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On asymptotic properties of certain orthogonal polynomials

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We compute the asymptotic distribution of zeros and the weak limit of orthogonal polynomials on the whole line whose weight contains a large parameter in the exponent. The techniques used and the results are motivated by recent studies on the eigenvalue statistics of random matrices.

1. Introduction

The study of asymptotic properties of orthogonal polynomials is a branch of analysis which goes back to classics, has numerous links with various areas of mathematics and related fields, and which is still actively developing, especially for the case of polynomials that are orthogonal on the whole real axis (see, e.g., the books [1–6] and references therein).

One newer useful link is with the theory of random matrices, where orthogonal polynomials provide a powerful tool for the study of the eigenvalue statistics of random Hermitian matrices whose probability distribution has the form

$$p_n(M)dM = Z_n^{-1}\exp\{-n\mathrm{Tr}V(M)\}dM\,, \tag{1.1}$$

where M is a $n\times n$ Hermitian matrix,

$$dM=\prod_{j=1}^{n} dM_{jj}\prod_{j<k} d\Im M_{jk}d\Re M_{jk}$$

is the "Lebesgue" measure for Hermitian matrices, the symbols $\Re z$ and $\Im z$ denote the real and imaginary parts of z, Z_n is the normalization factor and $V(\lambda)$ is a real valued function (see Theorems 1,2 below for explicit conditions and [7] for the physical motivation of (1.1)).

We denote by $p_n(\lambda_1, ..., \lambda_n)$ the joint eigenvalue probability density, which we assume to be symmetric without loss of generality. From random matrix theory [8] it is known that

$$p_n(\lambda_1, ..., \lambda_n) = Q_n^{-1} \prod_{1 \le j < k \le n} (\lambda_j - \lambda_k)^2 \exp\{-n \sum_{j=1}^{n} V(\lambda_j)\}, \tag{1.2}$$

where Q_n is the respective normalization factor. Let

$$p_k^{(n)}(\lambda_1, ..., \lambda_k) = \int p_n(\lambda_1, ..., \lambda_k, \lambda_{k+1}, ..., \lambda_n) d\lambda_{k+1} ... d\lambda_n \tag{1.3}$$

be the k-th marginal distribution density of (1.2). The link with orthogonal polynomials is provided by the formula (see [8])

$$p_k^{(n)}(\lambda_1, ..., \lambda_l) = \frac{(n-k)!}{n!} \det \|k_n(\lambda_i, \lambda_j)\|_{j,k=1}^{l}, \tag{1.4}$$

where

$$k_n(\lambda, \mu) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda) \psi_l^{(n)}(\mu) \tag{1.5}$$

is the reproducing kernel for the orthonormalized system

$$\psi_l^{(n)}(\lambda) = \exp\{-nV(\lambda)/2\} P_l^{(n)}(\lambda), \; l = 0, 1, ..., \tag{1.6}$$

and $P_l^{(n)}(\lambda), l = 0, 1, ...$ are orthogonal polynomials on $\mathbf{R}$ associated with the weight

$$w_n(\lambda) = e^{-nV(\lambda)}, \tag{1.7}$$

i.e.,

$$\int P_l^{(n)}(\lambda) P_m^{(n)}(\lambda) w_n(\lambda) d\lambda = \delta_{l,m}. \tag{1.8}$$

Our goal is to study asymptotic properties of the polynomials $P_n^{(n)}(\lambda)$ and related quantities.

Let us notice, that the weight (1.7) has an unusual form from the point of view of the traditional theory of orthogonal polynomials associated with weights on the whole $\mathbf{R}$. Indeed, in that theory the study of asymptotic properties of the orthogonal polynomial P_n for $n \to \infty$ is carried out for a weight of the form

$$w(x) = e^{-Q(x)} \tag{1.9}$$

that does not contain the large parameter n.* In this case the most nontrivial asymptotic properties of $P_n(x)$ manifest themselves for $x = O(a_n)$ where $a_n \to \infty$ as $n \to \infty$ (a_n are known as the Mhaskar–Rakhmanov–Saff numbers, e.g., $a_n = const\sqrt{n}$ for the Hermite polynomials corresponding to $Q(x) = x^2$). Therefore the parts of $Q(x)$ in (1.9) that contribute to the asymptotic behaviour of $P_n(x)$ are the "tails" of $Q(x)$ and that is why one needs to impose certain regularity conditions on the tails behaviour, requiring roughly their power law form (see, e.g., [2]). In our case of (1.7) it suffices to consider $\lambda = O(1)$ as $n \to \infty$ and no conditions on the tails $V(\lambda)$ are needed in order to study the asymptotic properties of $P_n^{(n)}$. Therefore the mathematical mechanisms which determine the asymptotic properties of the orthogonal polynomials associated with (1.7) and (1.9) are different, especially for "strictly" nonpower law (e.g., nonconcave) V's.

