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Projections of k-dimensional subsets of $\mathbf{R}^n$ onto k-dimensional planes

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Some properties of projections of sets with non-vanishing Hausdorff k-measure onto k-planes are studied. It is stated that there is a wide class of k-planes in $\mathbf{R}^n$ such that a projection of a closed k-dimensional set onto any plane of that class has dimension equal to k.

1. Introduction and statement of the main results

It is well-known that there is a subset X in $\mathbf{R}^2$ of positive Hausdorff 1-measure such that an orthogonal projection of X onto any liner is a set of zero measure [1]. In this paper we consider projections of k-dimensional subsets of $\mathbf{R}^n$ onto k-dimensional planes (here a k-dimensional set is a set whose topological dimension is equal to k [2]). It must be pointed out that any k-dimensional set is a set of positive Hausdorff k-measure. The inverse statement is not true.

Here we show that for any k-dimensional F_σ-subset X of $\mathbf{R}^n$ the set of k-dimensional projections of X onto k-dimensional planes is large (Theorem 1.2). Moreover, in the case $k = 1, n-1$ the complement to this set is a nowhere dense set of zero measure in the Grassmannian manifold G^n_k (Theorem 1.3). In the last two sections we consider applications to studying of a class of maps of $\mathbf{R}^n$ into $\mathbf{R}^m$ (Section 5) and Cartesian products of Baire spaces (Section 6).

Let $X \subset \mathbf{R}^n$, l, and s be k-dimensional and $(n-k)$-dimensional planes, respectively. Denote by $p^s_l(X)$ the projection of the set X onto the plane l along the plane s. The projection $p^s_l(X)$ is well-defined if and only if the planes l and s are transverse.

Theorem 1.1. *Let* $\dim X \geq k$. *Then in any coordinate system in* $\mathbf{R}^n$ *there are k-dimensional and $(n-k)$-dimensional coordinate planes l and s such that the projection $p^s_l(X)$ is a set of second category in l.*

Theorem 1.1 implies the following

Theorem 1.2. *Let X be a k-dimensional F_σ-subset of $\mathbf{R}^n$. Then in any coordinate system in $\mathbf{R}^n$ there are k-dimensional and $(n-k)$-dimensional coordinate planes l and s such that*

$$\dim p_l^s(X) = k \ .$$

It is easy to see that if X is an F_σ-subset of $\mathbf{R}^n$, then the projections $p_l^s(X)$ is an F_σ-set. Any F_σ-subset of second category has a non-empty interior, and Theorem 1.2 is a consequence of Theorem 1.1.

For k-dimensional sets which are not F_σ-subsets of $\mathbf{R}^n$ Theorem 1.2 fails. In Section 3, we construct an $(n-1)$-dimensional subset X of $\mathbf{R}^n$ such that for any $k > [\frac{n}{2}]$ any projection of X onto k-dimensional coordinate plane of a fixed coordinate system in $\mathbf{R}^n$ along an $(n-k)$-dimensional coordinate plane has an empty interior (i.e., dimension of this projection is less than k).

Let $X \subset \mathbf{R}^n$. Denote by $V_k^n(X)$ the set of all $l \in G_k^n$ such that for any $s \in G_{n-k}^n$ the projection $p_l^s(X)$ is a set of first category in l. It is easy to see that if X is an F_σ-subset of $\mathbf{R}^n$, then $l \in V_k^n(X)$ if and only if for any $s \in G_k^n$ we have $\dim p_l^s(X) < k$.

