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A result on polynomials and its relation to another, concerning entire functions of exponential type

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The Beurling–Malliavin multiplier theorem is deduced from the first result stated in the introduction, on polynomials. Work is largely based on de Branges' description of the extremal annihilating measures corresponding to certain spaces of bounded functions generated by weighted imaginary exponentials.

To Lars Hedberg, for his sixtieth birthday.

Introduction

The following two results have been known since the 1960's.

I. *There are numerical constants* $\eta_0 > 0$ *and* k *such that, for any polynomial* $P(z)$ *with*

$$\sum_{-\infty}^{\infty} \frac{\log^+ |P(n)|}{1+n^2} = \eta < \eta_0$$

one has, in the whole complex plane,

$$|P(z)| \le C_\eta e^{k\eta|z|},$$

where C_η *depends only on* η.

II. *Let* $W(x) \ge 1$ *be a function defined on* $\mathbb{R}$, *with either*
a) $\log W(x)$ *uniformly* Lip 1 *there,*

or

b) $W(x) = |F(x)|$ *for some entire function* $F(z)$ *of exponential type, and suppose that*

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx < \infty.$$

Then there are entire functions $\varphi(z) \not\equiv 0$ *of arbitrarily small exponential type making* $\varphi(x)W(x)$ *bounded on* $\mathbb{R}$.

The second result is the celebrated *theorem on the multiplier* due to Beurling and Malliavin, and we propose here to deduce it from the first one. That has indeed already been done in an earlier paper published in this journal ([1]), but I am no longer satisfied with the procedure followed there. Essential use was made in that paper of a lemma on Poisson integrals going back to Beurling, and we know now (see [2]) that II can be obtained *directly* from the lemma, without appeal to I. Here the passage from I to II will be effected without resorting to Beurling's lemma; in place of the latter a result of de Branges will play a key rôle. (Note: Although [1] was published after [2], the work described in it was done earlier, at the end of 1991. A badly corrupted version of [1] appeared in 1994, in a different journal.)

1. Result I is readily extended to entire functions of *small* exponential type.

Theorem 1 *Let* $f(z)$ *be entire, of exponential type* α, *and such that*

$$\sum_{-\infty}^{\infty} \frac{\log^+ |f(n)|}{1+n^2} = \eta < \infty.$$

Provided that α *and* η *are both less than a certain numerical constant* $c_0 > 0$ *we have, for all* z,

$$|f(z)| \le C_{\alpha,\eta} e^{\kappa(\alpha+\eta)|z|}.$$

Here, $C_{\alpha,\eta}$ *depends only on* α *and* η, *and* κ *is a numerical constant.*

Since a proof of this is given in § 1 of [1], we merely indicate its main steps.

One first considers even entire functions $f(z)$ of exponential type α given in the form

$$f(z) = \prod_k \left(1 - \frac{z^2}{\lambda_k^2}\right),$$

where the λ_k are real and > 0. Taking the *polynomials*

$$P_N(z) = \prod_{\lambda_k \le N} \left(1 - \frac{z^2}{\lambda_k^2}\right),$$

one shows by calculation that

$$\sum_{-\infty}^{\infty} \frac{\log^+ |P_N(n)|}{1+n^2} \le \sum_{-\infty}^{\infty} \frac{\log^+ |f(n)|}{1+n^2} + C\alpha + o(1)$$

with a numerical constant C, and $o(1)$ term tending to 0 as $N \to \infty$. When $\eta + C\alpha < \eta_0$, the constant figuring in I, that result holds for the $P_N(z)$ with N large, and yields our theorem for functions $f(z)$ of the above form. Passage from these to *general* entire functions $f(z)$ having the properties in question is standard; see [1].

Corollary 1 *Let $W(n) \ge 1$ be a function defined on $\mathbb{Z}$ with $W(n) \to \infty$ for $n \to \pm\infty$ and*

$$\sum_{-\infty}^{\infty} \frac{\log W(n)}{1+n^2} < \infty.$$

Then, for an $h > 0$ independent of W, finite linear combinations of the functions $e^{i\lambda n}/W(n)$, $-h \le \lambda \le h$, are not uniformly dense in $\mathcal{C}_0(\mathbb{Z})$.

For the proof, see p. 219 and then p. 218 of [1]. It is not explicitly stated there that h can be taken independent of W, but that clearly follows from the arguments given – any value $< c_0$, the constant in Theorem 1, will work.

Corollary 2 *If $g(z)$, with $|g(x)| \ge 1$ on $\mathbb{R}$, is entire and of exponential type $< c_0$, the constant appearing in Theorem 1, and if*

$$\int_{-\infty}^{\infty} \frac{\log |g(x)|}{1+x^2} dx < \infty, \tag{1}$$

there is an entire function $\varphi(z) \not\equiv 0$ of exponential type $< \pi$ with $\varphi(x)g(x)$ bounded on $\mathbb{R}$.

A somewhat elaborate proof of this corollary was given in § 2 of [1]; here is a shorter argument.