There is, however, a class of weights that belongs to both cases and for which we can reduce (1.7) to (1.9) and vice versa. This class consists of monomial weights

$$Q(x) = |x|^\alpha, \qquad V(\lambda) = |\lambda|^\alpha, \qquad \alpha > 0\,. \tag{1.10}$$

In this case the rescaling

$$n^{1/\alpha}\lambda = x \tag{1.11}$$

transforms (1.9) in (1.7) and gives the simple correspondence

$$P_l^{(n)}(\lambda) = P_l(\lambda n^{1/\alpha})n^{1/2\alpha}$$

between the orthogonal polynomials associated with the weights (1.9) and (1.7).

We shall now return back to the discussion of random matrices, in particular, to formula (1.4).

The simplest case of $p_n^{(1)}(\lambda_i)$ is already of considerable interest. Indeed, if $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ are eigenvalues of a random Hermitian matrix M, then N_n defined by

$$N_n(\Delta) = \frac{1}{n}\sum_{j=1}^{n} \chi_\Delta(\lambda_j^{(n)}), \qquad \Delta = (a, b), \qquad a \le b, \tag{1.12}$$

is their normalized counting measure (empirical eigenvalue distribution). Here and below the function $\chi_\Delta(\lambda)$ is the indicator of the interval Δ and the symbol $E\{\ldots\}$ denotes the expectation with respect to the probability measure (1.1). We have

$$E\{N_n(\Delta)\} = \int_\Delta p_1^{(n)}(\lambda)d\lambda \equiv \int_\Delta \rho_n(\lambda)d\lambda\,, \tag{1.13}$$

*See, however, the book [5], devoted to polynomial approximations with weights of the form (1.7).

where, according to (1.4) and (1.5),

$$\rho_n(\lambda) = \frac{1}{n} k_n(\lambda, \lambda) = \frac{1}{n} \sum_{j=0}^{n-1} [\psi_j^{(n)}(\lambda)]^2 . \tag{1.14}$$

The function

$$\Lambda(\lambda) = [k_n(\lambda, \lambda)]^{-1}$$

is known in the theory of orthogonal polynomials as the Christoffel function.

In the recent paper [9] it was proved that if $V(\lambda)$ is bounded from below for all $\lambda \in \mathbf{R}$ and satisfies the conditions

$$|V(\lambda)| \geq (2 + \epsilon) \log |\lambda|, \ |\lambda| \geq L_1 \tag{1.15}$$

for some L_1, $\epsilon > 0$, and

$$|V(\lambda_1) - V(\lambda_2)| \leq C(L_2)|\lambda_1 - \lambda_2|^{\gamma}, \quad |\lambda_{1,2}| \leq L_2 \tag{1.16}$$

for any $0 < L_2 < \infty$ and some $\gamma > 0$, then:

(i) $\rho_n(\lambda)$ converges to the limiting density $\rho(\lambda)$ (called the density of states of the random matrix ensemble (1.1)) in the Hilbert space defined by the energy norm

$$\left(- \int \log |\lambda - \mu| \rho(\lambda) \rho(\mu) d\lambda d\mu \right)^{1/2} ; \tag{1.17}$$

(ii) $\rho(\lambda)$ can be found either as the unique solution of the equation

$$\mathrm{supp} \rho \in \{ \lambda : u(\lambda) = \max_{\mu} u(\mu) \} , \tag{1.18}$$

where

$$u(\lambda) = 2 \int d\mu \rho(\mu) \log |\lambda - \mu| - V(\lambda) , \tag{1.19}$$

or as the density of the unique minimum of the functional (the electrostatic energy) defined on the set of all probability measures ν by the formula

$$U(\nu) = - \int \nu(d\lambda) \nu(d\mu)) \log |\lambda - \mu| - \int \nu(d\lambda) V(\lambda) . \tag{1.20}$$

R e m a r k. Note that equation (1.18) after simple transformations gives the singular integral equation

$$\int \frac{\rho(\mu) d\mu}{\lambda - \mu} = V'(\lambda)/2, \qquad \lambda \in \mathrm{supp} \rho . \tag{1.21}$$

If we know the support of ρ, we can find it by using well-known formulae of the theory of singular integral equations [11]. Unfortunately, information about the support of ρ cannot be obtained easily from equations (1.18)–(1.20). We can say only that the number of intervals of the support is not more then the number of extremal points of function V. To find the endpoints of these intervals we have to solve the system of certain algebraic equations.

