Matematicheskaya fizika, analiz, geometriya
2000, v. 5, No. 1/2, p. 49–67

Splitting of some non-localized solutions of the Korteweg–de Vries equation into solitons

Evgenii Ya. Khruslov and Holger Stephan*

Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering
47 Lenin Ave., 310164, Kharkov, Ukraine
E-mail: khruslov@ilt.kharkov.ua

* *Weierstrass Institute for Applied Mathematics and Stochastics,*
39 Mohrenstrasse, 10117, Berlin, Germany
E-mail: stephan@wias-berlin.de

Received July 7, 1996

The long-time asymptotic behaviour of non-localized solutions of Korteweg–de Vries (KdV) equation is studied and is proved that these solutions split in infinite sequence of solitons.

Introduction

One of the remarkable results obtained by the inverse scattering transform method [10] is that for any localized initial data (i.e., rapidly decreasing with $x \to \pm\infty$) the solution of the Korteweg–de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0$$

splits into a finite number n $(0 \leq n < \infty)$ of solitons as time tends to infinity $(t \to \infty)$ [9]. Other integrable nonlinear evolution equations exhibit similar splitting, though the authors do not know the exact references. This effect is an additional argument in favor of the physical interpretation of solitons as stable "long-living" particles.

The Cauchy problem for the KdV equation with non-localized initial data $u_0(x)$, namely, in the "step-like" form $u_0(x) \sim \dfrac{c^2}{2}(\pm 1 - 1)$ as $x \to \pm\infty$, was solved in 1975 [3]. It was proved, that $\left[\dfrac{N+1}{2}\right]$ soliton-like objects appear in some neighbourhoods $G_N^+(t)$ $(N = 1, 2, ...)$ of the leading edge (the front of the solution).

At large times these domains are $G_N^+(t) = \left\{x \in \mathbf{R},\ x > 4c^2t - \dfrac{N+1}{2c}\ln t\right\}$. The form of these objects is similar to ordinary solitons, but their velocities depend on t. In contrast to ordinary solitons, they are not exact solutions of the KdV equation, however they satisfy it with increasing accuracy when $t \to +\infty$. For this reason such objects are called asymptotic solitons. The number of these asymptotic solitons infinitely increases when $t \to +\infty$, if the observation domain in the neighbourhood of the solution front is extended correspondingly. In a more general form the same phenomenon is observed also for other non-localized initial data, as well as for other KdV-like equations [4]. Physically, one can consider this phenomenon as a manifestation of the fact that any non-localized initial data consists of an infinite number of solitons, which are gradually ejected at the front. The existence of a sufficiently wide living space for solitons is a natural (but not the unique) condition for this phenomenon. As usual, a beam $(T(t), \infty)$, belonging to the positive half-axis x, is taken as the domain, where the solution vanishes or tends to a constant (if there exist solitons on the background for the corresponding equation). As it was shown in [4], the existence of a continuous spectrum of multiplicity one of the L-operator of the corresponding Lax pair is a sufficient (nearly necessary) condition of the splitting. Moreover, the structure of the simple continuous spectrum of the operator L depends only on the behaviour of the initial data as $x \to -\infty$. Thus a wide class of initial data generate the same asymptotic formulae.

In this paper we study one non-localized solution of the KdV equation, which vanishes as $x \to +\infty$. Its behaviour at $x \to -\infty$ is not well understood yet. This solution belongs to the closure of the class of reflectionless potentials, which was introduced by V.A. Marchenko and D.S. Lundina in [6, 7] and H. Stephan in [8]. We derive a determinant asymptotic formula (1.5), which describes the oscillation structure of the solution in the neighbourhood of the front, and prove that as $t \to \infty$ the solution splits into an infinite series of solitons moving along the x-axis to the right. Our method to find and prove asympotic formula, being close conceptually to the method proposed in [3], is based on the reduction of the problem to the solution of an integral equation with a suitable degenerate kernel. However, as distinct from [3], where the well-known Marchenko integral equation is utilized, in the present paper we consider an integral operators acting on functions depending on the spectral parameter [5, 8]. To implement the idea mentioned above new techniques were developed which can be applied also to the solution of other nonlinear evolution equations within the framework of the Riemann–Hilbert problem ([10]).

The proposed method allows one also to investigate the asymptotic behaviour of the solution in the neighbourhood of the trailing edge (back of solution): $G_N(t)$

as $t \to -\infty$, and to show that $\left[\frac{N+1}{2}\right]$ solitons are selected from the solution in this domain. These solitons move along the x-axis to the left as $t \to -\infty$ and have smaller amplitudes than those of the solitons at the front.

1. Formulation of the problem and main results

Let $\varphi(\mu)$ be an arbitrary positive function defined on the interval $]a, b[$ $(b > a > 0)$ of the form

$$\varphi(\mu) = (\mu - a)^{\alpha}(b - \mu)^{\beta}\varphi_0(\mu)\,,$$

where

$$\varphi_0(\mu) \in \mathbf{C}^{\infty}[a, b],\ \varphi_0(a) > 0,\ \varphi_0(b) > 0, \quad \alpha, \beta > -\frac{1}{2}\,.$$

Consider a Fredholm integral equation with respect to the unknown function $g(\lambda; x, t)$ of the variable $\lambda \in]a, b[$:

$$g(\lambda; x, t) + e^{-\lambda(x-4\lambda^2 t)} \int_a^b \frac{e^{-\mu(x-4\mu^2 t)}}{\lambda + \mu} \varphi(\mu) g(\mu; x, t) d\mu = e^{-\lambda(x-4\lambda^2 t)}, \quad a < \lambda < b\,. \tag{1.1}$$

Here the variables $x, t \in \mathbf{R}$ are considered to be parameters. This equation has an unique solution $g(\lambda; x, t) \in \mathbf{C}^{\infty}(a, b)$, smoothly depending on $x \in \mathbf{R}$ and $t \in \mathbf{R}$. The methods developed in [5, 8] allow us to show that the function

$$u(x, t) = 2\frac{d}{dx}\int_a^b e^{-\mu(x-4\mu^2 t)}\varphi(\mu) g(\mu; x, t) d\mu, \tag{1.2}$$

which is a functional of the solution g of (1.1), satisfies the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

It is easy to see the solution (1.2) tends exponentially to zero as $x \to +\infty$, however its behaviour is unknown as $x \to -\infty$.

R e m a r k. Taking into account results of [1], one can suppose that in the case of $\varphi(\mu) = \sqrt{(b-\mu)(\mu-a)}\varphi_0(\mu)$, (i.e., the special case $\alpha = \beta = 1/2$), the solution, defined by (1.2), asymptotically tends to the periodic one-gap solution of the KdV equation as $x \to -\infty$.