Theorem 1.3. *Let* $\dim X \geq k$ *and* $k = 1, n-1$. *Then* $V_k^n(X)$ *is a nowhere dense set of zero measure.*

2. Proof of Theorem 1.1

2.1. Notation. Let

$$I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$$

and

$$a = (a_1, \dots, a_k) \in \mathbf{R}^k \ .$$

Denote by $l_I(a)$ the $(n-k)$-dimensional plane in $\mathbf{R}^n$ defined by the following conditions:

$$x_{i_1} = a_1, \dots, x_{i_k} = a_k \ .$$

Let

$$\{j_1, \dots, j_{n-k}\} = \{1, \dots, n\} \setminus I \ .$$

Consider the projection

$$p_I : \mathbf{R}^n \to \mathbf{R}^{n-k} \ ,$$

$$p_I(x_1, \dots, x_n) = (x_{j_1}, \dots, x_{j_{n-k}}) \ .$$

Then

$$p_I : l_I(a) \to \mathbf{R}^{n-k}$$

is a homeomorphism.

2.2. Essential dense sets. In this subsection we consider a relation between Baire category and a class of everywhere dense subsets of $\mathbf{R}^n$.

Definition 2.1. *A set $A \subset \mathbf{R}^n$ is called essential dense if it satisfies one of the following conditions:*

(i) $n = 1$ and the set A is everywhere dense;

(ii) $n > 1$ and there exists everywhere dense subset $C(A)$ of $\mathbf{R}$ and essential dense subset $B(A)$ of $\mathbf{R}^{n-1}$ such that

$$p_i(l_i(x) \cap A) = B(A)$$

for any $x \in C(A)$ and any $i = 1, \dots, n$.

Lemma 2.1. *Let X be a set of first category in $\mathbf{R}^n$. Then there exists an essential dense subset A of $\mathbf{R}^n$ such that $A \cap X = \emptyset$.*

We exploit the following lemma to prove Lemma 2.1.

Lemma 2.2. *Let X be a subset of $\mathbf{R}^n$ with an empty interior. Then there exists an everywhere dense subset A of $\mathbf{R}^n$ such that $X \cap A = \emptyset$.*

P r o o f o f L e m m a 2.2. Let B be an everywhere dense subset of $\mathbf{R}^n$ and

$$B_1 = B \cap X \ , B_2 = B \setminus B_1 \ , B_1 \neq \emptyset \ .$$

It is easy to see that for any $x \in X$ there exists a sequence $\{y_i(x)\}_{i=1}^{\infty}$ such that

$$\lim_{i \to \infty} y_i(x) = x \quad \text{and} \quad y_i(x) \notin X, \ \forall i = 1, 2, \dots .$$

Then the set

$$\bigcup_{x \in B_1} \{y_i(x)\}_{i=1}^{\infty} \bigcup B_1$$

is desired.

P r o o f o f L e m m a 2.1. In the case $n = 1$ Lemma 2.1 is a consequence of Lemma 2.2.

Let $n > 1$ and

$$E = \bigcup_{i=1}^{n} \{ \ x \in \mathbf{R} \mid l_i(x) \cap X \text{ is a set of second category in } l_i(x) \ \} \ .$$

The Kuratovsky–Ulam theorem [3] shows that E is a set of first category in $\mathbf{R}$. Therefore it has an empty interior and Lemma 2.2 implies the existence of a denumerable everywhere dense subset C of $\mathbf{R}$ such that $C \cap E = \emptyset$. Consider

$$X_{n-1} = \bigcup_{i=1}^{n} \bigcup_{x \in C} p_i(l_i(x) \cap X) \; .$$

It is a set of first category in $\mathbf{R}^{n-1}$. The inductive hypothesis implies the existence of an essential dense subset B of $\mathbf{R}^n$ such that $B \cap X_{n-1} = \emptyset$. Let $B_i(x)$ be a subset of $l_i(x)$ such that $B = p_i(B_i(x))$. Then the set

$$A = \bigcup_{i=1}^{n} \bigcup_{x \in C} B_i(x)$$

is as desired.

2.3. (k,n)**-sets.** Let $X \subset \mathbf{R}^n$ and

$$X_k = \bigcup_{|I|=n-k} p_I(X) \; .$$

Let $A \subset \mathbf{R}^k$ and

$$Y_k^n(A) = \bigcup_{|I|=k} \bigcup_{x \in A} l_I(x) \; .$$

Definition 2.2. *A set* Y *is called a* (k,n)*-set if there exists an essential dense subset* A *of* $\mathbf{R}^k$ *such that* $Y = Y_k^n(A)$.