Let us denote by $\lambda_1^*, \dots, \lambda_n^*$ the zeros of the orthogonal polynomial $P_n^{(n)}(\lambda)$ and introduce their normalized counting measure

$$N_n^*(\Delta) = \frac{1}{n}\sum_{j=1}^{n} \chi_\Delta(\lambda_j^*), \qquad \Delta = (a, b) . \tag{1.22}$$

Theorem 1. *The normalized counting measure (1.22) converges weekly as $n \to \infty$ to a limiting measure which is absolutely continuous with respect to the Lebesgue measure on* $\mathbf{R}$ *and whose density coincides with the function $\rho(\lambda)$, described above, i.e. can be found from formulae (1.18)–(1.20) (the density of states of the random matrix ensemble (1.1)).*

R e m a r k. A similar statement is known for orthogonal polynomials with n-independent weight [1]. Its proof is however based on different ideas.

Theorem 2. *Let the function $V(\lambda)$ satisfy condition (1.15) and (1.16) σ be the support of limiting distribution $\rho(\lambda)$. Assume that*

$$|V'''(\lambda)| \le C < \infty, \qquad \lambda \in \sigma , \tag{1.23}$$

and

$$\int_\sigma \frac{d\lambda}{\rho(\lambda)} \le C_1 < \infty \tag{1.24}$$

for some constants C and C_1. Then:

(i) if σ consists only of one interval (a, b), then $[\psi_n^{(n)}(\lambda)]^2$ converges in the energy norm (1.17), as $n \to \infty$, and

$$\lim_{n\to\infty} [\psi_n^{(n)}(\lambda)]^2 = \frac{1}{\pi\sqrt{(b-\lambda)(\lambda-a)}} \chi_{(a,b)}(\lambda) , \tag{1.25}$$

where $\chi_{(a,b)}(\lambda)$ is the characteristic function of the interval (a, b);

(ii) if $V(\lambda)$ is an even function and the support σ consists of two intervals $\sigma = (a, b) \cup (-b, -a)$, then $[\psi_n^{(n)}(\lambda)]^2$ converges in the energy norm as $n \to \infty$ and

$$\lim_{n\to\infty} [\psi_n^{(n)}(\lambda)]^2 = \frac{|\lambda|}{\pi\sqrt{(b^2-\lambda^2)(\lambda^2-a^2)}} \left(\chi_{(a,b)}(\lambda) + \chi_{(-b,-a)}(\lambda)\right) . \tag{1.26}$$

R e m a r k. From equation (1.21) and the standard facts of theory of singular integral equations (see, e.g., [11]) it is easy to obtain that the density $\rho(\lambda)$ in the neighbourhood of any endpoint a_i of the spectrum σ has the form

$$\rho(\lambda) = \sqrt{|\lambda - a_i|}\phi(\lambda)$$

with some bounded (even smooth if we assume (1.23)) function $\phi(\lambda)$, depending on V. For certain values of the parameters, that determine V, one can obtain the case when $\phi(a_i) = 0$. Thus the second condition in (1.23)) in fact means that we consider only generic case, when $\phi(a_i) \neq 0$.

2. Proof of Theorems 1, 2

P r o o f o f T h e o r e m 1. It follows from the orthogonality relations (1.8) that for $j = 0, 1, 2, \ldots$

$$\lambda P_j(\lambda) = a_j P_j(\lambda) + r_j P_{j+1}(\lambda) + r_{j-1}P_{j-1}(\lambda), \qquad r_{-1} = 0\,, \tag{2.1}$$

where

$$a_j = \int \lambda P_j^2(\lambda) e^{-nV(\lambda)} d\lambda, \qquad r_j = \int \lambda P_j(\lambda) P_{j+1}(\lambda) e^{-nV(\lambda)} d\lambda\,. \tag{2.2}$$

Here and below we omit the superscript (n). Denote by $J = \{J_{j,k}\}_{j,k=0}^{\infty}$ the Jacobi matrix, corresponding to (2.1)

$$J_{j,k} = a_j \delta_{j,k} + r_j \delta_{j+1,k} + r_{j-1}\delta_{j-1,k}. \tag{2.3}$$

Let

$$J_n = J - r_{n-1}A_n\,, \tag{2.4}$$

where $[A_n]_{j,k} = \delta_{j,n}\delta_{k,n-1} + \delta_{j,n-1}\delta_{k,n}$. Then J_n consists of two blocks $J_n^{(1)}$ and $J_n^{(2)}$, where the upper block $J_n^{(1)}$ is an $n \times n$ Jacobi matrix and the lower block $J_n^{(2)}$ is the semi-infinite Jacobi matrix whose first row consists of a_n and r_n.