From (1) it follows that

$$\sum_{-\infty}^{\infty} \frac{\log |g(n)|}{1+n^2} < \infty$$

(see p. 217 of [1] or [3], Ch.VIII, § B.11); we can thus apply Corollary 1 with $W(n) = (1+n^2)|g(n)|$, and from its conclusion we arrive, *by duality*, at the existence of a complex sequence $\{\gamma_n\}$, *not identically zero*, with $\sum_{-\infty}^{\infty} |\gamma_n| < \infty$ and

$$\sum_{-\infty}^{\infty} \gamma_n \frac{e^{i\lambda n}}{W(n)} = 0 \quad \text{for } -h \le \lambda \le h,$$

i.e.,

$$\sum_{-\infty}^{\infty} (-1)^n \gamma_n \frac{e^{int}}{W(n)} = 0, \quad \pi - h \le |t| \le \pi.$$

For h we may take *any* number $< c_0$, as mentioned above.

Denote by $\Phi(t)$ the last left-hand sum and put

$$\varphi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-izt} \Phi(t) dt;$$

then $\varphi(z)$ is $\not\equiv 0$, entire, of exponential type $\le \pi - h$ (sic!), and we have

$$\varphi(n) = (-1)^n \frac{\gamma_n}{W(n)}, \quad n \in \mathbb{Z}.$$

The product $\varphi(n)|g(n)| = (-1)^n \gamma_n/(1+n^2)$ is in particular bounded on $\mathbb{Z}$.

Suppose now that $g(z)$ has exponential type $< h$; then the product $\varphi(z)g(z)$ will be entire and of exponential type $< \pi$. Being bounded on $\mathbb{Z}$, that product must then be bounded on $\mathbb{R}$ according to a well known theorem of Miss Cartwright ([4], p. 180).

2. During the remainder of this paper we shall frequently have to deal with entire functions of (sometimes unrestricted) exponential type enjoying the other properties required of $g(z)$ in Corollary 2 to Theorem 1. Quite generally, the entire functions $f(z)$ of exponential type with

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2} dx < \infty$$

are said to belong to the *Cartwright class*, and that term will be used from now on. Various well known results about such functions will be invoked where needed, often without specific reference. Most of them are established in Chapters III and VI of [3].

Lemma *If $g(z)$, of Cartwright class and with $|g(x)| \geq 1$ on $\mathbb{R}$, is of sufficiently small exponential type, there is an entire function $F(z)$, also of Cartwright class but having only simple zeros, all in $\mathbb{Z}$, such that*

$$|g(x)| \leq |F(x+i)| \quad \text{for } x \in \mathbb{R}.$$

Proof There is no loss of generality in supposing that $|g(x)| \to \infty$ for $x \to \pm\infty$; otherwise replace $g(z)$ by $(z+i)g(z)$. That being the case, we see as in the proof of Corollary 2 to Theorem 1 that Corollary 1 of that result is applicable to $W(n) = |g(n)|$. Finite linear combinations of the $e^{i\lambda n}/|g(n)|$, $-h \leq \lambda \leq h$, are therefore *not* uniformly dense in $\mathcal{C}_0(\mathbb{Z})$, where h is a certain number > 0. This puts a theorem of de Branges at our disposal.

That result (see Ch. VI, § F.2 in [3]) furnishes an entire function $F(z)$ of exponential type h and Cartwright class whose zeros, all simple, lie in $\mathbb{Z}$ and for which

$$\sum_{\nu\in\Lambda} \left| \frac{g(\nu)}{F'(\nu)} \right| \; < \; \infty,$$

Λ denoting the set of those zeros.

More information about the growth of $F(z)$ is obtained in the course of proving de Branges' theorem. Then it is seen that

$$\frac{e^{i\lambda z}}{F(z)} \; = \; \sum_{\nu\in\Lambda} \frac{e^{i\lambda\nu}}{F'(\nu)(z-\nu)} \quad \text{for } -h \leq \lambda \leq h,$$

and from this and the preceding relation it readily follows (on taking $\lambda = \pm h$) that

$$|F(re^{i\vartheta})| \geq e^{hr|\sin\vartheta|} \tag{2}$$

for large enough r (depending on ϑ), as long as $\vartheta \neq 0, \; \pi \pmod{2\pi}$.

Assume now that $g(z)$ is of exponential type $< h$. Then it is claimed that

$$g(z) \; = \; F(z) \sum_{\nu\in\Lambda} \frac{g(\nu)}{F'(\nu)(z-\nu)}. \tag{3}$$

To see this, observe that the right side represents an entire function of exponential type – the series converges absolutely and uniformly when $\mathrm{dist}(z, \mathbb{Z}) \geq 1/4$. That makes the *difference* of the two sides – *call it $f(z)$* – *also* entire and of exponential type. For $\nu \in \Lambda$ we have $f(\nu) = 0$, so the *ratio* $f(z)/F(z)$ is *entire*, and indeed of exponential type by a theorem of Lindelöf ([3], Ch. III, § B).

Look at the relation

$$\frac{f(z)}{F(z)} \; = \; \frac{g(z)}{F(z)} \; - \; \sum_{\nu\in\Lambda} \frac{g(\nu)}{F'(\nu)(z-\nu)}.$$

Since $g(z)$ is of exponential type $< h$, we see from (2) that the *first term* on the right goes to zero as $z \to \infty$ along each of the 4 rays $\arg z = \pm\frac{\pi}{2} \pm \delta$, $\delta > 0$ being sufficiently small. The *second term* on the right clearly has the same behaviour (dominated convergence). So $f(z)/F(z)$ is bounded on each of those 4 rays, hence (by Phragmén–Lindelöf) bounded in the complex plane and thus finally identically zero. This proves (3).