The principal goal of the present paper is to investigate the asymptotic behaviour of the solution $u(x, t)$ which is determined by (1.1) and (1.2), in the neighbourhood of the leading edge when $t \to \infty$. We define the front leading edge as the domain

$$\Omega_N^+(t) = \left\{ x \in \mathbf{R} : \quad x > 4b^2 t - \frac{1}{2b} \ln t^{[N+\beta+1]} \right\}, \tag{1.3}$$

where N is an arbitrary positive integer and $t > 1$.

Let us introduce the notations:

$$C_{ij} = \frac{(i+j)!}{i!j!(2b)^{i+j+1}}, \quad i,j = 0,1,\dots; \tag{1.4}$$

$C^{(N)} = \{C_{ij}\}_{i,j=0}^N$ is the matrix of order $N+1$ with the entries C_{ij};

$$I_k(x,t) = \int_a^b e^{-2\mu(x-4\mu^2 t)}(b-\mu)^k \varphi(\mu)d\mu;$$

$I^{(N)}(x,t) = \{I_{i+j}(x,t)\}_{i,j=0}^N$ and $A^{(N)}(x,t) = C^{(N)} I^{(N)}(x,t)$ are the matrix-functions of order $N+1$ with the entries $I_{i+j}(x,t)$ and $A_{ij}(x,t) = \sum_{k=0}^N C_{ik} I_{k+j}(x,t)$, respectively.

For fixed parameter b, the numbers C_{ik} are fixed too, but in Theorem 1 we consider C_{00} to be a freely varying parameter.

Theorem 1. *The solution $u(x,t)$, which is determined by (1.1) and (1.2) everywhere in $\mathbf{R}^2$, is represented in the form*

$$u(x,t) = 2\frac{\partial^2}{\partial x \partial C_{00}} \log\det\left[E^{(N)} + A^{(N)}(x,t)\right] + \Delta^{(N)}(x,t) \tag{1.5}$$

in the domain $G_N^+ = \bigcup_{t>1} \Omega_N^+(t) \subset \mathbf{R}^2$, where $E^{(N)}$ is the identity matrix of order N, and the function $\Delta^{(N)}(x,t)$ satisfies the inequalities

$$\left|\Delta^{(N)}(x,t)\right|$$

$$\leq \begin{cases} \frac{K_N}{t}, & \text{as} \quad -\frac{[N+\beta+1]}{2b} \ln\sqrt{\frac{b}{b-a}} > x - 4b^2 t \geq -\frac{[N+\beta+1]}{2b} \ln t, \ t > 1, \\ \min\left\{ \frac{K_N}{t}, K\left(\frac{b-a}{b}\right)^{\frac{N+1}{2}} N \right\}, & \text{as} \quad x - 4b^2 t \geq -\frac{[N+\beta+1]}{2b} \ln\sqrt{\frac{b}{b-a}}, \ t > 1. \end{cases}$$

The constants K_N and K depend on the parameters a, b, α, β and the function $\varphi_0(\lambda)$. Taking into account inequalities obtained in Section 2, it is easy to carry out upper estimates for K_N and K, however they are rather combersome. What is only important for us is that $\lim\limits_{N\to\infty} \Delta^{(N)}(x,t) = 0$ uniformly with respect to $x \geq 4b^2t - \frac{[N+\beta+1]}{2b} \ln\sqrt{\frac{b}{b-a}}$, $t > 1$, and $\lim\limits_{t\to+\infty} \Delta^{(N)}(x,t) = 0$ uniformly with respect to $x \in \Omega_N^+(t)$ for any fixed N.

The asymptotic analysis of (1.5) allows us to prove that solution $u(x,t)$ splits into $\left[\dfrac{N+1}{2}\right]$ solitons in the domain $\Omega_N^+(t)$ as $t \to +\infty$. The following theorem holds.

Theorem 2. *The solution $u(x,t)$ can be represented in the form*

$$u(x,t) = -\sum_{k=1}^{\left[\frac{N+1}{2}\right]} \frac{2b^2}{\cosh^2\left[b\left(x - 4b^2t + \frac{1}{2b}\ln t^{2k-1+\beta} + x_k^0\right)\right]} + O\left(\frac{1}{\sqrt{t}}\right)$$

in the domain $\Omega_N^+(t)$ as $t \to +\infty$. The numbers x_k^0 are constant phases, which are given by

$$x_k^0 = \frac{1}{2b}\ln\frac{[(k-1)!]^2\Delta_1^{(k-1)}\Delta_2^{(k-1)}b^{6k-3+2\beta}2^{10k-5+4\beta}}{\varphi_0(b)(b-a)^\alpha\Delta_1^{(k)}\Delta_2^{(k)}},$$

where Δ_1^n and Δ_2^n are the determinants of the matrices with the entries $(i+k)!$ and $\Gamma(i+k+1+\beta)$, $i,k = 0,1,...,n-1$, respectively.

2. Proof of Theorem 1

Let us introduce the Hilbert space $L_{2\varphi}[a,b]$ of the real functions $g(\mu)$ on the interval (a,b) with the norm

$$||g|| = \left\{\int_a^b g^2(\mu)\varphi(\mu)d\mu\right\}^{1/2},$$

where $a,b > 0$. The function $\varphi(\mu) > 0$ was introduced in Section 1. In this space let us consider the operator A that depends on parameters $x,t \in \mathbf{R}$:

$$[Ag](\lambda) = \int_a^b \frac{e^{-(\lambda+\mu)x+4(\lambda^3+\mu^3)t}}{\lambda+\mu}\varphi(\mu)g(\mu)d\mu, \qquad \lambda \in (a,b). \tag{2.1}$$

Lemma 1. *The operator $(E+A)^{-1}$ exists and satisfies the relation*

$$||(E+A)^{-1}|| \leq 1$$

for all $x,t \in \mathbf{R}$ (E is the identity operator in $L_{2\varphi}[a,b]$).

