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Some new generalizations of the Lyapunov convexity theorem

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It is proved that Schreier's space, Lorentz sequence spaces, and Baernstein's spaces, which contain no subspaces isomorphic to l_2, have the Lyapunov property.

The famous Lyapunov theorem ([8; 9, Theorem 5.5] , or [3, p. 264]) states that the range of a non-atomic vector measure valued in a finite dimensional Banach space is convex and compact. The assertion is valid only in finite dimensional spaces. Many generalizations of it to infinite dimensional case have been obtained under various additional restrictions on a measure and a space. In 1992 V.M. Kadets and G. Schechtman [6] discovered that the closure of every non-atomic l_p $(p \neq 2, 1 \leq p < \infty)$ or c_0-valued vector measure range is convex. This statement holds in Orlicz sequence spaces, which contain no subspaces isomorphic to l_2 (this result has been obtained by the author but has not been published). We shall show that it is true in Lorentz sequence spaces, Schreier's space, and Baernstein's spaces. The idea of the proofs here is very close to [6], in particular Theorem 1 below almost coincides with analogous statement in [6], but the new trick which we employ considering auxiliary c_0-valued operator allows us to generalize the Lyapunov theorem for more spaces.

Throughout the paper by "X-valued measure" we mean a countably additive X-valued measure μ defined on a σ-field Σ of subsets of a set Ω. A Banach space X is said to have the Lyapunov property ($X \in \mathrm{LPr}$) if the closure of every non-atomic X-valued measure range is convex. In the sequel we will use standard notation of the classical Banach space theory (see [4, 3]).

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1. Preliminaries

First we summarize some material concerning Lorentz sequence spaces, Schreier's space, and Baernstein's spaces.

Let $1 \le p < \infty$. For any $a = (a_1, a_2, ...) \in c_0 \backslash l_1$, $a_1 \ge a_2 \ge ... \ge 0$, let $d(a,p) = \{x = (x_1, x_2, ...) \in c_0 : \sup_{\sigma \in \pi} \sum_{n=1}^{\infty} \left| x_{\sigma(n)} \right|^p a_n < \infty\}$, where π is the set of all permutations of the natural numbers $\mathbf{N}$. Then $d(a,p)$ with the norm $\|x\| = \left(\sup_{\sigma \in \pi} \sum_{n=1}^{\infty} \left| x_{\sigma(n)} \right|^p a_n \right)^{\frac{1}{p}}$ for $x \in d(a,p)$ is a Banach space and the sequence of unit vectors $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of $d(a,p)$. For the basic properties of the Lorentz spaces $d(a,p)$ we refer to [4, 5]. In particular, it is known that every infinite-dimensional subspace of $d(a,p)$ has a complemented subspace isomorphic to l_p [1].

A finite subset $E = \{n_1 < n_2 < ... < n_k\}$ of $\mathbf{N}$ is said to be admissible if $k \le n_1$. We denote by L the class of all admissible subsets of $\mathbf{N}$. Let $\mathbf{R}^{(N)}$ denote the vector space of all real sequences which are eventually 0. Schreier's space S is the $\|.\|_S$-completion of $\mathbf{R}^{(N)}$, where $\|x\|_S = \sup_{E \in L} \sum_{k \in E} |x(k)|$, $x(k)$ is a k-th coordinate of x, and $x \in \mathbf{R}^{(N)}$.

If E and F are finite non-void subsets of $\mathbf{N}$, we write "$E < F$" for $\max E < \min F$. For $x \in \mathbf{R}^{(N)}$ we write Ex to indicate vector defined by

$$(Ex)(k) = \begin{cases} x(k), & \text{if } \; k \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Fix $1 < p < \infty$. For $x \in \mathbf{R}^{(N)}$, we define

$$\|x\|_{B_p} = \sup \left\{ \left(\sum_{k=1}^{n} \|E_k x\|_{l_1}^p \right)^{\frac{1}{p}} : E_k \in L \text{ and } E_1 < E_2 < ... < E_n, \; n = 1, 2, ... \right\}.$$

Baernstein's space B_p is $\|.\|_{B_p}$-completion of $\mathbf{R}^{(N)}$. It is known that every infinite-dimensional subspace of B_p has a complemented subspace isomorphic to l_p. See [2] for details.