Lemma 2.3. *Let* X *be a subset of* $\mathbf{R}^n$ *such that* X_k *is a set of first category. Then there exists a* (k,n)*-set* Y *such that* $X \cap Y = \emptyset$.

P r o o f o f L e m m a 2.3. Lemma 2.1 guarantees the existence of an essential dense subset A of $\mathbf{R}^k$ such that $A \cap X_k = \emptyset$. Then $X \cap Y_k^n(A) = \emptyset$.

Theorem 1.1 is a consequence of Lemma 2.3 and the following

Lemma 2.4. *Let* $X \subset \mathbf{R}^n$ *and* $\dim X \geq k$. *Then* X *intersects any* (k,n)*-set.*

P r o o f o f L e m m a 2.4. We show that the dimension of the complement to any (k,n)-set in $\mathbf{R}^n$ is less than k. In the case $n = 1$ it is trivial.

Let $n > 1$ and Y be a (k,n)-set. Then there exists an essential dense subset A of $\mathbf{R}^{k+1}$ such that $Y = Y_{k+1}^n(A)$. Let $X = \mathbf{R}^n \setminus Y$ and $C(A)$ be the subset of $\mathbf{R}$ from Definition 2.1. Then

$$p_i(l_i(x) \cap Y) = Y_k^{n-1}(B(A)), \quad \forall i = 1, \dots, n, \quad \forall x \in C(A) \; ,$$

where $B(A)$ is the essential subset of $\mathbf{R}^k$ from Definition 2.1. Therefore

$$p_i(l_i(x) \cap X) = \mathbf{R}^{n-1} \setminus Y_k^{n-1}(B(A)), \quad \forall i = 1, \dots, n, \ \ \forall x \in C(A) .$$

It is easy to see that $Y_k^{n-1}(B(A))$ is a $(k, n-1)$-set and the inductive hypothesis implies that

$$\dim l_i(x) \cap X \leq k - 1, \quad \forall i = 1, \dots, n, \quad \forall x \in B . \tag{2.1}$$

Let

$$C(A)^n = \underbrace{C(A) \times \dots \times C(A)}_{n}$$

and Δ be an n-dimensional cube of $\mathbf{R}^n$ such that any vertex of Δ is a point of the set $C(A)^n$. Equation (2.1) shows that

$$\dim \partial\Delta \cap X \leq k - 1 .$$

The set $C(A)^n$ is everywhere dense, and $\dim X \leq k$ [2].

3. Example

Now we construct an $(n-1)$-dimensional subset X of $\mathbf{R}^n$ such that for any $k > [\frac{n}{2}]$ and any $I \subset \{1, \dots, n\}$, $|I| = n - k$, we have

$$\dim p_I(X) < k . \tag{3.1}$$

Let $m = [\frac{n}{2}]+1$. For any $I \subset \{1, ..., n\}$ such that $|I| = m$ consider denumerable everywhere dense subsets $\{A_i^I\}_{i=1,\,I}^m$ of $\mathbf{R}$ such that $A_i^I \cap A_j^J = \emptyset$, if $I \neq J$ or $i \neq j$. Let

$$A_I = A_1^I \times \dots \times A_m^I$$

and

$$Y = \bigcup_{|I|=m} \bigcup_{a \in A_I} l_I(a) .$$

The set Y is the denumerable union of non-intersecting $(n-m)$-dimensional planes and $\mathbf{R}^n \setminus Y = n - 1$ [4]. Let $X = \mathbf{R}^n \setminus Y$. Then equation (3.1) is satisfied for $k = m$. An immediate verification shows that it is satisfied for any $k \geq m$.