P r o o f. Since $a,\, b > 0$, A is a completely continuous operator in $L_{2\varphi}[a,b]$. Let $g \in L_{2\varphi}[a,b]$ be the solution of the equation

$$(E + A)g = f\,, \tag{2.2}$$

where $f \in L_{2\varphi}[a,b]$. After scalar multiplication of (2.2) by g and application of (2.1) and equality

$$\frac{1}{\mu+\lambda} = \int\limits_0^\infty e^{-(\mu+\lambda)s} ds,$$

we obtain

$$\int\limits_a^b g^2(\mu)\varphi(\mu)d\mu + \int\limits_0^\infty ds \left(\int\limits_a^b e^{-\mu s - \mu(x-4\mu^2 t)} \varphi(\mu) g(\mu) d\mu \right)^2 = \int\limits_a^b f(\mu) g(\mu)\varphi(\mu)d\mu. \tag{2.3}$$

Hence the homogeneous equation, which corresponds to (2.2), has only the trivial solution, and consequently equation (2.2) is uniquely solvable for any $f \in L_{2\varphi}[a,b]$. In addition, (2.3) yields

$$||g|| \le ||f||.$$

Thus the Lemma is proved. ∎

Using the expansion

$$\frac{1}{\lambda+\mu} = \sum_{i,j=0}^{\infty} \frac{(i+j)!}{i!j!(2b)^{i+j+1}} (b-\lambda)^i (b-\mu)^j,$$

let us write the operator A as a sum of the two operators

$$[A_N g](\lambda) = \int\limits_a^b e^{-(\lambda+\mu)x + 4(\lambda^3+\mu^3)t} \sum_{i,j=0}^{N} C_{ij}(b-\lambda)^i (b-\mu)^j \varphi(\mu) g(\mu) d\mu,$$

$$[B_N g](\lambda) = \int\limits_a^b e^{-(\lambda+\mu)x + 4(\lambda^3+\mu^3)t} \sum_{(i,j)\in \mathbf{R}^{(N)}} C_{ij}(b-\lambda)^i (b-\mu)^j \varphi(\mu) g(\mu) d\mu\,, \tag{2.4}$$

where the numbers C_{ij} are defined by (1.4) and $\mathbf{R}^{(N)}$ is the following set of the couples (i,j):

$$\mathbf{R}^{(N)} = \{(i,j): \quad 0 \le i < \infty,\ 0 \le j < \infty\} \setminus \{(i,j): \quad 0 \le i \le N,\ 0 \le j \le N)\}\,.$$

Since $0 < a \leq \lambda,\ \mu \leq b,\ C_{ij} > 0$ and $\mathbf{R}^{(N)} \subseteq \bigcup_{k=N+1}^{\infty} \{(i,j): \quad i+j=k\}$ the following inequalities hold:

$$0 < \sum_{(i,j)\in \mathbf{R}^{(N)}} C_{ij}(b-\lambda)^i(b-\mu)^j \leq \sum_{i+j=N+1}^{\infty} C_{ij}(b-\lambda)^i(b-\mu)^j$$

$$= \sum_{k=N+1}^{\infty}\sum_{i=0}^{k} \frac{k!}{i!(k-i)!(2b)^{k+1}}(b-\lambda)^i(b-\mu)^{k-i} = \frac{1}{2b}\sum_{k=N+1}^{\infty}\left[\frac{(b-\lambda)+(b-\mu)}{2b}\right]^k$$

$$\leq \frac{1}{2b}\left[\frac{(b-\lambda)+(b-\mu)}{2b}\right]^{N+1}\frac{1}{1-\frac{b-a}{b}} \leq \frac{1}{2a}\left(\frac{b-a}{b}\right)^{N+1}. \tag{2.5}$$

These inequalities allow us to estimate the norm of the operator B_N at $t>0$:

$$||B_N||^2 \leq \int_a^b\int_a^b e^{-2(\lambda+\mu)x+8(\lambda^3+\mu^3)t}\frac{1}{(2a)^2}\left(\frac{b-a}{b}\right)^{2(N+1)}\varphi(\lambda)\varphi(\mu)d\lambda d\mu$$

$$\leq \frac{\hat{\varphi}_0^2}{(2a)^2}\left(\frac{b-a}{b}\right)^{2(N+1)}\left(\int_a^b e^{-2\mu\xi-8\mu(b^2-\mu^2)t}(b-\mu)^\beta(\mu-a)^\alpha d\mu\right)^2$$

$$\leq \frac{\hat{\varphi}_0^2}{a^2}\left(\frac{b-a}{b}\right)^{2(N+1)}(b-a)^{2(1+\alpha+\beta)}e^{-2p(\xi)},$$

where

$$\hat{\varphi}_0 = \max_{[a,b]}\varphi_0(\lambda), \quad \xi = x-4b^2t, \quad p(\xi) = \xi(b+a)-|\xi|(b-a).$$

It follows from this that

$$||B_N|| \leq \left(\frac{b-a}{b}\right)^{\frac{N+1}{2}}\frac{b^{\frac{\beta}{2}}(b-a)^{1+\alpha+\frac{\beta}{2}}}{a}\hat{\varphi}_0 \tag{2.6}$$

at $t \geq 0$ and $\xi \geq -\frac{[N+1+\beta]}{2b}\ln\sqrt{\frac{b}{b-a}}$.

Let us estimate the norm of B_N when

$$-\frac{[N+1+\beta]}{2b}\ln\sqrt{\frac{b}{b-a}} > \xi > -\frac{[N+1+\beta]}{2b}\ln t, \quad t>1, \quad \xi = x-4b^2t.$$

Using the inequalities (2.5), we can write

$$||B_N||^2 \leq \int_a^b\int_a^b e^{-2(\lambda+\mu)x+8(\lambda^3+\mu^3)t}\frac{1}{(2a)^2}\left[\frac{(b-\lambda)+(b-\mu)}{2b}\right]^{2(N+1)}\varphi(\lambda)\varphi(\mu)d\lambda d\mu$$

$$= \frac{1}{(2a)^2} \sum_{k+j=2(N+1)} C_{kj} I_k(x,t) I_j(x,t), \tag{2.7}$$

where

$$I_k(x,t) = \int_a^b e^{-2\mu(x-4\mu^2 t)} (b-\mu)^k \varphi(\mu) d\mu. \tag{2.8}$$

Lemma 2. *The integrals $I_k(x,t)$ have the following asymptotic representation:*

$$I_k(x,t) = \frac{\varphi_1 \nu_1^{k+\beta+1} \Gamma(k+\beta+1)}{t^{k+\beta+1}} e^{-2b\xi} + \delta_k(t,\xi),$$

at $t \to \infty$ and $\xi = x - 4b^2 t > -t^{1/2}$. Here $\varphi_1 = \varphi_0(b)(b-a)^\alpha$, $\nu_1 = \left(\frac{1}{4b}\right)^2$ and $\Gamma(z)$ is Euler's Γ-function. The functions $\delta_k(t,\xi)$ can be estimated by

$$|\delta_k(t,\xi)| \le \frac{\Gamma(k+\beta+2)}{t^{k+\beta+3/2}} D^{k+\beta+2} e^{-p(\xi)},$$

where the constant D depends on parameters of the problem and $p(\xi) = \xi(b+a) - |\xi|(b-a)$.