Now we consider some lemmas. The following two lemmas are due to V.M. Kadets and G. Schechtman [6].

Lemma 1. *Let X be a Banach space. The following statements are equivalent:*

$(*)$ $X \notin \mathrm{LPr}$.

$(**)$ *There exists a triple $(\Omega, \Sigma, \lambda)$, where $\lambda : \Sigma \to \mathbf{R}_+$ is a non-atomic measure and a linear bounded map $T : L_\infty(\Omega, \Sigma, \lambda) \to X$ such that*

(a) T is $\sigma(L_\infty, L_1) - \sigma(X, X^*)$-continuous,

(b) there exists $\varepsilon > 0$ such that $\|Tf\| \ge \varepsilon \lambda (supp\, f)$ for any "sign" $f \in L_\infty$, i.e., any function taking only values 0 and ± 1.

We need some more notation. Let $r_i = r_i(\varpi)$ be a sequence of independent random variables taking values $+1$ and -1 with probability $\frac{1}{2}$. Fix a number $N \in \mathbf{N}$ and denote the random variable

$$T(\varpi) = \inf\left\{ j : \left| \sum_{i=1}^{j} r_i(\varpi) \right| \geq \sqrt{N} \right\}.$$

Further define the stopped martingale $\sum_{i=1}^{k} s_i$ by the rule

$$s_i(\varpi) = \begin{cases} r_i(\varpi), & \text{if } \; i < T(\varpi), \\ 0, & \text{if } \; i \geq T(\varpi). \end{cases}$$

The constructed martingale is evidently symmetric, and its absolute value is bounded by $\sqrt{N} + 1$.

Lemma 2. *Let k be a function of N. If $N = o(k)$, then*

$$\lim_{N \to \infty} P\left(\left| \sum_{i=1}^{k} s_i \right| < \sqrt{N} \right) = 0.$$

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$$\lim_{N \to \infty} P\left(\left| \sum_{i=1}^{k} s_i \right| < \sqrt{N} \right) = 1.$$

The next lemma has a technical nature and was proved in [7].

Lemma 3. *Let $X \in \mathrm{LPr}$, μ be an X-valued non-atomic measure, $\lambda : \Sigma \to \mathbf{R}_+$ be non-atomic. Then for every $A \in \Sigma$ with $\lambda(A) \neq 0$ and $\varepsilon > 0$ there exist $G' \in \Sigma|_A$, $G'' = A \backslash G'$ such that*

(i) $\lambda(G') = \lambda(G'') = \frac{1}{2}\lambda(A)$,

(ii) $\left\| \mu(G') - \frac{1}{2}\mu(A) \right\| < \varepsilon$.

The following lemma is a strengthening of the previous one and will be employed in the proof of Theorem 1.

Lemma 4. *Let $X \in \mathrm{LPr}$, μ be an X-valued non-atomic measure, $\lambda : \Sigma \to \mathbf{R}_+$ be non-atomic. Then for every $A \in \Sigma$, $\lambda(A) \neq 0$ there exist $G'_n \in \Sigma|_A$, $G''_n = A \backslash G'_n$ $(n = 1, 2, \ldots)$ such that*

(i) $\lambda(G'_n) = \lambda(G''_n) = \frac{1}{2}\lambda(A)$,

(ii) $z_n = \chi_{G'_n} - \chi_{G''_n}$ *are independent random variables on the measure space* $\left(A, \Sigma|_A, \frac{1}{\lambda(A)}\lambda\right)$,

(iii) $\left\| \mu(G'_n) - \frac{1}{2}\mu(A) \right\| \leq \frac{1}{2^n}$.