4. Proof of Theorem 1.3 and irregular subsets of the Grassmannian manifolds

In this section we consider a class of subsets in the Grassmannian manifold G_k^n. We exploit properties of these sets to prove Theorem 1.3.

4.1. Definition and elementary properties of irregular sets. Let

$$L = \{l_1, \dots, l_{c_k^n}\}$$

be a collection of k-dimensional planes, where

$$c_k^n = \frac{n!}{k!(n-k)!}\ .$$

Definition 4.1. *The collection L is called regular if there exists a coordinate system in $\mathbf{R}^n$ such that any l_i $(i = 1, \dots, c_k^n)$ is a coordinate plane of this system.*

It must be pointed out that any coordinate system in $\mathbf{R}^n$ has c_k^n distinct k-dimensional planes.

Definition 4.2. *A set $V \subset G_k^n$ is called irregular if for any $l_1 \in G_k^n, \dots, l_{c_k^n} \in G_k^n$ the collection $\{l_1, \dots, l_{c_k^n}\}$ is not regular.*

Definition 4.3. *An irregular set $V \subset G_k^n$ is called maximal if for any irregular set W such that $V \subset W$ one has $V = W$.*

It is not difficult to see that for any irregular set V there exists maximal irregular set W such that $V \subset W$. In what follows we exploit the following simple

Lemma 4.1. *An irregular set $V \subset G_k^n$ is maximal if and only if for any $l \in G_k^n \setminus V$ there exists $l_1 \in V, \dots, l_{c_k^n - 1} \in V$ such that the collection*

$$\{l_1, \dots, l_{c_k^n - 1}, l\}$$

is regular.

Let $\varphi_k^n : G_k^n \to G_{n-k}^n$ be the canonical homeomorphism; i.e., $\varphi_k^n(l) = l^\perp$ (where $l^\perp$ is the orthogonal complement to l). Then we have the following

Lemma 4.2. *The canonical homeomorphism φ_k^n maps any irregular subset V of G_k^n into an irregular subset of G_{n-k}^n. Moreover, if V is a maximal irregular set, then $\varphi_k^n(V)$ is maximal.*

P r o o f. An immediate verification shows that φ_k^n maps any regular collection of k-dimensional plane into a regular collection of $(n-k)$-dimensional plane. Lemma 4.2 is a consequence of this statement.

4.2. Irregular subset of G_k^n $(k = 1, n-1)$. Let $s \in G_m^n$ and

$$G_k^n(s) = \begin{cases} \{\, l \in G_k^n \mid l \subset s \,\}, & m \geq k\,, \\ \{\, l \in G_k^n \mid s \subset l \,\}, & m \leq k\,. \end{cases}$$

Then $G_k^n(s)$ is irregular.

Proposition 4.1. *Let $V \subset G_k^n$ be a maximal irregular set and $k = 1, n-1$. Then there is $s \in G_{n-k}^n$ such that $V = G_k^n(s)$.*

P r o o f. Consider the case $k = 1$. Let and $l \in G_1^n \backslash V$. Lemma 4.1 guarantees the existence of $l_1 \in V, \ldots, l_{n-1} \in V$ such that the collection

$$\{l_1, \ldots, l_{n-1}, l\}$$

is regular. Consider the $(n-1)$-dimensional plane s generated by $l_1, \ldots, l_{n-1}$. It is not difficult to see that $V = G_1^n(s)$. In the case $k = n-1$ the statement is a consequence of Lemma 4.2 and the following equation:

$$\varphi_k^n(G_k^n(s)) = G_{n-k}^n(\varphi_m^n(s))\ .$$

Any irregular set is a subset of a maximal irregular set. Therefore we have the following

Proposition 4.2. *Any irregular subset of G_k^n $(k = 1, n-1)$ is a nowhere dense set of zero measure.*

Theorem 1.1 shows us that if $X \subset \mathbf{R}^n$ and $\dim X \geq k$, then the set $V_k^n(X)$ is irregular. Therefore Theorem 1.3 is a consequence of Proposition 4.2.