From (3) we get, for $x \in \mathbb{R}$,

$$|g(x+i)| \;\leq\; |F(x+i)| \sum_{\nu \in \Lambda} \left| \frac{g(\nu)}{F'(\nu)} \right|,$$

with the sum on the right equal, without loss of generality, to 1. Here, since $g(z)$ is of Cartwright class, we may just as well take it to have all its zeros in the lower half plane, and *then* (after multiplication of $g(z)$ by an imaginary exponential, should that be necessary) $|g(x+iy)|$ will be an increasing function of y for $y \geq 0$ (see [4], p. 226). Thus,

$$|g(x)| \leq |g(x+i)| \leq |F(x+i)| \quad \text{for } x \in \mathbb{R},$$

and we are done.

Theorem 2 *Corresponding to any entire function $g(z)$ of Cartwright class and sufficiently small exponential type with $|g(x)| \geq 1$ on $\mathbb{R}$ there is a function $\omega(x) \geq 0$ defined there such that*

$$|\omega(x) - \omega(x')| \leq M|x - x'|, \quad x,\ x' \in \mathbb{R},$$

$$|g(x)| \;\leq\; e^{\omega(x)}, \quad x \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2} dx \;<\; \infty.$$

Here, M is a numerical constant.

Remark The result holds as long as $g(z)$ is of exponential type $< c_0$, the constant figuring in Theorem 1.

Proof of theorem (cf. [1], § 4). According to the lemma, if the exponential type of $g(z)$ is small enough, an entire function $F(z)$ with the properties enumerated there is available. For the entire function

$$G(z) = F(z+i), \tag{4}$$

like $F(z)$, *of Cartwright class*, we have in particular

$$1 \leq |g(x)| \leq |G(x)|, \quad x \in \mathbb{R},$$

and it suffices to exhibit a majorant $\omega(x)$ of $\log|G(x)|$ with the required properties.

Start by fixing a number $M > 0$ (in a way to be described presently), and form the open subset

$$\mathcal{O}_+ = \{x \in \mathbb{R}; \quad \log|G(\xi)| - \log|G(x)| > M(\xi - x) \text{ for some } \xi > x\} \qquad (5)$$

of $\mathbb{R}$. $\mathcal{O}_+$ is a countable union of certain disjoint open intervals, and it is first necessary to verify that none of those can be unbounded. Since that point was dispatched somewhat hastily in [1] (and in [2] !), we treat it more carefully here.

Take any component (a, b) of $\mathcal{O}_+$, $-\infty \leq a < b \leq \infty$. For $x \in (a, b)$ we denote by ξ_x the *supremum* of the ξ corresponding to it as in the right side of (5). Then $\xi_x \geq b$. Indeed, if that were not so, ξ_x would belong to (a, b), and there would be a $\xi' > \xi_x$ with $\log|G(\xi')| - \log|G(\xi_x)| > M(\xi' - \xi_x)$. At the same time ($\xi_x$ being, in the present circumstances, *finite*), we would have $\log|G(\xi_x)| - \log|G(x)| \geq M(\xi_x - x)$, making $\log|G(\xi')| - \log|G(x)| > M(\xi' - x)$ with $\xi' > \xi_x$, contradicting the specification of ξ_x.

Knowing that $\xi_x \geq b$ for $x \in (a, b)$, it is easy to see that b cannot be infinite. Otherwise, for fixed $x \in (a, b)$, we would have a sequence of ξ's tending to ∞ for which the relations on the right side of (5) hold. With these ξ's, we would eventually have

$$\log|G(\xi)| \geq M\xi/2,$$

contradicting the known fact that functions of Cartwright class have zero exponential growth along the real axis ([3], Ch. VI, §E.2; [4], p. 97).

Finally, a cannot be $-\infty$. If that were so, we would have a sequence $x_n \to -\infty$ and (since $\xi_{x_n} \geq b$) corresponding $\xi_n \geq b - 1$ with

$$\log|G(\xi_n)| - \log|G(x_n)| > M(\xi_n - x_n).$$

Since $\log|G(\xi_n)|$ is either bounded above (if the ξ_n are) or, *at worst*, $o(\xi_n)$ (should the ξ_n be unbounded – see above), this would make $\log|G(x_n)| \longrightarrow -\infty$, contradicting the relation $|G(x)| \geq 1$.

We can thus write

$$\mathcal{O}_+ = \bigcup_k (a_k, b_k)$$

with the (a_k, b_k) of finite length and disjoint; these intervals are obtained by the familiar construction of F. Riesz shown in figure 1.

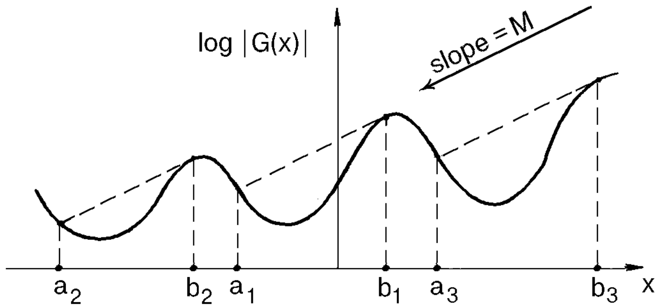

figure 1

If $x \in \mathcal{O}_+$ with, say, $a_k < x < b_k$, we have

$$\log|G(x)| < \log|G(b_k)| - M(b_k - x); \tag{6}$$

for $x = a_k$ and $x = b_k$ this relation goes over to an *equality*. On the other hand, when $x \notin \mathcal{O}_+$,

$$\log|G(x')| - \log|G(x)| \le M(x' - x)$$

for all $x' > x$, according to (5).