P r o o f. We shall construct by induction on k the sets G'_k, G''_k $(k = 1, 2, ...)$. Let $k = 1$. By Lemma 3, there is a $G'_1 \in \Sigma|_A$ such that

$$\left\| \mu \left(G'_1 \right) - \frac{1}{2} \mu \left(A \right) \right\| \leq \frac{1}{2} \quad \text{and} \quad \lambda \left(G'_1 \right) = \frac{1}{2} \lambda \left(A \right). \tag{1}$$

Let $k = k + 1$. Suppose G'_j, G''_j $(j = 1, ..., k)$, satisfying the required conditions, have been constructed. By independence of $\{r_j\}_{j=1}^{k}$, there are mutually disjoint sets $\{D_i\}_{i=1}^{2^k} \subset \Sigma|_A$ such that

$$\begin{aligned} r_1 &= \sum_{i=1}^{2^{k-1}} \chi_{D_i} - \sum_{i=2^{k-1}+1}^{2^k} \chi_{D_i}; \\ r_2 &= \sum_{i=1}^{2^{k-2}} \chi_{D_i} - \sum_{i=2^{k-2}+1}^{2^{k-1}} \chi_{D_i} + \sum_{i=2^{k-1}+1}^{3 \cdot 2^{k-2}} \chi_{D_i} - \sum_{i=3 \cdot 2^{k-2}+1}^{2^k} \chi_{D_i}; \\ \ldots &\ \ldots \ \ldots \ \ldots \\ r_k &= \sum_{i=1}^{2^k} (-1)^{i+1} \chi_{D_i}. \end{aligned}$$

By Lemma 3, there exists $D'_i \in \Sigma|_{D_i}$ such that

$$\left\| \mu \left(D'_i \right) - \frac{1}{2} \mu \left(D_i \right) \right\| \leq \frac{1}{4^{k+1}} \quad \text{and} \quad \lambda \left(D'_i \right) = \frac{1}{2} \lambda \left(D_i \right) \quad \left(k = 1, ..., 2^k \right).$$

Put $G'_{k+1} = \bigcup_{i=1}^{2^k} D'_i$. It is easy to verify that G'_{k+1}, G''_{k+1} satisfy conditions (i)–(iii). ■

2. A_p-property

Definition 1. *A Banach space X with a basis is said to have A_p-property $(X \in A_p)$ if for some basis $\{e_n\}_{n=1}^{\infty}$ in X there exists a map $\widetilde{T} \in L(X, c_0)$ such that for every $x = \sum_{i=1}^{N} x(i) e_i \in X$, with $x(i) \neq 0$ $(1 \leq i \leq N)$, $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y = \sum_{i=N+1}^{M} y(i) e_i \in X$ with $\left\| \widetilde{T} y \right\|_{\infty} \leq \delta$*

$$\left| \|x + y\|^p - \|x\|^p - \|y\|^p \right| < \varepsilon,$$

if $1 \leq p < \infty$, or

$$\left| \|x + y\| - \max \left\{ \|x\|, \|y\| \right\} \right| < \varepsilon,$$

if $p = \infty$.

Lemma 5. *If $X \in A_p$, then there is a map $\widetilde{T} \in L(X, c_0)$ such that for every $x \in X$, $x \neq 0$, for every sequence $x_n \underset{n\to\infty}{\overset{w}{\to}} 0$ in X with $\left\| \widetilde{T} x_n \right\|_\infty \underset{n\to\infty}{\to} 0$, and for every $\varepsilon > 0$ there exists n such that*

$$\left| \|x + x_n\|^p - \|x\|^p - \|x_n\|^p \right| < \varepsilon ,$$

if $1 \le p < \infty$, or

$$\left| \|x + x_n\| - \max\{\|x\|, \|x_n\|\} \right| < \varepsilon ,$$

if $p = \infty$.