5. Hausdorff maps

5.1. Recall the following

Theorem 5.1. (A.Y. Dubovitsky [5]). *Let $f \in C^1(\mathbf{R}^n, \mathbf{R}^m)$ and $n \geq m$. Then dimension of a typical level set of the map f is not greater than $n - m$.*

It must be pointed out that in the case $n < m$ a typical level set is empty. In this paper we prove an extension of Theorem 5.1. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a map.

Definition 5.1. *We say that*

(i) a typical level set of the map f satisfies to a condition A, if

$$mes\{\ y \mid f^{-1}(y) \ \ do\ not\ satisfy\ to\ A\ \} = 0\ ;$$

(ii) a condition B is k-nontypical for level sets of the map f, if

$$mes_k\{\ y \mid f^{-1}(y) \ \ satisfies\ to\ B\ \} = 0;$$

Here mes is Lebesgue measure and mes_k is the Hausdorff k-measure.

Definition 5.2. *Let k and l be natural numbers such that $k \le n$ and $l \le m$. Then the map f is called Hausdorff (k,l)-map if it maps any set of zero Hausdorff k-measure into a set of zero Hausdorff l-measure.*

Theorem 5.2. *Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a Hausdorff (k,l)-map. Then level sets of the map f whose dimension is greater than $n-k$ are l-nontypical.*

P r o o f. Let Y_i^n be an (i,n)-set. Then Lemma 2.4 implies that

$$\{y \mid dim f^{-1}(y) \ge i\ \} \subset f(Y_i^n)\ . \tag{5.1}$$

It is easy to see that Y_{n-k+1}^n is a denumerable union of $(k-1)$-dimensional planes; i.e., it is a set of zero Hausdorff k-measure. Therefore Theorem 5.2 is a consequence of equation (5.1) if $i = n-k+1$.

Now we consider examples of Hausdorff map.

(i) Any Lipschitzian map of $\mathbf{R}^n$ into $\mathbf{R}^m$ is the Hausdorff (i,i)-map if $i \le \min(n,m)$.

(ii) Any Gelderian map of $\mathbf{R}^n$ into $\mathbf{R}^m$ is the Hausdorff (k,l)-map if $l\sigma \le k$, where σ is Gelderian number of this map.

Theorem 5.2 could be used to find the dimension of typical level set of these maps.

5.2. Now we take advantage of Lemma 2.4 to prove the following

Theorem 5.3. (A.Y. Dubovitsky [5, 6]). *Let $f \in C^i(\mathbf{R}^n;\mathbf{R}^m)$ and $i = n - m - k + 1$. Then*

$$mes\{y \mid dim f^{-1}(y) \cap \Sigma(f) \ge k\} = 0\ ;$$

i.e., dimension of an intersection of a typical level set of the map f with $\Sigma(f)$ is not greater than $k-1$.

P r o o f. Lemma 2.4 shows that

$$\{y \mid \dim f^{-1}(y) \cap \Sigma(f) \geq k\} \subset f(Y_k^n \cap \Sigma(f)) \ . \tag{5.2}$$

We prove that

$$mesf(Y_k^n \cap \Sigma(f)) = 0. \tag{5.3}$$

Theorem 5.3 is a consequence of equations (5.2) and (5.3). Let $x \in Y_k^n \cap \Sigma(f)$. Then Y_k^n is a denumerable union of $(n-k)$-dimensional planes, and there exists an $(n-k)$-dimensional plane l lying in Y_k^n such that $x \in \Sigma(f|_l)$. The map $f|_l$ satisfies the conditions of the Sard theorem and

$$mesf(\Sigma(f|_l)) = 0 \ .$$

The set Y_k^n is a denumerable union of $(n-k)$-dimensional planes. It implies (5.3).

It must be pointed out that in [6] Theorem 5.3 was represented as a consequence of a more strong statement.