P r o o f. Changing variables $x = 4b^2 t + \xi$ and $\mu = b - \nu$, we find

$$I_k(x,t) = e^{-2b\xi} \int_0^{b-a} e^{2\nu\xi - 8\nu(b-\nu)(2b-\nu)t} \nu^k \varphi(b-\nu) d\nu. \tag{2.9}$$

Let us consider the function $\rho = \rho(\nu) = 8\nu(b-\nu)(2b-\nu)$. It can be shown, that there exists the inverse function $\nu = \nu(\rho)$ with $\nu(0) = 0$ and the series

$$\nu = \nu(\rho) = \nu_1 \rho + \nu_2 \rho^2 + \dots, \quad \nu_1 = \left(\frac{1}{4b}\right)^2,$$

$$\frac{d\nu}{d\rho} = \nu_1 + 2\nu_2 \rho + \dots,$$

$$(b-a-\nu(\rho))^\alpha \varphi_0(b-\nu(\rho)) = \varphi_1 + \varphi_2 \rho + \dots, \quad \varphi_1 = (b-a)^\alpha \varphi_0(b) \tag{2.10}$$

converge absolutely and uniformly at $|\rho| \le \tilde{\rho} < \frac{16b^3}{3\sqrt{3}}$. Let us choose such a number δ_1 that $0 < \delta_1 < \min\left\{\frac{16b^3}{3\sqrt{3}}, \rho(b-a)\right\}$ $(\rho(b-a) = a(b^2-a^2) > 0)$, and set $\delta = \nu(\delta_1)$ $(\delta < b-a)$. Then, taking into account the equality $\varphi(b-\nu) = \nu^\beta(b-a-\nu)^\alpha\varphi_0(b-\nu)$ and (2.9), we write

$$I_k(x,t) = e^{-2b\xi}\int_0^\delta e^{2\nu\xi}e^{-\rho(\nu)t}\nu^{k+\beta}(b-a-\nu)^\alpha\varphi_0(b-\nu)d\nu$$

$$+e^{-2b\xi}\int_\delta^{b-a} e^{2\nu\xi}e^{-\rho(\nu)t}\nu^{k+\beta}(b-a-\nu)^\alpha\varphi_0(b-\nu)d\nu = I_k'(x,t) + I_k''(x,t)\,.$$

It's easy to see that

$$I_k''(x,t) \le \hat{\varphi}_0 e^{-p(\xi)}e^{-\delta_1 t}\frac{(b-a)^{k+1+\beta+\alpha}}{k+1+\beta}, \tag{2.11}$$

where

$$\hat{\varphi}_0 = \max_{\lambda\in[a,b]}\varphi_0(\lambda), \qquad p(\xi) = \xi(b+a) - |\xi|(b-a).$$

Let us estimate the integral $I_k'(x,t)$. Using (2.10), we obtain

$$\nu^{k+\beta}(\rho)(b-a-\nu(\rho))^\alpha\varphi_0(b-\nu(\rho))\frac{d\nu}{d\rho} = \varphi_1\nu_1^{k+\beta+1}\rho^{k+\beta} + \rho^{k+\beta+1}\Phi_k(\rho),$$

$$e^{2\nu(\rho)\xi} = 1 + \rho E(\rho,\xi) \tag{2.12}$$

at $|\rho| < \delta_1$. The functions $\Phi_k(\rho)$ and $E(\rho,\xi)$ can be estimated as follows:

$$|E(\rho,\xi)| \le A|\xi|e^{\delta(|\xi|+\xi)},$$

$$\Phi_k(\rho) \le (k+\beta+1)B^{k+\beta+1}\varphi_1. \tag{2.13}$$

The constants A and B depend only on a, b, and $\varphi_0(\lambda)$, and are determined by

$$A = \max_{\rho<\delta_1}\left|\frac{2\nu(\rho)}{\rho}\right|,$$

$$B = \max_{\rho<\delta_1}\max\left\{\left|\frac{\nu(\rho)}{\rho}\right|, \left|\left(\frac{\nu(\rho)}{\rho}\right)'\right|, |G(\rho)|, |G'(\rho)|\right\},$$

where

$$G(\rho) = \nu'(\rho)\frac{\varphi_0(b-\nu(\rho))}{\varphi_0(b)}\frac{(b-a-\nu(\rho))^\alpha}{(b-a)^\alpha}.$$

According to (2.12), the integral $I_k'(\xi,k)$ can be represented in the form

$$\begin{aligned} I_k'(\xi,t) &= e^{-2b\xi}\int\limits_0^{\delta_1} e^{2\nu(\rho)\xi}e^{-\rho t}\nu^{k+\beta}(\rho)(b-a-\nu(\rho))^{\alpha}\varphi_0(b-\nu(\rho))\frac{d\nu}{d\rho}d\rho \\ &= \varphi_1\nu_1^{k+\beta+1}e^{-2b\xi}\int\limits_0^{\infty} e^{-\rho t}\rho^{k+\beta}d\rho - \varphi_1\nu_1^{k+\beta+1}e^{-2b\xi}\int\limits_{\delta_1}^{\infty} e^{-\rho t}\rho^{k+\beta}d\rho \\ &+ e^{-2b\xi}\int\limits_0^{\delta_1} e^{-\rho t}\rho^{k+\beta+1}(E(\rho,\xi)+\Phi_k(\rho))d\rho \\ &+ e^{-2b\xi}\int\limits_0^{\delta_1} e^{-\rho t}\rho^{k+\beta+2}E(\rho,\xi)\Phi_k(\rho)d\rho = \sum_{j=0}^{3} I_k^{(j)}(\xi,t). \end{aligned}$$

It is evident that

$$I_k^{(0)}(\xi,t) = \varphi_1\nu_1^{k+\beta+1}\Gamma(k+\beta+1)\frac{e^{-2b\xi}}{t^{k+\beta+1}}.$$

Taking into account (2.13), we obtain estimates for the other summands:

$$|I_k^{(1)}(\xi,t)| \le \varphi_1(2\nu_1)^{k+\beta+1}\Gamma(k+\beta+1)e^{-\frac{\delta_1}{2}t}\frac{e^{-2b\xi}}{t^{k+\beta+1}},$$

$$|I_k^{(2)}(\xi,t)| < \varphi_1\left(\nu_1^{k+\beta+1}A|\xi|e^{\delta(|\xi|+\xi)} + (k+\beta+1)B^{k+\beta+1}\right)\Gamma(k+\beta+2)\frac{e^{-2b\xi}}{t^{k+\beta+2}},$$

$$|I_k^{(3)}(\xi,t)| < (k+\beta+1)\varphi_1|\xi|AB^{k+\beta+1}\Gamma(k+\beta+3)\frac{e^{-2b\xi+\delta(|\xi|+\xi)}}{t^{k+\beta+3}}.$$