P r o o f. Let $\{e_n\}_{n=1}^\infty$ be a basis in X, for which the conditions of Definition 1 are satisfied. Fix $x = \sum_{i=1}^\infty x(i) e_i \in X$, $x \neq 0$, $x_n = \sum_{i=1}^\infty x_n(i) e_i \in X$, $n = 1, 2, \ldots$, $x_n \underset{n\to\infty}{\overset{w}{\to}} 0$, $\left\| \widetilde{T} x_n \right\|_\infty \underset{n\to\infty}{\to} 0$, $\varepsilon > 0$. Put $C = \sup_n \|x_n\|$. Choose $\theta \in (0,1)$ such that for any $a, b \in [0, \|x\| + 1 + C]$ with $|a - b| \le \theta$ the inequality $|a^p - b^p| < \frac{\varepsilon}{4}$ holds. Select $N \in \mathbf{N}$ such that $\left\| \sum_{i=N+1}^\infty x(i) e_i \right\| \le \frac{\theta}{4}$. Denote $x^\varepsilon = \sum_{i=1}^N x^\varepsilon(i) e_i$, where

$$x^\varepsilon(i) = \begin{cases} x(i), & \text{if} \quad x(i) \neq 0, \\ \frac{\theta}{4N}, & \text{if} \quad x(i) = 0. \end{cases}$$

It is obvious that

$$\left| \|x\|^p - \|x^\varepsilon\|^p \right| < \frac{\varepsilon}{4} . \tag{2}$$

Find $\delta > 0$ for x^ε and $\frac{\varepsilon}{4}$. There is $n \in \mathbf{N}$ such that $\left\| \widetilde{T} x_n \right\|_\infty \le \delta$ and $\left\| \sum_{i=1}^N x_n(i) e_i \right\| \le \frac{\theta}{4}$. Select $M \in \mathbf{N}$ such that $\left\| \sum_{i=M+1}^\infty x_n(i) e_i \right\| \le \frac{\theta}{4}$. Put $x_n^\varepsilon = \sum_{i=N+1}^M x_n(i) e_i$. It is easy to see that $\|x_n - x_n^\varepsilon\| \le \frac{\theta}{2}$ and $\left\| \widetilde{T} x_n^\varepsilon \right\|_\infty \le \delta$ and consequently

$$\left| \|x_n\|^p - \|x_n^\varepsilon\|^p \right| < \frac{\varepsilon}{4} , \tag{3}$$

$$\left| \|x^\varepsilon + x_n^\varepsilon\|^p - \|x^\varepsilon\|^p - \|x_n^\varepsilon\|^p \right| < \frac{\varepsilon}{4} . \tag{4}$$

Note that $\left| \|x + x_n\| - \|x^\varepsilon + x_n^\varepsilon\| \right| \le \theta$. Therefore

$$\left| \|x + x_n\|^p - \|x^\varepsilon + x_n^\varepsilon\|^p \right| \le \frac{\varepsilon}{4} . \tag{5}$$

Combining (2)–(5), we get

$$\begin{aligned} \left| \|x + x_n\|^p - \|x\|^p - \|x_n\|^p \right| \ &\le \ \left| \|x^\varepsilon + x_n^\varepsilon\|^p - \|x^\varepsilon\|^p - \|x_n^\varepsilon\|^p \right| \\ &+ \ \left| \|x + x_n\|^p - \|x^\varepsilon + x_n^\varepsilon\|^p \right| + \left| \|x\|^p - \|x^\varepsilon\|^p \right| \\ &+ \ \left| \|x_n\|^p - \|x_n^\varepsilon\|^p \right| < \varepsilon . \end{aligned}$$

The case $p = \infty$ can be shown in the same way. ∎

Theorem 1. *If $X \in A_p$ $(p \neq 2)$, then $X \in$ LPr.*

P r o o f. Let us consider two cases.