According to (2.1), the zeros $\left\{\lambda_j^*\right\}_{j=1}^{n}$ of $P_n(\lambda)$ are the eigenvalues of $J_n^{(1)}$. Thus, according to the spectral theorem, the Stieltjes transform $g_n^*(z)$ of the measure $N^*(\Delta)$ (1.22) can be written as

$$g_n^*(z) \equiv \frac{1}{n}\sum_{j=1}^{n}\frac{1}{\lambda_j^* - z} = \frac{1}{n}\mathbf{Tr}R_n^{(1)}(z) = \frac{1}{n}\sum_{j=1}^{n}[R_n^{(1)}]_{j,j}\,, \tag{2.5}$$

where $R_n^{(1)} = (J_n^{(1)} - z)^{-1}$ is the resolvent of $J^{(1)}$. Since J_n has a block structure, $R_n = (J_n - z)^{-1}$ has also a block structure, and then

$$\frac{1}{n}\sum_{j=1}^{n}[R_n]_{j,j} = \frac{1}{n}\sum_{j=1}^{n}[R_n^{(1)}]_{j,j}\,.$$

Set $R = (J - z)^{-1}$. According to the resolvent identity, we have

$$R - R_n = R(J - J_n)R_n \,.$$

Thus

$$\left| \frac{1}{n}\sum_{j=1}^{n} R_{j,j} - \frac{1}{n}\sum_{j=1}^{n}[R_n^{(1)}]_{j,j} \right|$$

$$= \frac{r_{n-1}}{n}\left|\sum_{j=1}^{n}(RA_nR_n)_{j,j}\right| = \frac{r_{n-1}}{n}\left|\sum_{j=1}^{n}\left((RR_n)_{j,n} + (RR_n)_{n-1,j}\right)\right|$$

$$\leq \frac{r_{n-1}}{2n}\sum_{j=1}^{n}\left(|R_{j,n}|^2 + |(R_n)_{n-1,j}|^2 + |R_{n,j}|^2 + |(R_n)_{j,n-1}|^2\right) \leq \frac{2|r_{n-1}|}{n(\Im z)^2}\,, \quad (2.6)$$

because $\sum_{j=1}^{n}|R_{j,n}|^2 = (RR^*)_{n,n} \leq ||R||^2 \leq |\Im z|^{-1}$ and because the analogous inequality is valid for R_n.

On the other hand, according to representation (1.14), the Stieltjes transform $g_n(z)$ of the measure $E\{N_n(\Delta)\}$ can be written as

$$g_n(z) \equiv \int \frac{\rho_n(\lambda)d\lambda}{\lambda - z} = \frac{1}{n}\sum_{j=1}^{n}\int\frac{\psi_k^2(\lambda)d\lambda}{\lambda - z} = \frac{1}{n}\sum_{j=1}^{n}(R_n)_{j,j}\,. \qquad (2.7)$$

Thus, we obtain from (2.5)–(2.7) that

$$|g_n(z) - g_n^*(z)| \leq \frac{2|r_{n-1}|}{n(\Im z)^2}\,. \qquad (2.8)$$

To estimate r_{n-1} we use the result of [9], according to which there exist positive numbers L_0, L, A, and a such that

$$\rho_n(\lambda) \leq Ae^{-naV(\lambda)}, \quad |\lambda| > L\,. \qquad (2.9)$$

This bound and identity (1.14) imply

$$\psi_{n-1}^2(\lambda) \leq n\rho_n(\lambda) \leq nAe^{-naV(\lambda)}, \quad |\lambda| \geq L\,. \qquad (2.10)$$

Then, by using the Schwarz inequality, we get from (2.2)

$$|r_{n-1}| \leq \left[\int \lambda^2\psi_{n-1}^2(\lambda)d\lambda\right]^{1/2}\left[\int\psi_n^2(\lambda)d\lambda\right]^{1/2} \leq C\,. \qquad (2.11)$$

Relations (2.8), (2.10) and results of [9] prove Theorem 1.

