +\infty$

$$(h, R_\lambda Bu) \geq 0, \quad \forall u \in \Omega\,. \tag{22}$$

Since $\mu \leq \omega_0(a) < \lambda$, $\lambda > 0$, then at least one of the numbers $\frac{\mu}{\lambda+\mu}$ or $\frac{-\mu}{\lambda-\mu}$ is non-negative. From this and due to (21), (22) the following unequalities hold:

$$(h, R_\lambda T^n_\lambda Bu) = (T^{*^n}_\lambda h, R_\lambda Bu) \geq 0, \quad n = 1,2,\ldots$$

at least for one of the operators $T^*_\lambda = T^*_\lambda(A)$ or $T^*_\lambda = T^*_\lambda(-A)$. Let further denote $G = G_{\lambda,A}$, if $T_\lambda = T_\lambda(A)$ and $G = G_{\lambda,-A}$, if $T_\lambda = T_\lambda(-A)$.

Hence, for every $u^\infty = (u_0, u_1, \ldots) \in (\bar{u} - \Omega)^\infty$, we have

$$\begin{aligned}(h, Gu^\infty) &= \lim_{n\to+\infty}\left[-(h, \sum_{k=0}^{n} R_\lambda T^k_\lambda B(\bar{u}-u_k)) + (h, \sum_{k=0}^{n} R_\lambda T^k_\lambda B\bar{u})\right] \\ &\leq \lim_{n\to+\infty}(h, \sum_{k=0}^{n} R_\lambda T^k_\lambda B\bar{u}) = (h, \frac{1}{\lambda}Bu)\,.\end{aligned}$$

Therefore $\frac{1}{\lambda}Bu \in \operatorname{int} G(\bar{u}-\Omega)^\infty$. That contradicts the initial assumptions. The theorem is proved.

Corollary 1. *The space $\tilde{U}^\infty$ can be interpreted as the space of arbitrary sequences $\tilde{v}^\infty = (v_0, v_1, \ldots)$, $v_n \in U$, $n = 0, 1, \ldots$, such that:*

i) there exist $\lim_{n\to+\infty} v_n = \mu\bar{u}$, $\mu \in R$;
ii) $\sum_{n=0}^{\infty} ||v_n - \lim_{k\to+\infty} v_k||^2 < \infty$
with the norm

$$||\tilde{v}^\infty|| = ||\lim_{n\to+\infty} v_n||/||\bar{u}|| + \left(\sum_{n=0}^{\infty} ||v_n - \lim_{k\to+\infty} v_k||^2\right)^{1/2}.$$

In so doing, we will write $\tilde{u}^\infty = (u_0 + \mu\bar{u}, u_1 + \mu\bar{u}, \ldots)$ instead of $\tilde{u}^\infty = (u^\infty, \mu\bar{u})$, $u^\infty \in U^\infty$.

Consider set $\tilde{\Omega}^\infty(\bar{u}) = \{\tilde{v}^\infty \in \tilde{U}^\infty : v_n \in \Omega, n = 0, 1, \ldots; \lim_{n\to+\infty} v_n = \bar{u}\}$. It is clear that

$$\tilde{\Omega}^\infty(\bar{u}) = \bar{u}^\infty + (\Omega - \bar{u})^\infty.$$

Therefore we can give another statement of Theorem 3.

The system (1), (2) is the locally controllable iff

$$0 \in \operatorname{int} \tilde{G}_{\lambda,A}\tilde{\Omega}^\infty(\bar{u}), \quad 0 \in \operatorname{int}\tilde{G}_{\lambda,-A}\tilde{\Omega}^\infty(\bar{u}). \tag{23}$$

Corollary 2. *Let Ω be a cone. Then the conditions of Theorem 3 are necessary and sufficient for the global controllability of the system (1), (2).*

3. Another form of the criterion of the controllability

Now we will give conditions of the local controllability in terms of the conjugate operator A^*.