Hence

$$I_k'(\xi,t) = \varphi_1\nu_1^{k+\beta+1}\Gamma(k+\beta+1)\frac{e^{-2b\xi}}{t^{k+\beta+1}} + \delta_k'(\xi,t),$$

where it is implied that the function $\delta_k'(\xi,t)$ obeyes the estimation

$$|\delta_k'(\xi,t)| \le \frac{\Gamma(k+\beta+2)}{t^{k+\beta+3/2}}D_1^{k+\beta+2}e^{-p(\xi)}$$

at $\xi > -t^{1/2}$. The constant D_1 depends on A, B, φ_1, ν_1, and δ_1, i.e., on parameters of the problem. The last statements and inequality (2.11) conclude the proof. ■

Inserting the asymptotic expressions for the integrals $I_n(x,t)$ obtained in Lemma 2 into (2.7), we get

$$||B_N||^2 \le \frac{B^{2(N+2+\beta)}\Gamma(2(N+2+\beta))}{t^{2(N+2+\beta)}}e^{-2p(\xi)},$$

and consequently,

$$||B_N|| \leq \tilde{K}_N t^{-1} \tag{2.14}$$

at $\xi \geq -\frac{[N+1+\beta]}{2b}\ln t$, where $\tilde{K}_N$ is a constant depending on N, $\varphi_0(\lambda)$ and the problem parameters a, b, α, and β. The inequalities (2.6) and (2.14) give us a necessary estimations of the norm of B_N.

Now let us return to the equation (1.1) and rewrite it in the form

$$g + A_N g + B_N g = f \tag{2.15}$$

in $L_{2\varphi}[a,b]$, where the operators A_N and B_N are determined by (2.4). Here A_N is the operator with the degenerate kernel. According to (2.6) and (2.14), the norm of the operator B_N becomes small if $N \to \infty$ or $t \to \infty$ in the corresponding range of x. The solution of (2.15) is represented in the form

$$g = g_N + \delta_N, \tag{2.16}$$

where g_N is the solution of the equation

$$g_N + A_N g_N = f. \tag{2.17}$$

Therefore

$$\delta_N = -(E+A)^{-1} B_N g_N. \tag{2.18}$$

According to (1.2),

$$u(x,t) = 2\frac{d}{dx}(f,g), \tag{2.19}$$

where $f = f(\mu) = e^{-\mu(x-4\mu^2 t)}$. The parentheses denote the scalar product in $L_{2\varphi}[a,b]$. The self-adjointness of A in $L_{2\varphi}[a,b]$, (2.16), and (2.18) allow us to write

$$(f,g) = (f,g_N) + (f,\delta_N), \tag{2.20}$$

$$(f,\delta_N) = \Big(f, (E+A)^{-1} B_N g_N\Big) = \Big((E+A)^{-1} f, B_N g_N\Big)$$
$$= (g_N + \delta_N, B_N g_N) = (g_N, B_N g_N) - \Big((E+A)^{-1} B_N g_N, B_N g_N\Big). \tag{2.21}$$

In Section 3 we will show that

$$|(f,g_N)| < CN \tag{2.22}$$

for $\xi \geq -\frac{[N+1+\beta]}{2b}\ln t$, $t > 1$, where the constant C does not depend on N, ξ, and t. Therefore, by the virtue of the positiveness of A_N, from equation (2.17) follows that

$$||g_N||^2 < CN. \tag{2.23}$$

Applicating this inequality, (2.6), (2.14), and Lemma 1 to (2.21), we obtain

$$|(f,\delta_N)| \le \begin{cases} \min\left\{\frac{K'_N}{t}, NK'\left(\frac{b-a}{b}\right)^{\frac{N+1}{2}}\right\}, & \text{as} \quad \xi \ge -\frac{[N+1+\beta]}{2b}\ln\sqrt{\frac{b}{b-a}}, \\ \frac{K'_N}{t}, \quad \text{as} \quad -\frac{[N+1+\beta]}{2b}\ln\sqrt{\frac{b}{b-a}} > \xi \ge -\frac{[N+1+\beta]}{2b}\ln t, & t>1, \end{cases} \tag{2.24}$$

where the constants K'_N and K' depend on the function $\varphi_0(\lambda)$ and the problem parameters a, b, α, and β. Taking into account (2.18) and Lemma 1, it is easy to show that the function $F(x) = (f, \delta_N)$ can be continuated up to a function $F(z)$ of exponential type $3b$ from the real axis into the complex plane $\mathbf{C}$. In a similar way one can prove that the estimates (2.24) hold for the function $F(z)$ when z belongs to any beam $z = \xi + i\eta, \xi \ge -\frac{[N+1+\beta]}{2b}\ln t$. Hence, due to the Cauchy theorem, we conclude that estimates of the type given by (2.24) are valid for the function

$$\Delta^{(N)}(x,t) = 2\frac{d}{dx}(f,\delta_N), \tag{2.25}$$

i.e., the last conclusion of Theorem 1 concerning the residual $\Delta^{(N)}(x,t)$ is proved.

According to (2.19) and (2.20), we have to show that

$$(f, g_N) = \frac{\partial}{\partial C_{00}} \ln\det\left[E^{(N)} + A^{(N)}(x,t)\right]. \tag{2.26}$$

Let us find the solution $g_N(\lambda; x,t)$ of the integral equation (2.17) with the degenerate kernel

$$A_N(\lambda,\mu;x,t) = e^{-(\lambda+\mu)x+4(\lambda^3+\mu^3)t}\sum_{i,j=0}^{N} C_{ij}(b-\lambda)^i(b-\mu)^j\varphi(\mu),$$

which corresponds to the operator A_N (2.4). Taking into account the specific form of the kernel, we seek for the solution in the form

$$g_N(\lambda;x,t) = \sum_{k=0}^{N} g_k^{(N)}(x,t)(b-\lambda)^k e^{-\lambda(x-4\lambda^2 t)}. \tag{2.27}$$