P r o o f o f T h e o r e m 2. The proof of Theorem 2 is based on a well known identity of random matrix theory (see, e.g., [7])

$$E\left\{\left(\frac{1}{n}\mathbf{Tr}G(z)\right)^2\right\}+E\left\{\frac{1}{n}\mathbf{Tr}G(z)V'(M)\right\}=0\,, \qquad (2.12)$$

where $G(z)=(M-z)^{-1}$ and M is distributed according to the distribution (1.1). We use also the result [10]

$$E\left\{\left(\frac{1}{n}\mathbf{Tr}G(z)\right)^2\right\}-E^2\left\{\left(\frac{1}{n}\mathbf{Tr}G(z)\right)\right\}=O\left(\frac{1}{n^2(\Im z)^4}\right)$$

and the spectral theorem, according to which for any integrable and polynomially bounded function $f(\lambda)$

$$E\left\{\left(\frac{1}{n}\mathbf{Tr}f(J)\right)\right\}=\int f(\lambda)\rho_n(\lambda)d\lambda\,.$$

Therefore (2.12) can be rewritten as follows

$$g_n^2(z)+\int\frac{V'(\mu)}{\mu-z}\rho_n(\mu)d\mu=O\left(\frac{1}{n^2(\Im z)^4}\right)\,. \qquad (2.13)$$

Set $z=\lambda+i\eta$, $\lambda,\eta\in\mathbf{R}$. Then it is convenient to rewrite (2.13) as

$$g_n^2(z)+V'(\lambda)g_n(z)+Q_n(z)=O\left(\frac{1}{n^2\eta^4}\right)\,, \qquad (2.14)$$

where

$$Q_n(z)=\int\frac{V'(\mu)-V'(\lambda)}{\mu-z}\rho_n(\mu)d\mu\,.$$

According to [9], for any differentiable function $\phi(\mu)$ growing not faster then $e^{nV(\mu)}$ as $|\mu|\to\infty$, we have the inequality

$$\left|\int\phi(\mu)\rho_n(\mu)d\mu-\int\phi(\mu)\rho(\mu)d\mu\right|\le C||\phi'||_2^{1/2}||\phi||_2^{1/2}n^{-1/2}\log^{1/2}n\,.$$

Here the symbol $||...||_2$ denotes the L_2-norm on the interval $(-L,L)$ and L is some fixed number depending on the function V and such that $\mathrm{supp}\rho\in(-L,L)$. Thus we get for $\eta=O(n^{-1/3})$

$$Q_n(z)=Q(\lambda)+O(\eta)\,, \qquad (2.15)$$

where

$$Q(\lambda)=\int\frac{V'(\mu)-V'(\lambda)}{\mu-\lambda}\rho(\mu)d\mu\,. \qquad (2.16)$$

Therefore for $\eta = O(n^{-1/3})$ (2.14) can be rewritten as

$$g_n^2(z) + V'(\lambda)g_n(z) + Q_n(z) = O(\eta)\,,$$

which implies

$$g_n(\lambda + i\eta) = -\frac{V'(\lambda)}{2} + \sqrt{\left(\frac{V'(\lambda)}{2}\right)^2 - Q(\lambda)} + O(\eta)\,, \qquad (2.17)$$

where the branch of the square root is real and positive as $\lambda \to \infty$. Since, according to the the inversion formula for the Stieltjes transform [11], $\rho(\lambda) = \lim_{\eta\to 0}\lim_{n\to\infty}(i\pi)^{-1}\Im g_n(\lambda + i\eta)$, we get from (2.17) that

$$\mathrm{supp}\rho \subset \{\lambda : \left(\frac{V'(\lambda)}{2}\right)^2 - Q(\lambda) \le 0\}$$

and

$$\rho(\lambda) = \pi^{-1}\sqrt{Q(\lambda) - \left(\frac{V'(\lambda)}{2}\right)^2}, \qquad if \quad \left(\frac{V'(\lambda)}{2}\right)^2 - Q(\lambda) < 0\,. \qquad (2.18)$$

Thus (2.17) takes the form

$$g_n(\lambda + i\eta) + \frac{V'(\lambda)}{2} = i\pi\rho(\lambda) + O(\eta)\,. \qquad (2.19)$$

To proceed further we introduce the density

$$p_n^+(\lambda_1, ..., \lambda_{n+1}) = \frac{1}{Z_n^+}\exp\left\{-n\sum_{j=1}^{n+1}V(\lambda_j)\right\}\prod_{1\le j<k\le n+1}(\lambda_j - \lambda_k)^2\,. \qquad (2.20)$$

The difference of this density from the density (1.2) written for the $n+1$ variables $\lambda_1, ...\lambda_{n+1}$ is that in the former we have the factor n in the exponent while in the latter we would then have $n+1$. Set

$$\rho_n^+(\lambda) = \frac{n+1}{n}\int p_n^+(\lambda, \lambda_2, ..., \lambda_{n+1})d\lambda_2...d\lambda_{n+1} = \frac{1}{n}\sum_{j=0}^{n}\psi_j^2(\lambda)\,. \qquad (2.21)$$