Theorem 4. *The system (1), (2) is locally controllable iff:*

i) the equation (1) is exactly controllable;
ii) for the operator A^* there not exist an eigenvector h with a real responding eigenvalue such that

$$(h, Bu) \geq 0, \quad when\, u \in \Omega. \tag{24}$$

P r o o f. N e c e s s i t y. Let the system (1), (2) be local controllable. Then it is clear that the equation (1) is exact controllable. Next we argue by contradiction. Let there exist the eigenvector h: $A^*h = \mu h$, $\mu \in R$ such that condition (24) holds. Then this vector will be an eigenvector for the operators $T^*_\lambda(\mp A)$ and

$$T^*_\lambda(A)h = \frac{\mu}{\lambda+\mu}h, \quad T^*_\lambda(-A)h = \frac{-\mu}{\lambda-\mu}h.$$

As it has been noted (proof of Theorem 2), at least one of the numbers $\nu = \frac{\mu}{\lambda+\mu}$ or $\nu = \frac{-\mu}{\lambda-\mu}$ in non-negative. Let further denote by T_λ^* one of the operators $T_\lambda^*(\mp A)$ for which eigenvalue ν is non-negative. Then we have for every $u \in \Omega$

$$(h, R_\lambda T_\lambda^n Bu) = (T_\lambda^{*^n} R_\lambda^* h, Bu) = \gamma \nu^n (h, Bu), \quad n = 0, 1, 2, \dots ,$$

where $\gamma = \frac{1}{\lambda+\mu}$, if $T_\lambda^* = T_\lambda^*(A)$, and $\gamma = \frac{1}{\lambda-\mu}$, if $T_\lambda^* = T_\lambda^*(-A)$. This equalities together with (24) contradict the conditions (23).

S u f f i c i e n c y. Let the system (1), (2) is not locally controllable. Then at least one of the conditions (23) is not true. For example, $0 \notin \mathrm{int}\tilde{G}_{\lambda,A}\tilde{\Omega}^\infty(\bar{u})$. Hence the cone K generated by the set $\tilde{G}_{\lambda,A}\tilde{\Omega}^\infty(\bar{u})$ is convex, has non-empty interior (due to the Theorem 2) and $K \neq X$. Since the cone K is invariant with respect to the operator $T_\lambda(A)$ then for the operator $T_\lambda^*(A)$ there exists eigenvector h with real eigenvalue such that $(h, x) \geq 0$, $x \in \tilde{G}_{\lambda,A}\tilde{\Omega}^\infty(\bar{u})$. As in the proof of Theorem 3, the vector h will be also an eigenvector for the operator A^* and the corresponding eigenvalue will be real. To show that $(h, Bu) \geq 0$, $u \in \Omega$, we consider the sequence $\tilde{v}_n^\infty = (v_0^n, \dots, v_k^n, \dots) \in \tilde{\Omega}^\infty(\bar{u})$, where $v_k^n = u \in \Omega$, $k = 0, \dots, n$, $k = n+1, \dots$. Then

$$(h, Bu) = \lambda \lim_{n\to+\infty} (h, \tilde{G}_{\lambda,A}\tilde{v}_n^\infty) \geq 0 .$$

This contradicts the condition ii). The theorem is proved.

Corollary 3. *i) Let the equation (1) be exactly controllable and the operator A^* has no real eigenvalues, then the system (1), (2) is locally controllable.*

ii) If, besides, the set Ω is a cone with non-empty interior, then, moreover, the system is globally controllable.

It is clear that the obtained criterion of the local controllability (Theorem 3) is of the same type as the criterion [3] in the case of a bounded operator A. But it is rather difficult to establish immediately a connection between these conditions (as it has been done for the conditions of exact controllability in Section 1). At the same time that can be established in the following way. The condition (12) of the exact controllability in the case of a bounded operator A takes the form (14). Hence, in this case the conditions of the local controllability become:

i) $\exists m :\ \mathrm{Lin}\,\{Bu, ABu, \dots, A^m Bu\} = X$;

ii) $\exists h \neq 0 :\ A^* h = \lambda h,\ \lambda \in R;\ (h, Bu) \geq 0, u \in \Omega$.