Put

$$\omega_+(x) = \begin{cases} \log|G(x)| & \text{for } x \notin \mathcal{O}_+, \\ \log|G(b_k)| - M(b_k - x) & \text{if } a_k < x < b_k. \end{cases}$$

Then $\omega_+(x)$ is continuous, and

$$\log|G(x)| \le \omega_+(x) \quad \text{for } x \in \mathbb{R}$$

by (6).

But it is also claimed that

$$\omega_+(x) \le 3\log|G(x)|, \quad x \in \mathbb{R},$$

provided that our numerical constant M is chosen suitably. Since that is obvious for $x \notin \mathcal{O}_+$, we need only look at the x belonging to the intervals (a_k, b_k).

For that purpose, we consider the Hadamard factorization of $F(z)$ (related to $G(z)$ by (4)), which, here, takes the form

$$F(z) = ce^{az}z^l \prod_{\substack{\nu\in\Lambda \\ \nu\ne 0}} \left(1 - \frac{z}{\nu}\right) e^{z/\nu}, \tag{7}$$

with $l = 0$ or 1 and $\Lambda \subseteq \mathbb{Z}$. Moreover, *a is real* in the present circumstances. That follows from (2) and the fact that $F(z)$ is of exponential type h. These two properties, taken together, and the particular form of (7) imply that $|e^{aiy}|$ has the same behaviour for $y \to +\infty$ as for $y \to -\infty$, which can only happen when $a \in \mathbb{R}$. That being the case, logarithmic differentiation of (7) yields

$$\frac{\partial \log |F(x+iy)|}{\partial y} \;=\; \sum_{\nu\in\Lambda} \frac{y}{(x-\nu)^2+y^2} \;. \tag{8}$$

The functions $|F(x+iy)|$ and $|G(x+iy)| = |F(x+iy+i)|$ are, in particular, *increasing with* y when $y > 0$.

Let, now, x lie in one of the intervals (a_k, b_k); the idea is to move from b_k to x along the path shown in figure 2.

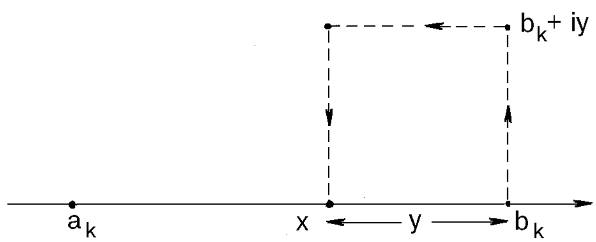

figure 2

Take

$$y = b_k - x,$$

and observe first of all that

$$|G(b_k+iy)| \ge |G(b_k)| \tag{9}$$

(see above). Again, $\log |G(z)| \ge \log |G(\operatorname{Re} z)| \ge 0$ for $\operatorname{Im} z > 0$, so, $\log |G(z)|$ being harmonic there, we get, by Harnack,

$$\log |G(x+iy)| = \log |G(b_k - y + iy)| \ge \frac{1}{3} \log |G(b_k+iy)|.$$

Referring to (9), we see that

$$\log |G(x+iy)| \ge \frac{1}{3} \log |G(b_k)| \;. \tag{10}$$

From (4) and (8) we now have, for $\eta > 0$,

$$\frac{\partial \log |G(x+i\eta)|}{\partial \eta} \;=\; \sum_{\nu\in\Lambda} \frac{\eta+1}{(x-\nu)^2+(\eta+1)^2} \,,$$

with the sum on the right

$$\leq \sum_{-\infty}^{\infty} \frac{\eta+1}{(x-n)^2+(\eta+1)^2}$$

because $\Lambda \subseteq \mathbb{Z}$. The last sum is evidently bounded by a *numerical* constant C when $\eta > 0$ (comparison with $\int_{-\infty}^{\infty}((\eta+1)/((x-t)^2+(\eta+1)^2))dt$), so finally

$$\frac{\partial \log|G(x+i\eta)|}{\partial \eta} \leq C, \quad \eta > 0. \tag{11}$$

Thence, $\log|G(x)| \geq \log|G(x+iy)| - Cy$, so by (10),

$$\log|G(x)| \ \geq \ \frac{1}{3}(\log|G(b_k)| - 3C(b_k - x)).$$

Take now $M = 3C$. According to the above definition of $\omega_+(x)$, that makes the right side of the last relation equal to $\frac{1}{3}\omega_+(x)$, since $a_k < x < b_k$. This shows that

$$\omega_+(x) \leq 3\log|G(x)|$$

for $x \in \mathcal{O}_+$, and hence for all $x \in \mathbb{R}$, as we had claimed. An immediate consequence is that

$$\int_{-\infty}^{\infty} \frac{\omega_+(x)}{1+x^2}dx \leq 3\int_{-\infty}^{\infty} \frac{\log|G(x)|}{1+x^2}dx < \infty, \tag{12}$$

$G(z)$ being of Cartwright class.