Then

$$\psi_n^2(\lambda) = n[\rho_n^+(\lambda) - \rho_n(\lambda)].$$

Furthermore, by using the analogue of identity (2.12) for the density p_n^+ and arguments similar to those used for proving (2.13), we obtain the relation

$$[g_n^+(z)]^2 + \int\frac{V'(\mu)\rho_n^+(\mu)}{\mu - z}d\mu = O(\frac{1}{n^2\eta^4}) \qquad (2.22)$$

for the Stieltjes transform $g_n^+(z)$ of $\rho_n^+(\mu)$ and $z = \lambda + i\eta,\ \eta > 0$. Set

$$\Delta_n(z) = n(g_n^+(z) - g_n(z)) = \int \frac{\psi_n^2(\mu)}{\mu - z} d\mu \,. \tag{2.23}$$

Then, subtracting (2.13) from (2.22) and multiplying the result by n, we obtain that

$$\Delta_n(z)(g_n(z) + g_n^+(z)) + \int \frac{V'(\mu)\psi_n^2(\mu)}{\mu - z} d\mu = O(\frac{1}{n\eta^4}) \,. \tag{2.24}$$

Set $\eta = n^{-1/5}$. Then this relation takes the form

$$\Delta_n(z)(2g(z) + V'(\lambda) + O(\eta)) = \int \frac{(V'(\lambda) - V'(\mu))\psi_n^2(\mu)}{\mu - z} d\mu + O(\eta) \,. \tag{2.25}$$

Since the r.h.s. of (2.25) is real valued and according to (2.17) and (2.18) $2g(z) + V'(\lambda)$ is also real valued for $\lambda \notin \sigma$, we get from (2.25) that $\Im \Delta_n(z) \to 0$ as $n \to \infty$ and then derive easily that

$$\psi_n^2(\lambda) \to 0, \quad \lambda \notin \sigma \tag{2.26}$$

in the energy norm (1.17).

On the other hand, according to (2.19) for $\lambda \in \sigma$ relation (2.25) takes the form

$$\Delta_n(z)(2\pi i \rho(\lambda) + O(\eta)) = \int \frac{(V'(\lambda) - V'(\mu))\psi_{n-1}^2(\mu)}{\mu - z} d\mu + O(\eta)$$

which implies

$$|\Re \Delta_n(\lambda + i\eta)| \le \frac{Cn^{-1/5}}{\rho(\lambda)}, \qquad \lambda \in \sigma \,. \tag{2.27}$$

Assume now that σ consists of $k \ge 1$ intervals:

$$\sigma = \cup_{j=1}^k (a_j, b_j) \,. \tag{2.28}$$

Integrating (2.27) with respect to λ, we conclude that there exist $d_1, ..., d_k$ (depending on n) such that if we consider the function

$$d(\lambda) = \sum_{j=1}^k d_j \chi_{(a_j, b_j)}(\lambda) \,, \tag{2.29}$$

then

$$\Big|(A\psi_n^2)(\lambda) - d(\lambda)\Big| \le Cn^{-1/5}, \qquad \lambda \in \sigma \,, \tag{2.30}$$

where the operator A is defined by the formula

$$(Af)(\lambda) \equiv \int_\sigma f(\mu) \log \frac{1}{|\lambda-\mu|} d\mu, \qquad \lambda \in \sigma\,. \tag{2.31}$$

On the other hand, from the theory of singular integral equations [11] it is known that the solution of the equation

$$(Af)(\lambda) = d(\lambda), \qquad \lambda \in \sigma\,, \tag{2.32}$$

can be written as follows

$$f_n(\lambda) = \frac{q_k^{(n)}(\lambda)}{\sqrt{R(\lambda)}}, \qquad R(\lambda) \equiv \prod_{j=1}^{k} (\lambda - a_j)(b_j - \lambda)\,, \tag{2.33}$$

where $q_k^{(n)}(\lambda)$ is a polynomial of degree $(k-1)$. From (2.30) and (2.32) we obtain the inequality

$$(A(\psi_n^2 - f_n), (\psi_n^2 - f_n)) \le Cn^{-1/5}$$

which implies

$$|\psi_n^2(\lambda) - f_n(\lambda)| \to 0, \qquad \text{as} \qquad n \to \infty\,, \tag{2.34}$$

in the energy norm (1.17). Now, substituting ψ_n^2 in (2.24) by $f_n(\lambda)$, we get for z with $\Im z > 0$ and $n \to \infty$:

$$g(z) \int_\sigma \frac{q_k^{(n)}(\lambda) d\lambda}{\sqrt{R(\lambda)}(\lambda - z)}$$

$$= -\int_\sigma \frac{V'(\lambda) q_k^{(n)}(z) d\lambda}{\sqrt{R(\lambda)}(\lambda - z)} - \int_\sigma \frac{V'(\lambda)(q_k^{(n)}(\lambda) - q_k^{(n)}(z)) d\lambda}{\sqrt{R(\lambda)}(\lambda - z)} + o(1)\,. \tag{2.35}$$