And, as it has been shown in [3], the last conditions are equivalent to the conditions

$$\exists m :\ 0 \in \mathrm{int}\,\{B\Omega, AB\Omega, \dots, A^m B\Omega\} ,$$

$$0 \in \operatorname{int}\{B\Omega, -AB\Omega, \dots, (-1)^m A^m B\Omega\}.$$

E x a m p l e 2. Consider the equation from example 1. It is known that the spectrum of the operator A is of the form

$$\sigma(A) = \{2i\pi k,\, k = 0, \mp 1, \mp 2, \dots\}.$$

Besides, every point of this spectrum is eigenvalue. Hence there exists the unique real eigenvalue $\lambda = 0$, which responds to the eigenvector $v \equiv 1$.

Thus if Ω is an arbitrary convex set in $L_2[0.5, 1]$, which contains zero and satisfies the condition (2), then the system (1), (2) will be locally controllable iff there exist two function u_1, $u_2 \in \Omega$ such that

$$(\int_{1/2}^{1} u_1(t)\, dt > 0, \quad \int_{1/2}^{1} u_2(t)\, dt < 0.$$

For example, if

$$\Omega = \left\{ u(\cdot) \in L_2[0.5, 1] : \quad \int_{1/2}^{1} (u(t) - t)^2\, dt \le \frac{7}{24} \right\},$$

then the system is locally controllable, if

$$\Omega = \left\{ u(\cdot) \in L_2[0.5, 1] : \quad \int_{1/2}^{1} (u(t) - 1)^2\, dt \le \frac{1}{2} \right\},$$

then the system is not locally controllable.

4. The control wave process with the braking force

Let $G \subset R^n$ be a closed bounded domain in R^n with a boundary ∂G of the class C^2. Consider in G the control wave process, which is given by the equation

$$\psi_{tt}(t, s) = \Delta\psi(t, s) - \mu\psi_t(t, s) + u(t, s), \quad \psi_{\partial G} = 0, \tag{25}$$

where $s \in G$, $t \in [0, \infty)$. Here $-\mu\psi_t(t, x)$, $\mu > 0$, is the braking force which is proportional to the speed and it is the control force. By means of the substitution,

$$\psi(t, \cdot) = x_1(\cdot)(t), \quad \psi_t(t, \cdot) = x_2(\cdot)(t), \tag{26}$$

we reduce the equation (25) to the form

$$\dot{x} = Ax + Bu, \tag{27}$$

where $x = (x_1, x_2) \in H = H_1 \times H_2$, $H_1 = \overset{\circ}{W}{}_2^1 (G)$, $H_2 = L_2(G)$, $u \in U = L_2(G)$. The operator A is defined by the equality

$$A = \begin{pmatrix} 0 & I \\ \Delta & -\mu I \end{pmatrix},$$

$D(A) = \{(x_1, x_2) \in H : \Delta x_1 \in H_2,\ x_2 \in H_1\}$, and the operator is of the form

$$B = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Next we will also assume that the control u satisfies some restrictions

$$u \in \Omega \subset L_2(G)\,. \tag{28}$$

Observe that the value

$$\frac{1}{2}||x||^2 = \frac{1}{2}\int_G (|\nabla x_1(s)|^2 + |x_2(s)|^2)\, ds$$

coincides with the full inner energy of the wave. Thus the exact controllability of the system (27), (28) means the possibility to put out the inner energy of the wave for any initial values by choosing the corresponding control force $u(t,\cdot) \in \Omega \subset U$. In this part, we investigate the controllability problem for the system (27), (28). First of all consider the spectral properties of the operator A. It is known [11] that under the given suppositions on the domain G the Laplace operator Δ is negative defined. Spectrum of this operator consists of the countable set of eigenvalues $-\lambda_k^2$, $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le \ldots$, $\lambda_k \to \infty$. Moreover, the corresponding eigenvectors f_k $(\Delta f_k = -\lambda_k^2 f_k)$ form the orthogonal basis in the space $L_2(G)$. We shall assume that this basis is normed. Thus the spectrum $\sigma(A)$ of the operator A is of the form

$$\sigma(A) = \left\{ r_{\mp k} = -\frac{\mu}{2} \mp \sqrt{\frac{\mu^2}{4} - \lambda_k^2}\,, \quad k = 1, 2, \ldots \right\}.$$