Almost half of what is needed in the proof of our theorem is now in our possession. Fortunately, a good part of what remains follows from the work already done, and we do not need to go into much detail.

A repetition of the construction made to obtain $\omega_+(x)$ with, however, $-x$ playing the role of x, yields a second majorant $\omega_-(x)$ of $\log|G(x)|$. Here, the procedure is to form the open subset

$$\mathcal{O}_- = \{x \in \mathbb{R}; \quad \log|G(\xi)| - \log|G(x)| > M(x-\xi) \text{ for some } \xi < x\}$$

of $\mathbb{R}$, check (by an argument like one already made) that $\mathcal{O}_-$ is the union of certain disjoint bounded intervals (α_k, β_k), and then put

$$\omega_-(x) = \begin{cases} \log|G(x)|, & x \notin \mathcal{O}_-, \\ \\ \log|G(\alpha_k)| - M(x-\alpha_k) & \text{if } \alpha_k < x < \beta_k. \end{cases}$$

The function $\omega_-(x)$ has properties analogous (and symmetric) to those of $\omega_+(x)$; some of them are apparent from figure 3. They can be verified most rapidly by simply invoking the results already obtained for $\omega_+(x)$ after replacing $G(z)$ by the entire function $\overline{G(-\bar{z})}$. In particular, we have

$$\int_{-\infty}^{\infty} \frac{\omega_-(x)}{1+x^2} dx \leq 3 \int_{-\infty}^{\infty} \frac{\log|G(x)|}{1+x^2} dx < \infty,$$

provided that M is taken equal to $3C$.

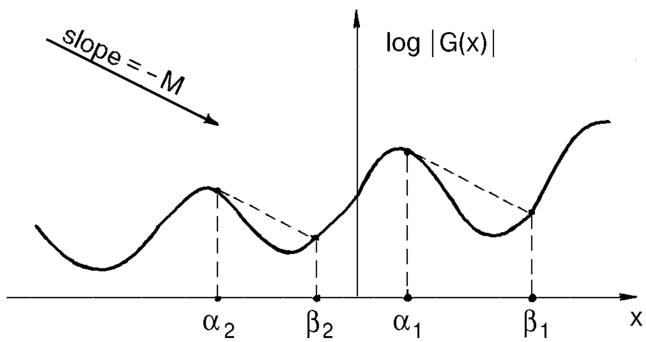

figure 3

It now remains only to put

$$\omega(x) = \max(\omega_+(x),\ \omega_-(x)).$$

This function is continuous, and clearly a *majorant* of $\log|G(x)| \geq \log|g(x)|$. Since $\omega(x) \leq \omega_+(x) + \omega_-(x)$, we have, by the last inequality and (12),

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2} dx < \infty.$$

Finally,

$$|\omega(x') - \omega(x)| \leq M(x' - x) \quad \text{for} \quad x,\ x' \in \mathbb{R}.$$

This property, which seems evident when one imagines the superposition of figures 1 and 3, is carefully verified in § 4 of [1], but the procedure adopted there cannot be used in the circumstances of Theorem 3, to be given below. That is why we include here the following argument which is always valid.

We have, then, to prove that

$$-M(x' - x) \leq \omega(x') - \omega(x) \leq M(x' - x) \quad \text{for } x' > x,$$

and consider first the *right-hand* inequality. That relation is implied by a *local version* of it asserting the existence, for each $x \in \mathbb{R}$, of a $\delta_x > 0$ such that

$$\omega(x') - \omega(x) \le M(x' - x) \quad \text{for } x < x' < x + \delta_x,$$

and we proceed to verify this.

Given $x \in \mathbb{R}$ we have *two* possibilities: $\omega(x) > \log|G(x)|$ or $\omega(x) = \log|G(x)|$.

If the *first* is realized, one of *three things* can happen: either $\omega_+(x) > \omega_-(x)$ or $\omega_-(x) > \omega_+(x)$ or, finally, $\omega_+(x) = \omega_-(x) > \log|G(x)|$. In the first case, $\omega_+(x') > \omega_-(x')$ for the x' belonging to a certain interval $(x - \delta_x,\ x + \delta_x)$, $\delta_x > 0$. Then $\omega(x') = \omega_+(x')$ for such x', making $\omega(x') - \omega(x) = M(x' - x)$ for $x < x' < x + \delta_x$. In the second case, $\omega_-(x') > \omega_+(x')$ for $x - \delta_x < x' < x + \delta_x$ where $\delta_x > 0$, and we find that $\omega(x') - \omega(x) = -M(x' - x)$, $x < x' < x + \delta_x$. In the third case, there is a $\delta_x > 0$ with $\omega(x') = \omega_-(x')$ for $x - \delta_x < x' \le x$ and $\omega(x') = \omega_+(x')$ for $x \le x' < x + \delta_x$; here we again have $\omega(x') - \omega(x) = M(x' - x)$ for $x < x' < x + \delta_x$. The local relation in question thus *holds* when $\omega(x) > \log|G(x)|$.

There is still the possibility that $\omega(x) = \log|G(x)|$. When it is realized, $x \notin \mathcal{O}_+ \cup \mathcal{O}_-$, so in particular,

$$\log|G(\xi)| - \log|G(x)| \le M(\xi - x) \quad \text{for } \xi > x, \tag{13}$$

by (5). If $x' > x$ and $\omega(x') = \log|G(x')|$, this already implies our desired relation (*without* the local restriction on x').