According to [9, 11], the existence of a bounded $\rho(\lambda)$ implies that

$$\int_\sigma \frac{V'(\lambda)\lambda^j d\lambda}{\sqrt{R(\lambda)}} = 0, \qquad j = 0, 1, ..., k-1\,,$$

and then

$$g(z) = \pi^{-1} \sqrt{R(z)} \int_\sigma \frac{V'(\lambda) d\lambda}{\sqrt{R(\lambda)}(\lambda - z)}\,.$$

Therefore the second term of the r.h.s. of (2.35) is zero and

$$\pi^{-1} \int_\sigma \frac{q_k^{(n)}(\lambda) d\lambda}{\sqrt{R(\lambda)}(\lambda - z)} = \frac{q_k^{(n)}(z)}{\sqrt{R(z)}} + o(1), \qquad n \to \infty\,. \tag{2.36}$$

Thus, taking into account that $\int \psi_n^2(\lambda)d\lambda = 1$, we obtain

$$q_k^{(n)}(\lambda) = \pi^{-1} \prod_{j=1}^{k-1} (\lambda - \mu_j)\,, \tag{2.37}$$

where $\mu_j \in (b_j, a_{j+1})$ is a point from the jth gap of the support σ of ρ. Therefore, in the case of one-interval support $\sigma = (a, b)$, (1.25) follows from (2.26), (2.33), (2.34) and (2.37). In the case of the even $V(\lambda)$, $\psi_n^2(\lambda)$ also has to be an even function and thus if $k = 2$, $\sigma = (a, b) \cup (-b, -a)$, then $\mu_1 = 0$ and (1.26) also follows from (2.26), (2.33), (2.34) and (2.37). Theorem 2 is proved.

Remarks.

(i) In the one-interval case the limits of entries a_n and r_n of the Jacobi matrix J defined by (2.2) exist. Indeed, according to (2.2), the existence of the limit of a_n is a direct corollary of the weak convergence of $\psi_n^2(\lambda)$ and one has

$$\lim_{n\to\infty} a_n = \frac{a+b}{2}. \tag{2.38}$$

To obtain the limit of r_n let us note that it follows from (1.25) that the following limits exist

$$\lim_{n\to\infty} [r_n^2 + r_{n-1}^2 + a_n^2] = \lim_{n\to\infty} \int \lambda^2 \psi_n^2(\lambda) d\lambda$$
$$= \left(\frac{a+b}{2}\right)^2 + \frac{1}{2}\left(\frac{a-b}{2}\right)^2 \tag{2.39}$$

and

$$\lim_{n\to\infty} [(r_n^2+r_{n-1}^2+a_n^2)^2+(r_n a_n+r_n a_{n+1})^2+(r_{n-1}a_{n-1}+r_{n-1}a_n)^2+r_n^2 r_{n+1}^2+r_{n-1}^2 r_{n-2}^2]$$

$$= \lim_{n\to\infty} \int \lambda^4 \psi_n^2(\lambda) d\lambda = \frac{3}{8}\left(\frac{a-b}{2}\right)^4 + 3\left(\frac{a^2-b^2}{4}\right)^2 + \left(\frac{a+b}{2}\right)^4. \tag{2.40}$$

From the limiting relations (2.38)–(2.40) it is easy to derive that

$$\lim_{n\to\infty} r_n^2 = \left(\frac{b-a}{4}\right)^2. \tag{2.41}$$

(ii) In the two-interval case the limits of the entries of J with odd and even indices exist:

$$\lim_{n\to\infty} r_{2n+1}^2 = l_1, \qquad \lim_{n\to\infty} r_{2n}^2 = l_2\,, \tag{2.42}$$

and we have

$$l_1 = \left(\frac{b-a}{2}\right)^2, \; l_2 = \left(\frac{b+a}{2}\right)^2 \; or \; l_1 = \left(\frac{b+a}{2}\right)^2, \; l_2 = \left(\frac{b-a}{2}\right)^2. \qquad (2.43)$$