Furthermore, the following three cases are possible:

i) $k \in I_1 = \{l \in N : \frac{\mu}{2} < \lambda_i\}$, both eigenvalues $r_{\mp k}$ are complex, $\mathrm{Re}\, r_{\mp k} = -\mu/2$, and the corresponding eigenvectors are $\phi^{\mp k} = (f_k, r_{\mp k} f_k)$;

ii) $k \in I_2 = \{l \in N : \frac{\mu}{2} > \lambda_i\}$: both eigenvalues $r_{\mp k}$ are real negative, and the corresponding eigenvectors are $\phi^{\mp k} = (f_k, r_{\mp k} f_k)$;

iii) $k \in I_3 = \{l \in N : \frac{\mu}{2} = \lambda_i\}$: $r_k = r_{-k} = \mu/2$, and the eigenvector $\phi^k = (f_k, r_k f_k)$ and the principal vector $\phi^{-k} = (f_k, (1 + r_k) f_k)((A - r_k I)\phi^{-k} = \phi^k)$ corresponds to this eigenvalue.

It is easy to see that $\{\phi^m\}$, $m = \mp 1, \mp 2, \mp 3, \dots$, is the basis of the space H such that $(\phi^k, \phi^m) = 0$ when $|k| \neq |m|$. Moreover, operator A generates a strongly continuous group $\{e^{At}\}$, $-\infty < t < +\infty$, which is defined by the equality

$$e^{At}x = e^{At} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \alpha_m \phi^m = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} e^{r_m t}\alpha_m \phi^m + \sum_{k\in I_3} te^{-\mu t/2}\alpha_k \phi^k ,$$

where $\sum_{k=-\infty}^{\infty} ||\alpha_k \phi^k + \alpha_{-k}\phi^{-k}||^2 < \infty$.

Next we will prove that the equation (27) is the exactly controllable. To see this we verify that the operator $N(\lambda)$ is positive defined.

Denote by $L_k = \operatorname{Lin}\{\phi^k, \phi^{-k}\}$, $k = 1, 2, \dots$. Then it is clear that for $x^k \in L_k$

$$B^* e^{-A^* t} x^k \in \operatorname{Lin}\{f_k\}, \quad k = 1, 2, \dots .$$

Therefore, if $x \in H$, $x = \sum_{k=1}^{\infty} x^k$, $x^k \in L_k$, then

$$(N(\lambda)x, x) = \int_0^\infty e^{-\lambda t} ||B^* e^{-A^* t} x||^2 \, dt = \sum_{k=1}^{\infty} (N(\lambda)x^k, x^k) ,$$

and thus

$$\frac{(N(\lambda)x, x)}{||x||^2} \geq \inf_k \gamma_k, \quad \gamma_k = \inf_{x^k \in L_k} \frac{(N(\lambda)x^k, x^k)}{||x^k||^2} .$$

It is easy to see that all numbers γ_k, $k = 1, 2, \dots$, are positive and then it is sufficiently to show that there exist $\gamma_0 > 0$ such that $\gamma_k > \gamma_0$, $k = 1, 2, \dots$. Denote by $\{\hat\phi^m\}_{\substack{m=-\infty \\ m\neq 0}}^{\infty}$ the dual basis for the basis $\{\phi^m\}_{\substack{m=-\infty \\ m\neq 0}}^{\infty}$, i.e., $(\hat\phi^l, \phi^p) = \delta_{l,p}$, l, $p = \mp 1, \mp 2, \dots$. Then for $k \in I_1$, $x^k = \alpha\hat\phi^k + \beta\hat\phi^{-k} \in L_k$ we have

$$\begin{aligned}
(N(\lambda)x^k, x^k) &= \int_0^\infty e^{-\lambda t} ||B^* e^{-A^* t}(\alpha\hat\phi^k + \beta\hat\phi^{-k})||^2 \, dt \\
&= \int_0^\infty \frac{1}{4|\operatorname{Im} r_k|^2} e^{-\lambda t} \left| \alpha e^{-r_{-k}t} + \beta e^{-r_k t} \right|^2 dt \\
&= \frac{1}{4|\operatorname{Im} r_k|^2} \left(\frac{|\alpha|^2 + |\beta|^2}{\lambda - \mu} - \frac{\alpha\bar\beta}{\lambda + 2r_{-k}} - \frac{\bar\alpha\beta}{\lambda + 2r_k} \right) .
\end{aligned}$$