Substituting (2.27) into (2.17), we obtain a system of linear algebraic equations for the functions $g_k^{(N)} = g_k^{(N)}(x,t)$:

$$g_k^{(N)} + \sum_{j=0}^{N} A_{kj}^{(N)} g_j^{(N)} = \delta_{k0}, \quad k = 0, ..., N,$$

where $\delta_{00} = 1$ and $\delta_{k0} = 0$ for $k = 1, ..., N$,

$$A_{kj}^{(N)} = A_{kj}^{(N)}(x,t) = \sum_{i=0}^{N} C_{ki} I_{i+j}(x,t).$$

The integrals $I_k(x,t)$ were defined by (2.8). The solution of this system is given by

$$g_k^{(N)}(x,t) = \frac{D_k^{(N)}(x,t)}{D^{(N)}(x,t)}, \tag{2.28}$$

where $D^{(N)}(x,t) = \det\left[E^{(N)} + A^{(N)}(x,t)\right]$ is the determinant of the matrix $E^{(N)} + A^{(N)}(x,t)$ with the entries $\delta_{ij} + A_{ij}^{(N)}(x,t),\ i,j = 0,...,N$, and $D_k^{(N)}(x,t)$ is the determinant of the matrix obtained by replacing the k-th column of the matrix $E^{(N)} + A^{(N)}(x,t)$ by the column $(1,0,...,0)^{\perp}$.

It follows from (2.26) and (2.28) that

$$(f, g_N) = \frac{G^{(N)}(x,t)}{D^{(N)}(x,t)},$$

where $G^{(N)}(x,t)$ is the determinant of the matrix obtained by replacing the first line of the matrix $E^{(N)} + A^{(N)}(x,t)$ by the line $\{I_0(x,y), I_1(x,y), ..., I_N(x,y)\}$. Therefore, taking into account that $A^{(N)}(x,t) = C^{(N)}I^{(N)}(x,t)$ and setting C_{00} as a varying parameter, we obtain (2.18). Thus Theorem 1 is proved. ■

3. Proof of Theorem 2

Let us denote via $\tilde{D}^{(N)}(x,t;\lambda_0,...,\lambda_N)$ the determinant of the matrix $\Lambda^{(N)} + A^{(N)}(x,t)$, where $\Lambda^{(N)} = \mathrm{diag}(\lambda_0,...,\lambda_N)$ is the diagonal matrix depending on $N+1$ parameters $\lambda_0,...,\lambda_N$. It is evident that $\tilde{D}^{(N)}(x,t;1,...,1) = D^{(N)}(x,t) = \det\left[E^{(N)} + A^{(N)}(x,t)\right]$. This determinant is a polynom with respect to λ_k:

$$\begin{aligned}\tilde{D}^{(N)}(x,t;\lambda_0,...,\lambda_N) &= \lambda_0...\lambda_N + \hat{\lambda}_0\lambda_1...\lambda_N D_0^{(1)}(x,t) + \lambda_0\hat{\lambda}_1\lambda_2...\lambda_N D_1^{(1)}(x,t)\\ &+ ... + \lambda_0...\lambda_{N-1}\hat{\lambda}_N D_N^{(1)}(x,t) + \hat{\lambda}_0\hat{\lambda}_1\lambda_2...\lambda_N D_{01}^{(2)}(x,t)\\ &+ ... + \lambda_0...\hat{\lambda}_{i_1}...\hat{\lambda}_{i_k}\lambda_k D_{i_1...i_k}^{(k)}(x,t) + D_{0...N}^{(N)}(x,t),\end{aligned} \tag{3.1}$$

where $D_{i_1...i_k}^{(k)}(x,t)$ is the determinant of the matrix of the order k with the entries $A_{i_r i_p}(x,t)$, $r,p = 1,...,k$; the hat means that the corresponding parameter is absent. Taking into account that $A_{ik} = \sum\limits_{j=0}^{N} C_{ij}I_{j+k}(x,t)$, we obtain from Lemma 2 that

$$\begin{aligned}D_{i_1...i_k}^{(k)}(x,t) &= \det C^{(k)} \det I^{(k)}(x,t) + d^{(k)}(x,t)\\ &= \det C^{(k)} \det \Gamma^{(k)} \frac{e^{-2bk\xi}}{t^{k(k+\beta)}} + d_1^{(k)}(x,t)\,,\end{aligned}$$

when $i_1 = 0,\ i_2 = 1, ...,\ i_k = k-1$, i.e., as $i_1 + i_2 + ... + i_k = \frac{k(k-1)}{2}$, and

$$\left| D^{(k)}_{i_1...i_k}(x,t) \right| < C_k \frac{e^{-kp(\xi)}}{t^{k(k+\beta)+1}}\,,$$

when $i_1+i_2+...+i_k > \frac{k(k-1)}{2}$. Here $\xi = x - 4b^2 t$, $p(\xi) = \xi(b+a) - |\xi|(b-a)$, $C^{(k)}$, and $\Gamma^{(k)}$ are matrices of order k with the elements C_{ij} and $\Gamma^{(k)}_{ij} = \varphi_1 \nu_1^{i+j+\beta+1}\Gamma(i+j+\beta+1)$, $i,j = 0,...,k-1$, respectively, $\Gamma(z)$ is Euler's Γ-function, and the functions $d^{(k)}(x,t)$ and $d_1^{(k)}(x,t)$ satisfy the estimates

$$\left| d^{(k)}(x,t) \right|,\ \left| d_1^{(k)}(x,t) \right| \le C_k \frac{e^{-kp(\xi)}}{t^{k(k+\beta)+1/2}}.$$

Setting $\lambda_i = 1$, $i = 0,...,N$, in (3.1), it follows that

$$\det\left[E^{(N)} + A^{(N)}(x,t)\right] = 1 + \sum_{k=1}^{N} \det C^{(k)} \det \Gamma^{(k)} \frac{e^{-2bk\xi}}{t^{k(k+\beta)}}(1 + \delta_k(x,t)), \qquad (3.2)$$

where the functions $\delta_k(x,t)$ satisfy the estimates

$$|\delta_k(x,t)| < \frac{C_k}{\sqrt{t}}. \qquad (3.3)$$

A more detailed analysis of the determinants $D^{(k)}_{i_1...i_k}$ shows, that the derivatives of these functions with respect to x and the parameter C_{00} obey the same estimates:

$$\left|\frac{\partial \delta_k}{\partial C_{00}}\right|,\ \left|\frac{\partial \delta_k}{\partial x}\right|,\ \left|\frac{\partial^2 \delta_k}{\partial^2 x}\right| < \frac{C}{\sqrt{t}}. \qquad (3.4)$$

R e m a r k. The last estimates follow also from the possibility to continue analytically the functions $\delta_k = \delta_k(x,t,C_{00})$ into the strips $|\mathrm{Im} C_{00}| < C$ and $|\mathrm{Im} x| < C$ with respect to x and C_{00}. The estimates (3.3) remain valid there.