Indeed, since in this case we assume that $V(\lambda)$ is an even function we have $a_n = 0$ (see (2.2)). Moreover, on the basis of Theorem 2, we conclude that

$$\lim_{n\to\infty}[r_n^2 + r_{n-1}^2] = \lim_{n\to\infty}\int \lambda^2\psi_n^2(\lambda)d\lambda = \left(\frac{a^2+b^2}{2}\right)^2 \qquad (2.44)$$

and

$$\lim_{n\to\infty}[(r_n^2 + r_{n-1}^2)^2 + r_n^2 r_{n+1}^2 + r_{n-1}^2 r_{n-2}^2] = \lim_{n\to\infty}\int \lambda^4\psi_n^2(\lambda)d\lambda$$
$$= \left(\frac{a^2+b^2}{2}\right)^2 + \frac{1}{2}\left(\frac{a^2-b^2}{2}\right)^2. \qquad (2.45)$$

Combining the limiting relations (2.44) and (2.45), we get (2.42) and (2.43).

(iii)The above statements solve an analogue of the Freud conjecture for the weight (1.7). This conjecture concerns limits of entries of the Jacobi matrix associated with the corresponding orthogonal polynomials and was formulated by G. Freud for the n-independent weight (1.9) with the monomial $Q(x) = |x|^\alpha$. The conjecture was solved in a number of important papers (see, e.g., [3] for references, results and discussions). However, for the case where $Q(x)$ is a monomial only the case (i) in the above remark is possible, i.e., the limit of the properly normalized entries of the Jacobi matrix exists but not the different limits of the even and odd coefficients like we have in the case (ii) above.

(iv) Returning to our notations with superscript (n) which we used in the Introduction, we can rewrite our results as $\lim_{n\to\infty} c_n^{(n)} = c$, where $c^{(n)}$ is any of the l.h.s. of (2.38), (2.41), and (2.42), and c is the respective r.h.s. of the formulae. In fact we have proved more, namely that the limits of all $c_k^{(n)}$ such that $\lim_{n\to\infty} \dfrac{k}{n} = x \geq 0$ exist, i.e., the function

$$c(x) = \lim_{n\to\infty, k/n\to x} c_k^{(n)}$$

exists. To see this it suffices to replace $V(\lambda)$ in (1.7) by $\dfrac{1}{x}V(\lambda)$ and to repeate our arguments for this case.

(v) The results presented above were obtained in 1994-1995 and published in Preprint No. 277 of the E. Schrödinger Institute, Vienna, October 1995. Since that time there appeared two papers [13, 14] devoted to the asymptotic properties of orthogonal polynomials of the form (1.7) and (1.8). In the paper [13] the polynomials with the weight

$$V(\lambda) = \frac{t\lambda^2}{2} + \frac{g\lambda^4}{4}, \quad g > 0, \tag{2.46}$$

were considered. Here for $t < -2\sqrt{g}$ the support of the density of states consists of two intervals (see, e.g., [10]). For this case in [13] the asymptotic formula for the respective orthogonal polynomials is established. In particular, the formula yields relations (1.26) and (2.42)–(2.43) for the weight (2.46). In the paper [14] the asymptotic formulas for the polynomials (1.7) and (1.8) with an arbitrary real analytic function $V(\lambda)$ with sufficient growth at infinity are established. In the case when the support of the density of states consists of one or two intervals, these asymptotic formulas imply the limiting relations (1.25)–(1.26) and (2.42)–(2.43). The methods of these papers are completely different from ours and are based on the techniques of the theory of completely integrable systems and on a powerful version of the steepest descent method developed previously by authors of [14]. We note in the conclusion that our proofs of (1.25)–(1.26) and (2.42)–(2.43) are valid under more general conditions, requiring only local boundedness of the third derivative of $V(\lambda)$ (see (1.15), (1.16) and (1.23)).

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Асимптотические свойства некоторых ортогональных полиномов

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Найдено асимптотическое распределение нулей и слабый предел для некоторого семейства полиномов, ортогональных на всей оси относительно веса, который содержит большой параметр в экспоненте. Метод исследования и полученный результат мотивированы недавними исследованиями в области распределения собственных значений случайных матриц.

Асимптотичні властивості деяких ортогональних поліномів

С. Алберерiо, Л. Пастур, М. Щербина

Знайдено асимптотичний розподіл нулів та слабку границю для деякої сім'ї поліномів, що є ортогональними на всій вісі відносно ваги, яка має великий параметр в експоненті. Метод дослідження та здобуті результати мотивовано останніми дослідженнями в галузі розподілу власних значень випадкових матриць.