Suppose, then, that $x' > x$ and $\omega(x') > \log|G(x')|$. In that event we have *two* cases: $\omega(x') = \omega_+(x')$ or $\omega(x') = \omega_-(x')$. In the *first* of these, $x' \in \mathcal{O}_+$ and hence x' belongs to one of the components (a_k, b_k) of that set. Being in $\mathbb{R} \sim \mathcal{O}_+$, the point $x < x'$ cannot lie in that component, so $x \le a_k$ and by (13),

$$\log|G(a_k)| - \log|G(x)| \le M(a_k - x).$$

At the same time, $\omega_+(x') - \log|G(a_k)| = M(x' - a_k)$, making $\omega(x') - \omega(x) = \omega_+(x') - \log|G(x)| \le M(x' - x)$. In the *second* case, x' lies in one of the components (α_k, β_k) of $\mathcal{O}_-$, and $x \le \alpha_k$. Thence, by (13),

$$\log|G(\alpha_k)| - \log|G(x)| \le M(\alpha_k - x),$$

whilst $\omega_-(x') - \log|G(\alpha_k)| = -M(x' - \alpha_k)$, so that $\omega(x') - \omega(x) = \omega_-(x') - \log|G(x)| < M(x' - x)$. The desired relation thus holds (again with $\delta_x = \infty$) in both cases.

We still have to establish the inequality $\omega(x') - \omega(x) \ge -M(x' - x)$ for $x' > x$, or, what comes to the same thing, that

$$\omega(x') - \omega(x) \le M(x' - x) \quad \text{for } x' < x.$$

This is done by repeating the above reasoning after making the change of variable $x \to -x$ (which causes $\mathcal{O}_+$ and $\mathcal{O}_-$ to exchange rôles).

Verification of the property $|\omega(x') - \omega(x)| \le M|x' - x|$ is now completed. Our theorem is thus proved.

Remarks The idea of basing the work in this § on de Branges' theorem came to me after I had read a recent paper by M. Sodin ([5]). The argument suggested by figure 2 has been used by H. Pedersen in his thesis ([6]).

3. We need two more results like Theorem 2. The first is rather straightforward.

Theorem 3 *Let $w(x) \ge 0$, defined on $\mathbb{R}$, have the properties that*

$$\int_{-\infty}^{\infty} \frac{w(x)}{1+x^2} dx \ < \ \infty \tag{14}$$

and that

$$|w(x) - w(x')| \le L|x - x'| \quad \text{for } x,\ x' \in \mathbb{R}, \tag{15}$$

where L is a constant > 0. Corresponding to any l, $0 < l < L$, there is then a function $\omega(x) \ge w(x)$ defined on $\mathbb{R}$ with

$$|\omega(x) - \omega(x')| \le l|x - x'|, \quad x,\ x' \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2} dx \ < \ \infty.$$

Proof We are going to make the construction used to prove Theorem 2, with $w(x)$ now playing the rôle of $\log|G(x)|$, and begin by forming a function $\omega_+(x) \ge w(x)$ corresponding to the slope $l < L$ instead of M.

For that purpose, we take the set

$$\mathcal{O}_+ = \{x \in \mathbb{R};\quad w(\xi) - w(x) > l(\xi - x) \text{ for some } \xi > x\}.$$

As usual, $\mathcal{O}_+$ is a disjoint union of certain intervals (a_k, b_k), and it is first necessary to ascertain that those are all of finite length. That will follow just as in the proof of Theorem 2 provided that we know $w(x)$ to be $o(x)$ for $x \to \infty$.

Suppose that this were *false* and that we had $w(x_0) \ge \delta x_0$ for some *arbitrarily large* values of x_0, where $0 < \delta < 2L$. Then, by (15), we would have $w(x_0) \ge \dfrac{\delta}{2} x_0$

for $\left(1-\frac{\delta}{2L}\right)x_0 \leq x \leq x_0$, making

$$\int_{(1-\frac{\delta}{2L})x_0}^{x_0} \frac{w(x)}{x^2}dx \ \geq \ \frac{\delta^2}{4L} \ .$$

That, however, is incompatible with (14) for values of x_0 tending to ∞.

We are thus assured that the components (a_k, b_k) of $\mathcal{O}_+$ are bounded, and can put

$$\omega_+(x) = w(b_k) - l(b_k - x) \quad \text{for } a_k < x < b_k. \tag{16}$$

Outside of $\mathcal{O}_+$, $\omega_+(x)$ is taken equal to $w(x)$.

Looking, now, at an interval (a_k, b_k) with b_k *large*, we proceed to estimate

$$\int_{a_k}^{b_k} \frac{\omega_+(x)}{x^2}dx$$

in terms of

$$\int_{a'_k}^{b_k} \frac{w(x)}{x^2}dx,$$

where

$$a'_k = b_k - \frac{1}{L}(w(b_k) - w(a_k)). \tag{17}$$

A glance at figure 1 shows (taking $M = l$) that $w(b_k) - w(a_k) = l(b_k - a_k)$, so, since $0 < l < L$, we have

$$a_k \ < \ a'_k \ < \ b_k. \tag{18}$$

For $a_k < x < b_k$, $\omega_+(x)$ is given by (16) whereas, from (15),

$$w(x) \geq w(b_k) - L(b_k - x). \tag{19}$$

We can thus compare the values assumed by $\omega_+(x)$ on (a_k, b_k) with those taken by $w(x)$ on the smaller interval (a'_k, b_k); the manner of doing that can be seen in figure 4.