The case $1 \leq p < 2$. First we fix $N \in \mathbf{N}$ and $k = \left[\frac{N}{\ln N}\right]$. We shall prove ad absurdum. Assume that $X \notin$ LPr. By Lemma 1, there are $(\Omega, \Sigma, \lambda)$, $\varepsilon > 0$ and $T : L_\infty(\Omega, \Sigma, \lambda) \to X$, satisfying the conditions a) and b). Let us show by induction on j that there exist functions $\{t_i\}_{i=1}^j \in L_\infty$ jointly equidistributed with $\{s_i\}_{i=1}^j$ such that

$$\left\| T\left(\sum_{i=1}^{j} t_i\right)\right\|^p > \sum_{i=1}^{j} \|T(t_i)\|^p - 1\,. \tag{6}$$

For $j = 1$, there is nothing to prove. Suppose that $\{t_i\}_{i=1}^m$ satisfying (6) has been constructed. Now consider $A = \left\{\varpi \in \Omega : \left|\sum_{i=1}^m t_i\right| < \sqrt{N}\right\}$ and the auxiliary c_0-valued measure $\mu(D) = \widetilde{T}T\chi_D$, where $\widetilde{T}$ is the operator from Definition 1 for the space X. Using the LPr of c_0, we may apply Lemma 4 to the measure μ. Choose a sequence $\{z_n\}_{n=1}^\infty$, meeting the conditions of Lemma 4. By condition (iii), we get $\left\|\widetilde{T}Tz_n\right\|_\infty \underset{n\to\infty}{\longrightarrow} 0$. Since $z_n \underset{n\to\infty}{\overset{w^*}{\longrightarrow}} 0$, we have $Tz_n \underset{n\to\infty}{\overset{w}{\longrightarrow}} 0$. By Lemma 5, we can find a number $n \in \mathbf{N}$ such that

$$\left\| T\left(\sum_{i=1}^{m} t_i\right) + T(z_n)\right\|^p > \left\|\sum_{i=1}^{m} T(t_i)\right\|^p + \|Tz_n\|^p - \delta\,,$$

where δ is arbitrarily small. Fitting δ sufficiently small and putting $t_{m+1} = z_n$, we obtain the required inequality.

Suppose that $\{t_i\}_{i=1}^k$ are jointly equidistributed with $\{s_i\}_{i=1}^k$ and subject to the condition (6) with $j = k$. Then

$$\|T\|^p \left\|\sum_{i=1}^{k} t_i\right\|^p \geq \left\| T\left(\sum_{i=1}^{k} t_i\right)\right\|^p > \sum_{i=1}^{k} \|T(t_i)\|^p - 1\,.$$

In view of the condition $(**)$, we have $\|T(t_i)\| \geq \varepsilon\lambda(\mathrm{supp} t_i) \geq \varepsilon\lambda(\mathrm{supp} t_k)$ for $i \leq k$. Since $\{t_i\}_{i=1}^k$ and $\{s_i\}_{i=1}^k$ are equidistributed, this gives us by the choice of k that

$$\begin{aligned}\left(\sqrt{N}+1\right)^p \|T\|^p + 1 &\geq \sum_{i=1}^{k} \|T(t_i)\|^p \geq k\varepsilon^p \left(P(s_k \neq 0)\right)^p \\ &\geq \left(\frac{N}{\lg N} - 1\right)\varepsilon^p \left(P\left(\left|\sum_{i=1}^{k-1} s_i\right| < \sqrt{N}\right)\right)^p .\end{aligned}$$

By Lemma 2, the last factor tends to 1, hence this inequality cannot hold for large N. This contradiction completes the verification of the first case.

The case $2 < p < \infty$. Assume that $X \notin \mathrm{LPr}$. By analogy with the first case, we fix $N \in \mathbf{N}$, let $k = [N \lg N]$, and construct functions $\{t_i\}_{i=1}^k \in L_\infty$ jointly equidistributed with $\{s_i\}_{i=1}^k$ such that

$$\left\| T\left(\sum_{i=1}^k t_i\right)\right\|^p \leq \sum_{i=1}^k \|T(t_i)\|^p + 1 \leq \|T\|^p k + 1. \tag{7}$$

Further the proof coincides with the same case in [6].