6. Residual subsets of Cartesian products of Baire spaces

6.1. We begin this section with the following

Definition 6.1. *A topological space X is called Baire if any subset of X with a non-empty interior is a set of second category.*

It is not difficult to see that a space X is Baire if and only if any residual subset of X (i.e., denumerable intersection of open everywhere dense subsets of X) is everywhere dense.

In this section we prove the following extension of Lemma 2.1.

Proposition 6.1. *Let $X = \times_{i=1}^{\infty} X_i$ be a Cartesian product and any X_i be a Baire space with a denumerable base. Then for any residual subset Y of X there exists everywhere dense subsets A_i of X_i such that $\times_{i=1}^{\infty} A_i \subset Y$.*

Proposition 6.1 is true if one of the spaces X_i is separable. Any space with a denumerable base is separable. The inverse statement fails. Proposition 6.1 shows that *a denumerable product of Baire spaces with a denumerable base is a Baire space*. It must be pointed out that this statement was proved in [7] for a Cartesian product of any family of Baire spaces with denumerable base.

6.2. P r o o f o f P r o p o s i t i o n 6.1.

Lemma 6.1. *Let $X = X_1 \times X_2$, X_2 be a space with a denumerable base and Y a residual subset of X. Then*

$$\{\ x \in X_1 \mid \text{ the set } (x \times X_2) \cap Y \text{ is residual }\ \}$$

is residual.

Lemma 6.2. *Let X be a separable space and Z be a subset of X with an empty interior. Then there exists a denumerable everywhere dense subset A of X such that $A \cap Z = \emptyset$.*

Lemma 6.1 is a consequence of the Ulam–Kuratovski theorem [3]. Lemma 6.2 was proved for the case $X = \mathbf{R}^n$ (Lemma 2.2). In the general case the proof is analogous.

Lemma 6.3. *Let $X = X_1 \times X_2$, X_1 be a separable Baire space and X_2 a Baire space with a denumerable base. Then for any residual subset Y of X there exists subsets A_i of X_i $(i = 1, 2)$ such that $A_1 \times A_2 \subset X$, A_1 is everywhere dense and A_2 is residual.*

Lemma 6.3 is a consequence of Lemmas 6.1 and 6.2 (see the proof of Lemma 2.1). It is not difficult to see that Lemma 6.3 implies Proposition 6.1.

6.3. R e m a r k t o P r o p o s i t i o n 6.1. Let $X = \times_{i=1}^{n} X_i$ be a Cartesian product of Baire spaces with a denumerable base. Then in Proposition 6.1 one of the sets $A_1, \dots, A_n$ is residual. If one of the sets $X_1, \dots, X_n$, for example X_1, is separable and has no a denumerable base, then one of the sets $A_2, \dots, A_n$ is residual.

Consider the residual set

$$B = \mathbf{R}^2 \setminus \{ (x, y) \in \mathbf{R}^2 \mid x = y \} .$$

Proposition 6.1 guarantees the existence of everywhere dense sets A_1 and A_2 such that $A_1 \times A_2 \subset B$. It is easy to see that $A_1 \cap A_2 = \emptyset$. Therefore one of these sets is not residual.

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О проекциях подмножеств $\mathbf{R}^n$ с ненулевой k-мерой Хаусдорфа на k-мерные плоскости

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Изучаются свойства проекций множеств с ненулевой k-мерой Хаусдорфа на k-мерные плоскости. Доказано, что в $\mathbf{R}^n$ есть достаточно большой класс k-мерных плоскостей таких, что проекция замкнутого k-мерного множества на эти плоскости имеет размерность k.

Про проекції підмножин $\mathbf{R}^n$ з ненульовою k-мірою Хаусдорфа на k-мірні площини

М.О. Панков

Досліджуються властивості проекцій множин з ненульовою k-мірою Хаусдорфа на k-мірні площини. Доведено, що в $\mathbf{R}^n$ існує досить широкий клас k-мірних площин таких, що проекція замкненої k-мірної множини на ці площини має розмірність k.