Let us take the advantage of the following equality, which is proved, for example, in [2]:

$$k \det C_0^{(k)} = \det C_1^{(k-1)},$$

where $C_0^{(k)}$ is the matrix of the order k with the elements $\frac{(i+j)!}{i!j!}$, $i,j = 0,...,k-1$, and $C_1^{(k-1)}$ is the matrix of the order $k-1$ with the elements $\frac{(i+j)!}{i!j!}$, $i,j = 1,...,k-1$. From this we obtain the relation

$$\frac{\partial}{\partial C_{00}} \det C^{(k)} = 2bk \det C^{(k)} \left.\right|_{C_{00}=(2b)^{-1}}. \qquad (3.5)$$

Now, taking into account Theorem 1 and (3.2)–(3.5), we obtain asymptotic formula for the solution:

$$u(x,t) = -2\frac{d^2}{d\xi^2}\ln\left[1+\sum_{k=1}^{N}\det C^{(k)}\det\Gamma^{(k)}\frac{e^{-2bk\xi}}{t^{k(k+\beta)}}\right]_{\xi=x-4b^2t} + O\left(\frac{1}{\sqrt{t}}\right) \quad (3.6)$$

in the domain $\Omega_N^+(t) = \left\{x > 4b^2t - \frac{1}{2b}\ln t^{[N+1+\beta]}\right\}$ as $t \to \infty$.

Introduce the notations:

$$\Delta_N(\xi,t) = 1+\sum_{k=1}^{N}P_k\frac{e^{-2bk\xi}}{t^{k(k+\beta)}}\,, \quad (3.7)$$

$$P_k = \det C^{(k)}\det\Gamma^{(k)} = \frac{\varphi_0^k(b)(b-a)^{2\alpha}\Delta_1^{(k)}\Delta_2^{(k)}}{\prod\limits_{i=0}^{k-1}(i!)^2 b^{k(3k+2\beta)}2^{k(5k+4\beta)}}\,, \quad (3.8)$$

where $\Delta_1^{(k)} > 0$ and $\Delta_2^{(k)} > 0$ are the determinants of the matrices of the order k with the entries $(i+j)!$ and $\Gamma(i+j+1+\beta)$, $i,j = 0,...,k-1$, respectively. (They are positive since both are Gramm's determinants.) Then from (3.6) and (3.7) follows that

$$u(x,t) \sim u_N(x,t) = -2\frac{d^2}{d\xi^2}\ln\Delta_N(\xi,t)\Big|_{\xi=x-4b^2t} = -\frac{\Delta_N''\Delta_N - (\Delta_N')^2}{\Delta_N^2} \quad (3.9)$$

and

$$\Delta_N''\Delta_N - (\Delta_N')^2 = 4b^2\sum_{i,k=0}^{N}\frac{(i-k)^2P_iP_ke^{-2(i+k)b\xi}}{t^{i(i+\beta)+k(k+\beta)}}. \quad (3.10)$$

Let us cover the domain $\xi > -(2b)^{-1}\ln t^{[N+1+\beta]}$ by the intervals

$$a_1(t) = \{-(2b)^{-1}\ln t^{2+\beta+\varepsilon} < \xi < \infty\},$$

$$a_n(t) = \{-(2b)^{-1}\ln t^{2n+\beta+\varepsilon} < \xi < -(2b)^{-1}\ln t^{2(n-1)+\beta-\varepsilon}\},$$

$$\cdots,$$

$$a_{\left[\frac{N+1}{2}\right]}(t) = \left\{-(2b)^{-1}\ln t^{[N+\beta+1]} < \xi < -(2b)^{-1}\ln t^{2\left[\frac{N-1}{2}\right]+\beta-\varepsilon}\right\}.$$

Taking into account (3.7) and (3.10), we obtain

$$\Delta_N^2 = \left[\frac{P_{n-1}e^{-2(n-1)b\xi}}{t^{(n-1)(n-1+\beta)}} + \frac{P_ne^{-2nb\xi}}{t^{n(n+\beta)}}\right]\left(1+O\left(t^{-1/2}\right)\right)$$

and

$$\Delta_N''\Delta_N - (\Delta_N')^2 = 4b^2 \frac{2P_n P_{n-1} e^{-2(2n-1)b\xi}}{t^{n(n+\beta)+(n-1)(n-1+\beta)}} \left(1 + O\left(t^{-1/2}\right)\right),$$

when $\xi \in a_n(t)$ and $t \to +\infty$. Hence, by virtue of (3.6) and (3.9), it follows that

$$u(x,t) = -\sum_{n=1}^{\left[\frac{N+1}{2}\right]} \frac{2b^2}{\cosh^2\left[b\left(x - 4b^2 t + \frac{1}{2b}\ln t^{2n-1+\beta} + x_n^0\right)\right]} + O\left(t^{-1/2}\right)$$

uniformly with respect to $\xi = x - 4b^2 t \in a_n(t)$, where $x_n^0 = \frac{1}{2b}\ln\frac{P_n}{P_{n-1}}$. Thus together with (3.8) we obtain the required asymptotics of the solution in the domain $\Omega_N(t)$. Since $\left[\frac{N+1}{2}\right]$ solitons are confined in $\Omega_N(t)$, and the integral of each of them convergs, we simultaneously obtain the inequality (2.22). This concludes the proof. ■

4. Asymptotic behaviour of solutions as $t \to -\infty$

The method developed in the previous sections allows us to study the asymptotic behaviour of the solution $u(x,t)$ (determined everywhere in $\mathbf{R}^2$ by (1.1) and (1.2)) in the domains

$$\Omega_N^-(t) = \left\{x \in \mathbf{R}^1 : \quad x > 4a^2 t - \frac{[N+1+\alpha]}{2a}\ln|t|\right\}, \qquad t < -1,$$

as $t \to -\infty$. We call these domains neighbourhoods of the trailing edge (back of the solution), since $u(x,t)$ exponentially vanish as $x \to +\infty$ and the graph of $u(x,t)$ is moving to the left as $t \to -\infty$. It turns out that $u(x,t)$ also splits into $\left[\frac{N+1}{2}\right]$ asymptotic solitons in the domain Ω_N^- as $t \to -\infty$. These solitons have the amplitude $2a^2 < 2b^2$ and move to the left. The exact result is contained in the following

Theorem 3. *The solution $u(x,t)$ of the KdV equation has the following asymptotic representation:*

$$u(x,t) = -\sum_{k=1}^{\left[\frac{N+1}{2}\right]} 2a^2 \cosh^2\left[a\left(x - 4a^2 t + \frac{1}{2a}\ln|t|^{2n-1+\alpha} + x_n^0\right)\right]$$

in the domains $\Omega_N^-(t)$ as $t \to -\infty$. Here x_n^0 are constant phases, which are determined by

$$x_n^0 = \frac{1}{2a}\ln\frac{[(n-1)!]^2\,\Delta_1^{(n-1)}\Delta_2^{(n-1)} a^{6n-3+2\alpha} 2^{10n-5+4\alpha}}{\varphi_0(a)(b-a)^{\beta}\Delta_1^{(n)}\Delta_2^{(n)}},$$

where $\Delta_1^{(n)} > 0$ and $\Delta_2^{(n)} > 0$ are the same determinants as in Theorem 2.