The case $p = \infty$ is analyzed just like previous one. ∎

In Lemmas 6–8 below the role of $\widetilde{T}$ from Definition 1 plays the natural coordinate embedding of the correspondent sequence space into c_0.

Lemma 6. $d(a,p) \in A_p$.

P r o o f. Note that if $x = (x_1, x_2, ...) \in d(a,p)$, $\|x\| = \left(\sum_{n=1}^\infty \widehat{x}_n^p a_n\right)^{\frac{1}{p}}$, where $(\hat{x}_1, \hat{x}_2, ...)$ is an enumeration of $\{|x_n|\}_{n=1}^\infty$ such that $\hat{x}_1 \geq \hat{x}_2 \geq ...$. Fix $x = \sum_{i=1}^N x_i e_i \in d(a,p)$, where $x_i \neq 0$ $(i = 1, 2, ..., N)$, $\varepsilon \in (0,1)$. Without loss of generality we may assume that $x_1 \geq x_2 \geq ... > 0$. Since $a = (a_1, a_2, ...) \in c_0$, there exists $k > N$ such that $\sum_{i=k}^{k+N} a_i < \frac{\varepsilon}{2}$. Take $\delta = \min\left\{\frac{\varepsilon}{2k}, \min\limits_{1\leq i\leq N} x_i\right\}$. Let $y = \sum_{i=N+1}^M y_i e_i \in d(a,p)$ with $\|y\|_\infty \leq \delta$. We may assume that $y_{N+1} \geq y_{N+2} \geq \ \ldots \geq 0$. Further we estimate the number $\left|\|x+y\|^p - \|x\|^p - \|y\|^p\right|$:

$$\begin{aligned}
\left|\|x+y\|^p - \|x\|^p - \|y\|^p\right| &= \sum_{i=N+1}^M y_i^p \left(a_{i-N} - a_i\right) \\
&= \sum_{i=N+1}^{N+k} y_i^p \left(a_{i-N} - a_i\right) + \sum_{i=N+k+1}^M y_i^p \left(a_{i-N} - a_i\right).
\end{aligned}$$

Since $1 \geq a_1 \geq a_2 \geq ... \geq 0$, we obtain

$$\sum_{i=N+1}^{N+k} y_i^p \left(a_{i-N} - a_i\right) \leq \sum_{i=N+1}^{N+k} y_i^p \leq \frac{\varepsilon}{2}.$$

The application of $\|y\|_\infty \leq 1$ yields

$$\begin{aligned}
\sum_{i=N+k+1}^M y_i^p \left(a_{i-N} - a_i\right) &\leq \sum_{i=N+k+1}^M \left(a_{i-N} - a_i\right) \\
&= \sum_{i=k+1}^{N+k} a_i - \sum_{i=M-N+1}^M a_i < \frac{\varepsilon}{2}.
\end{aligned}$$

So, $|\|x+y\|^p - \|x\|^p - \|y\|^p| < \varepsilon$. ■

Lemma 7. $S \in A_\infty$.

P r o o f. Let $x = \sum_{i=1}^N x_i e_i \in S$, where $x_i \neq 0$ $(i = 1, 2, ..., N)$, $\varepsilon > 0$. Denote $\delta = \frac{1}{N}\min_{1\le i\le N}|x_i|$. Choose $y = \sum_{i=N+1}^M y_i e_i \in S$ with $\|y\|_\infty \le \delta$. It is evident that $\|x\|, \|y\| \le \|x+y\|$, consequently, $\max\{\|x\|, \|y\|\} \le \|x+y\|$.