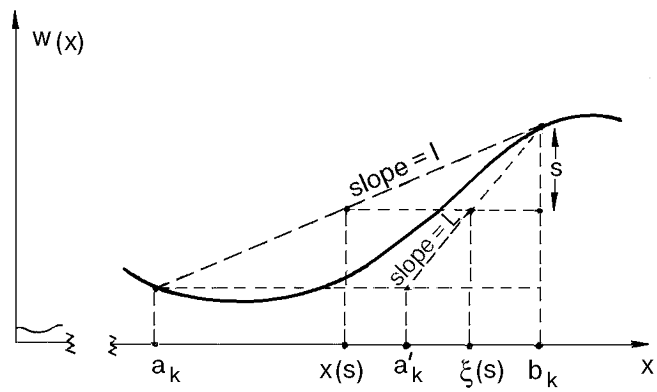

figure 4

It is convenient to introduce a new variable s ranging from 0 to $w(b_k) - w(a_k)$. Then, putting

$$x(s) = b_k - \frac{s}{l},$$

$$\xi(s) = b_k - \frac{s}{L},$$

we see from (16) and (19) that

$$w(\xi(s)) \geq w(b_k) - s = \omega_+(x(s)), \quad 0 \leq s \leq w(b_k) - w(a_k),$$

whence

$$\int_{a_k}^{b_k} \frac{\omega_+(x)}{x^2} dx \; = \; \frac{1}{l} \int_0^{w(b_k)-w(a_k)} \frac{\omega_+(x(s))}{(x(s))^2} ds \; \leq \; \frac{1}{l} \int_0^{w(b_k)-w(a_k)} \frac{w(\xi(s))}{(x(s))^2} ds. \tag{20}$$

For the values of s figuring in these integrals, we have, surely,

$$\frac{\xi(s)}{x(s)} \; < \; \frac{b_k - (w(b_k)/L)}{b_k - (w(b_k)/l)} \; .$$

Here, since we are taking b_k to be *large*, we can, as seen earlier, ensure that $w(b_k) \leq \delta b_k$ with $0 < \delta < l$. That will make

$$\frac{\xi(s)}{x(s)} \; < \; \frac{1 - (\delta/L)}{1 - (\delta/l)}$$

in (20), and the right-hand member of that relation will then be

$$\leq \left(\frac{l}{L}\right) \left(\frac{L - \delta}{l - \delta}\right)^2 \int_{a_k'}^{b_k} \frac{w(\xi)}{\xi^2} d\xi$$

by (17); this is the estimate spoken of above. Referring to (18) we get, for the (a_k, b_k) with b_k large,

$$\int_{a_k}^{b_k} \frac{\omega_+(x)}{x^2} dx \;\leq\; \left(\frac{l}{L}\right) \left(\frac{L-\delta}{l-\delta}\right)^2 \int_{a_k}^{b_k} \frac{w(x)}{x^2} dx \;.$$

Comparison of $\int_{a_k}^{b_k} (\omega_+(x)/x^2) dx$ with $\int_{a_k}^{b_k} (w(x)/x^2) dx$ for the intervals (a_k, b_k) with b_k *very negative* proceeds according to the same plan and is even easier. Finally, $\omega_+(x)$ is *bounded* on the intervals (a_k, b_k) lying in any finite portion of $\mathbb{R}$, and $\omega_+(x) = w(x)$ on $\mathbb{R} \sim \mathcal{O}_+$. Convergence of

$$\int_{-\infty}^{\infty} \frac{\omega_+(x)}{x^2} dx$$

thus follows from (14).

As in the proof of Theorem 2, our next step is to construct a majorant $\omega_-(x)$ of $w(x)$ with properties symmetric to those of $\omega_+(x)$, using, however, chords of slope $-\,l$ instead of $-\,M$ (see figure 3). A repetition of the arguments just made shows then that

$$\int_{-\infty}^{\infty} \frac{\omega_-(x)}{x^2} dx \;<\; \infty.$$

We now have all we need. It suffices (as for Theorem 2) to take

$$\omega(x) = \max(\omega_+(x),\ \omega_-(x)),$$

and the present proof is complete.

Theorem 4 *Let $\omega(x) \geq 0$, defined on $\mathbb{R}$, have the properties*

$$|\omega(x) - \omega(x')| \leq l|x - x'|, \quad x,\ x' \in \mathbb{R},$$

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2} dx \;<\; \infty.$$

Then there is an entire function $F(z)$ of Cartwright class and exponential type $\leq 2l$ with

$$|F(x)| \;\geq\; e^{\omega(x)} \quad \text{for } x \in \mathbb{R}.$$

This result is established by methods from the theory of weighted approximation going back to Akhiezer. The proof has been published at least 3 times: first in [7], then in [3] (Ch. X, § C.1), and more recently in [1].