Let us outline the key points of the proof. For simplicity we set $a < b < 2a$. Using expansion into a series

$$\frac{1}{\lambda+\mu} = \sum_{i,j=0}^{\infty} \frac{(i+j)!(-1)^{i+j}}{i!j!(2a)^{i+j+1}} (\lambda-a)^i (\mu-a)^j,$$

which converges absolutely and uniformly as $a \le \lambda$, $\mu \le b$, we approximate the operator A (see (2.1)) by an integral operator A_N with a degenerate kernel. We obtain estimates for the norm of the operator $B_N = A - A_N$, which are similar to those given by (2.6) and (2.7):

$$||B_N|| \le \left(\frac{b-a}{a}\right)^{\left[\frac{N+1}{2}\right]} \frac{(b-a)^\beta b}{2a(2a-b)} \hat{\varphi}_0 \quad \text{at } \xi > -\frac{[N+1+\alpha]}{2a} \ln\sqrt{\frac{a}{b-a}},\ t < -1,$$

and

$$||B_N|| \le \frac{1}{2a(2a-b)} \sum_{k+j=2(N+1)} C_{kj} I_k(x,t) I_j(x,t) \le C|t|^{-1}$$

$$\text{at} \quad -\frac{[N+1+\alpha]}{2a} \ln\sqrt{\frac{a}{b-a}} > \xi > -\frac{[N+1+\alpha]}{2a} \ln|t|,\ \ t < -1,\ \ \xi = x - 4a^2 t,$$

where $I_k(x,t) = \int\limits_a^b e^{-2\mu(x-4\mu^2 t)} (\mu-a)^k \varphi(\mu) d\mu$.

To estimate the integrals $I_k(x,t)$, we prove an analogue of Lemma 2

Lemma 3. *The integrals $I_k(x,t)$ have the following asymptotic representation*

$$I_k(x,t) = \frac{\varphi_1 \nu_1^{k+\alpha+1} \Gamma(k+\alpha+1)}{|t|^{k+\alpha+1}} e^{-2\alpha\xi} \left(1 + O\left(\frac{1}{\sqrt{|t|}}\right)\right)$$

as $t \to -\infty$ and $\xi = x - 4a^2 t > -\sqrt{|t|}$, where $\varphi_1 = \varphi_0(a)(b-a)^\beta$ and $\nu_1 = \left(\frac{1}{4a}\right)^2$.

Thus the problem is reduced to the solution of an integral equation with the degenerate kernel

$$A_N(\lambda,\mu;x,t) = \sum_{i,j=0}^{N} C'_{ij} e^{-(\lambda+\mu)x + 4(\lambda^3+\mu^3)t} (\lambda-a)^i (\mu-a)^j \varphi(\mu),$$

where

$$C'_{ij} = \frac{(i+j)!(-1)^{i+j}}{i!j!(2a)^{i+j+1}}.$$

After solving this equation the asymptotic behaviour of the solution at $t \to -\infty$ can be investigated in a same way as in the previous case. As the result we obtain the asymptotic formula of Theorem 3.

References

[1] *V.D. Ermakova*, An inverse scattering problem for one-dimension Schrödinger equation with nondecaying potential and its application to the KdV equation integration, Ph.D. Thesis, Kharkov State University (1983) (in Russian).

[2] *D.K. Faddeev and I.S. Sominsky*, Sbornik zadach po vysshej algebre. Nauka, Moskow (1977) (in Russian).

[3] *E.Ya. Khruslov*, Asymptotics of the solution of the Cauchy problem for the Korteweg–de Vries equation with step-like initial data. — Mat. sb. (1976), v. 99(141), No. 2, p. 261–281 (in Russian).

[4] *E.Ya. Khruslov and V.P. Kotlyarov*, Soliton asymptotics of nondecreasing solutions of nonlinear completely integrable evolution equations. — Adv. Soviet Math. (1994), v. 19, p. 129–180.

[5] *V.A. Marchenko*, Nonlinear equations and operator algebras. D. Reidel, Dordrecht (1987).

[6] *V.A. Marchenko and D.Sh Lundina*, Limits of n-soliton solutions of nonlinear Schrödinger equation. — Dokl. Akad. Nauk Ukraine (1992), No. 8, p. 21–24 (in Russian).

[7] *V.A. Marchenko and D.Sh Lundina*, Limits of reflectionless Dirac operators. — Adv. Soviet Math. Operator Spectral Theory and Related Topics. Ed. by V.A. Marchenko. Amer. Math. Soc., Providence, RI (1994).

[8] *H. Stephan*, Inverse problems for non–decreasing potentials. — Teor. Funktsii, Funktsional. Analiz i Prilozhen. (1986), v. 45, p. 123–132 (in Russian).

[9] *Sh. Tanaka*, Korteweg–de Vries equation asymptotic behaviour of solitons. — Publ. Res. Inst. Math Sci. (1975), v. 10, pp. 367–369.

[10] *V.E. Zakharov, S.V. Manakov, S.P. Novikov, and L.P. Pitaevsky*, Soliton theory. Nauka, Moscow (1980) (in Russian)

Распад некоторых нелокализованных решений уравнения Кортевега–де Фриза на солитоны

Е. Я. Хруслов, Х. Стефан

Изучается асимптотическое поведение нелокализованных решений уравнения Кортевега–де Фриза при больших временах и доказывается, что эти решения распадаются в бесконечную последовательность солитонов.

Розпад деяких нелокалізованих розв'язків рівнянння Кортевега–де Фріза на солітони

Є. Я. Хруслов, Х. Стефан

Вивчається асимптотична поведінка нелокалізованих розв'язків рівняння Кортевега–де Фріза за великим часом і доводиться, що ці розв'язки розпадаються у нескінченну послідовність солітонів.