Since

$$||\hat\phi^k|| \sim O\left(\frac{1}{|r_k|}\right) \sim O\left(\frac{1}{|\operatorname{Im} r_k|}\right) ,$$

$|\lambda + 2r_{-k}| = |\lambda + 2r_k| \to \infty$, when $k \to \infty$, then for sufficiently large k and $\lambda > \mu = 2\omega_0(-A)$ there exists $C > 0$:

$$(N(\lambda)x^k, x^k) \geq \frac{1}{8|\operatorname{Im} r_k|^2} \frac{|\alpha|^2 + |\beta|^2}{\lambda - \mu} \geq \frac{||\alpha\hat\phi^k + \beta\hat\phi^{-k}||^2}{16|\operatorname{Im} r_k|^2 ||\hat\phi^k||^2 (\lambda - \mu)} > \frac{C||x^k||^2}{(\lambda - \mu)} ,$$

$x^k \in L_k$. Thus the sequence γ_k is bounded from below and therefore the operator $N(\lambda)$ is positive defined.

Now we can give solution of the local controllability problem for the system (27), (28) under the supposition that the set $\Omega \subset L_2(G)$ has a non-empty interior and $0 \in \Omega$. Let us divide the set I_2 into subsets

$$I_2 = \bigcup_{p=1}^{m} J_p$$

such that for k_1, $k_2 \in I_2$ the equality $\lambda_{k_1} = \lambda_{k_2}$ is valid iff for some $p \in \overline{1,m}$: k_1, $k_2 \in J_p$ and let $I_3 = J_{m+1}$. Thus an arbitrary real eigenvector ϕ of the operator A^* is of the form

$$\phi = \sum_{k\in J_p} \alpha_k^p \hat{\phi}^k, \quad p = \overline{1,m}, \quad \text{or} \quad \phi = \sum_{k\in J_p} \beta_k^p \hat{\phi}^{-k}, \quad p = \overline{1,m+1}\,.$$

Therefore, due to the theorem 4, we obtain the following result: the system (27), (28) is locally controllable iff for every vector of the form

$$f = \sum_{k\in J_p} \gamma_k^p f_k, \quad p = \overline{1,m+1}\,,$$

where $\gamma_k^p \in R$, $k \in J_p$, $\sum_{k\in J_p} (\gamma_k^p)^2 \neq 0$, there exist two elements u_1, $u_2 \in \Omega$ such that

$$\int_G f(s) u_1(s)\, ds > 0, \quad \int_G f(s) u_2(s)\, ds < 0\,.$$

Finally, we note that in the case $\mu = 0$ (without a braking force) all eigenvalues of the operator A are not real and then the system (27), (28) is locally controllable.

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Локальная и глобальная точная управляемость в гильбертовом пространстве

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Исследуется проблема управляемости для линейных систем с генератором сильно непрерывной группы. Получены критерии точной и локальной точной управляемости. Основные результаты статьи опираются на некоторые фундаментальные теоремы теории операторов. Рассмотрено приложение полученных результатов к задаче управления волновым процессом при наличии тормозящей силы.

Локальна та глобальна точна керованість в просторі Гільберта

В.І. Коробов, Г.М. Скляр

Досліджується проблема керованості для лінійних систем з генератором сильно неперервної групи. Отримано критерії точної та локальної точної керованості. Основні результати статті спираються на деякі фундаментальні теореми теорії операторів. Розглянуто застосування одержаних результатів до задачі керування хвильовим процесом при наявності гальмуючої сили.