4. Let us now put together Theorems 2, 3 and 4.

The first of these can be adapted to entire functions of Cartwright class and *unrestricted* exponential type. *Fix*, indeed, an $h > 0$ (any value $<$ the c_0 of Theorem 1 will do) such that Theorem 2 holds, *as it stands*, for the entire functions $g(z)$ of Cartwright class and exponential type $\leq h$. Then, *by a simple change of scale* we deduce from that result the following:

If $f(z)$, entire and of Cartwright class, is of exponential type A, with $|f(x)| \leq 1$ on $\mathbb{R}$, there is a function $w(x) \geq 0$ defined thereon such that

$$e^{w(x)} \;\geq\; |f(x)|, \quad x \in \mathbb{R},$$

$$|w(x) - w(x')| \leq M\frac{A}{h}|x - x'|, \quad x,\ x' \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} \frac{w(x)}{1+x^2}dx \;<\; \infty.$$

Here, M is the constant appearing in Theorem 2.

This observation and Theorem 3 lead without further ado to

Theorem 5 *Given the entire function $f(z)$ of Cartwright class with $|f(x)| \geq 1$ on $\mathbb{R}$ we have, for any $l > 0$, a function $\omega(x) \geq 0$ on $\mathbb{R}$ with*

$$e^{\omega(x)} \;\geq\; |f(x)|, \quad x \in \mathbb{R},$$

$$|\omega(x) - \omega(x')| \leq l|x - x'|, \quad x,\ x' \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2}dx \;<\; \infty.$$

Thence, by Theorem 4:

Theorem 6 *If $f(z)$ is as in Theorem 5 there are entire functions $F(z)$ of Cartwright class and arbitrarily small exponential type with $|F(x)| \geq |f(x)|$ on $\mathbb{R}$.*

5. Returning to the two results stated in the introduction, we proceed to deduce II from I.

In the first place, I implies IIb). Given an entire function $F(z)$ of Cartwright class with $|F(x)| \geq 1$ on $\mathbb{R}$, we take any large constant K and apply Theorem 6 to $f(z) = F(Kz)$ so as to get $g(z)$, of Cartwright class and exponential type $< c_0$ (the constant figuring in Theorem 1), with

$$|g(x)| \;\geq\; |F(Kx)|, \quad x \in \mathbb{R}.$$

By Corollary 2 to Theorem 1 there is then an entire function $\varphi(z) \not\equiv 0$ of exponential type $< \pi$ with $\varphi(x)g(x)$ bounded on $\mathbb{R}$. The entire function $\psi(z) = \varphi(z/K)$ is now $\not\equiv 0$ and of exponential type $< \pi/K$, and $\psi(x)F(x)$ is bounded on $\mathbb{R}$.

IIa) is deduced from IIb) by simple appeal to Theorem 4.

Discussion

We have thus derived II from I. H. Pedersen has recently shown, in [6], how to get I, not from II but from a more specific intermediate result, obtained in the course of one of the *proofs* of II and frequently looked on as a *special version* of the latter. From this point of view, I and II seem to be roughly equivalent.

Pedersen's procedure has the advantage of yielding not only I, but a considerable *refinement* of Theorem 1. The argument referred to in § 1 can only work for *very small* values of the exponential type α of $f(z)$, but Pedersen shows that Theorem 1 holds for $\alpha \leq 0.44$ (and, by a modification of his method, one can see that this is still the case for values < 2 of α). It would be very good if this could be extended to all values $< \pi$ of α (no upper limit $> \pi$ can work), but it is not clear at present how that might be done.

References

[1] *P. Koosis*, A relation between two results about entire functions of exponential type. — Mat. Fiz., Analiz, Geom. (1995), v. 2, No. 2, p. 212–231.

[2] *P. Koosis*, Construction of a certain superharmonic majorant. — Ann. Inst. Fourier (1994), v. 44, p. 729–766.

[3] *P. Koosis*, The Logarithmic Integral. Vols. I and II. Cambridge University Press, Cambridge (1988 and 1992).

[4] *R.P. Boas*, Entire Functions. Academic Press, New York (1954).

[5] *M. Sodin*, Which perturbations of quasianalytic weights preserve quasianalyticity? How to use de Branges' theorem. — J. d'Analise Math. (1996), v. 69, p. 293–309.

[6] *H. Pedersen*, Uniform estimates of entire functions by logarithmic sums. — J. Funct. Anal. (1997), v. 146, No. 2, p. 517–556.

[7] *P. Koosis*, Harmonic estimation in certain slit regions and a theorem of Beurling and Malliavin.— Acta Math. (1979), v. 142, p. 275–304.

Один результат о полиномах и его связь с другим, относящимся к целым функциям экспоненциального типа

Пол Кусис

Теорема Берлинга–Малявена о мультипликаторах выводится из первого результата о полиномах, сформулированного во введении. Работа в значительной степени основана на описании де Бранжа экстремальных аннулирующих мер, которые соответствуют некоторым пространствам ограниченных функций, порожденных мнимыми экспонентами в пространствах с весом.

Один результат про поліноми та його зв'язок з іншим, який стосується цілих функцій экспоненціального типу

Пол Кусіс

Теорема Берлінга–Малявена про мультиплікатори виводиться з першого результату про поліноми, який сформульовано у вступі. Робота в значній мірі грунтується на описі де Бранжа екстремальних анулюючих мір, що відповідають деяким просторам обмежених функцій, які породжені уявними експонентами у просторах з вагою.