We shall prove that $\|x+y\| \le \max\{\|x\|, \|y\|\}$. Let $k \le N$, $E = \{n_1 < ... < n_k\}$ be an admissible set, i.e., $k \le n_1$. If $n_k \le N$, then

$$\sum_{i\in E}|x_i| + \sum_{i\in E}|y_i| = \sum_{i\in E}|x_i| \le \|x\|.$$

If $n_k > N$, then

$$\sum_{i\in E}|x_i| + \sum_{i\in E}|y_i| \le \sum_{i\in E}|x_i| + \min_{1\le i\le N}|x_i| \le \min_{i\in E'}|x_i| \le \|x\|,$$

where $E' = \{k_0\} \cup \{i \in E : i \le N\}$, $k_0 \in \overline{k, N}\backslash E$. Let $k > N$, then

$$\sum_{i\in E}|x_i| + \sum_{i\in E}|y_i| = \sum_{i\in E}|y_i| \le \|y\|.$$

Thus $\|x+y\| \le \max\{\|x\|, \|y\|\}$. ■

Lemma 8. $B_p \in A_p$.

P r o o f. Let $x = \sum_{i=1}^N x_i e_i \in B_p$, $x \neq 0$, $\varepsilon > 0$. Select $\theta \in (0,1)$ such that for any $a, b \in \left[0, \|x\|_{l_1} + 1\right]$ with $|a-b| \le \theta$ the inequality $|a^p - b^p| < \varepsilon$ holds. Take $\delta = \frac{\theta}{N}$. Choose $y = \sum_{i=N+1}^M y_i e_i \in S$ with $\|y\|_\infty \le \delta$. Evidently, $\|x\|^p + \|y\|^p \le \|x+y\|^p$. We shall prove that $\|x+y\|^p \le \|x\|^p + \|y\|^p + \varepsilon$. Let $E_1 < E_2 < ... < E_n$, $E_i \in L$, and

$$\|x+y\|^p = \sum_{i=1}^n \|E_i(x+y)\|_{l_1}^p.$$

We introduce the sets of indices

$$\begin{aligned} I_1 &= \left\{i \in \overline{1,n} : E_i \cap \overline{N+1, M} = \emptyset\right\}, \\ I_2 &= \left\{i \in \overline{1,n} : E_i \cap \overline{1, N} = \emptyset\right\}, \\ i_0 &= \overline{1,n}\backslash(I_1 \cup I_2). \end{aligned}$$

Then

$$\|x+y\|^p = \sum_{i\in I_1} \|E_i x\|_{l_1}^p + \sum_{i\in I_2} \|E_i y\|_{l_1}^p + \|E_{i_0}(x+y)\|_{l_1}^p .$$

Let us estimate the last item:

$$\|E_{i_0}(x+y)\|_{l_1}^p = \left(\left\|\left(E_{i_0}\cap\overline{1,N}\right)x\right\|_{l_1} + \left\|\left(E_{i_0}\cap\overline{N+1,M}\right)y\right\|_{l_1}\right)^p < \|E_{i_0}x\|_{l_1}^p + \varepsilon.$$

This implies that

$$\|x+y\|^p = \sum_{i\in I_1\cup i_0} \|E_i x\|_{l_1}^p + \sum_{i\in I_2} \|E_i y\|_{l_1}^p + \varepsilon \le \|x\|^p + \|y\|^p + \varepsilon.$$

The lemma is proved. ■

Corollary 1. *The Lorentz sequence spaces $d(a,p)$ $(p\ne 2)$, Baernstein spaces B_p $(p\ne 2)$, and Schreier space S have the Lyapunov property.*

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Некоторые новые обобщения теоремы Ляпунова

О. Владимирская

Доказано, что пространство Шриера, пространства Лоренца, пространства Бернштейна, не содержащие изоморфных копий пространства l_2, обладают свойством Ляпунова.

Деякі нові узагальнення теореми Ляпунова

О. Владимирська

Доведено, що простір Шрієра, простори Лоренца, простори Бернштейна, які не вміщають ізоморфних копій простору l_2, мають властивість Ляпунова.