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A characterization of some even vector-valued Sturm–Liouville problems

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We call "even" a Sturm–Liouville problem

$$-y'' + Q(x)y = \lambda\, y,\ \ 0 \le x \le \pi, \tag{1}$$

$$y'(0) - h\, y(0) = 0, \tag{2}$$

$$y'(\pi) + Hy(\pi) = 0 \tag{3}$$

in which $H = h$ and $Q(\pi - x) \equiv Q(x)$ on $[0, \pi]$. In this paper we study the vector-valued case, where the *potential* $Q(x)$ is a real symmetric $d \times d$ matrix for each x in $[0, \pi]$, and the entries of Q and their first derivatives (in the distribution sense) are all in $L^2[0, \pi]$. We assume that h and H are real symmetric $d \times d$ matrices.

We prove that a vector-valued Sturm–Liouville problem (1)–(3) is even if and only if, for each eigenvalue λ, whose multiplicity is $r = r_\lambda$ (where $1 \le r \le d$, and where $\varphi_1(x, \lambda), \ldots, \varphi_r(x, \lambda)$ denote orthonormal eigenfunctions belonging to λ), there exists an $r \times r$ matrix $A = (a_{ij})$ (which may depend on λ and on the choice of basis $\{\varphi_i(x, \lambda)\}_{i=1}^r$, but does not depend on x) such that

(1) A is orthogonal and symmetric,

and

(2) for $1 \le i \le r$, $\varphi_i(\pi, \lambda) = \sum_{j=1}^r a_{ij}\varphi_j(0, \lambda)$.

To some extent our theorem can be considered a generalization of N. Levinson's results in [2].

§ 1. Introduction

Suppose that $Q(x)$ is a real symmetric $d \times d$ matrix for each x in $[0, \pi]$, and that the entries of the *potential* Q and their first derivatives (in the distribution sense) are all in $L^2[0, \pi]$. Let h and H be real symmetric $d \times d$ matrices and λ a real number.

We consider the following Sturm–Liouville problem, in which y is a real d-dimensional vector-valued function:

$$-y'' + Q(x)y = \lambda\, y,\ 0 \le x \le \pi, \tag{1}$$

$$y'(0) - h\, y(0) = 0, \tag{2}$$

$$y'(\pi) + H y(\pi) = 0. \tag{3}$$

In the following definition, we use the term "even" to mean that the potential $Q(x)$ is "even about $\pi/2$", meaning $Q(\pi - x) \equiv Q(x)$, and that the "boundary-condition" matrices h and H are equal.

Definition. *The Sturm–Liouville problem (1)–(3) is called even if*

(i) $Q(x) \equiv Q(\pi - x)$

and

(ii) $H = h$.

In 1950 V. Marchenko [4], using transmutation operators, proved that, if two scalar-valued Sturm–Liouville problems on the half-line*,

$$-y'' + q_1(x)\, y = \lambda y,\ y'(0) = h_1\, y(0); \quad -y'' + q_2(x)\, y = \lambda y,\ y'(0) = h_2 y(0)\,,$$

have the same spectral function, then $q_2(x) \equiv q_1(x)$ and $h_2 = h_1$. We will refer to the version of this for a finite interval as Marchenko's uniqueness theorem.

In the scalar case ($d = 1$) the following known theorem is based on work of Levinson (see [2]). In the theorem, we let $\varphi(x, \lambda)$ denote the solution of equation (1) with the initial conditions $\varphi(0, \lambda) = c$, $\varphi'(0, \lambda) = hc$, where $c > 0$ is chosen so that $\int_0^\pi \varphi(x, \lambda)^2 dx = 1$. Then $\varphi(x, \lambda)$, for every λ, satisfies the first boundary condition (2). The *eigenvalues*, $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$, of the problem (1)–(3) are the roots of the equation

$$\varphi'(\pi, \lambda) + H\, \varphi(\pi, \lambda) = 0\,.$$

The corresponding *eigenfunctions* are $\varphi(x, \lambda_n)$, $n \ge 0$. We have normalized these functions in L^2. That is, $\|\varphi(\bullet, \lambda_n)\|_2 = 1$ for all $n \ge 0$.

Theorem 1.1. *A Sturm–Liouville problem is even if and only if, for all* $n \ge 0$,

$$\varphi(\pi, \lambda_n) = \pm\varphi(0, \lambda_n),\ n = 0, 1, 2, \ldots\,. \tag{1.2}$$

Moreover, the signs alternate.

*The proof of Marchenko's theorem on a finite interval is essentially the same.

We offer a proof that resembles, to some extent, the proof of the analogous theorem in the vector-valued case.

P r o o f of N e c e s s i t y. Let the potential in (1)–(3) be denoted $q(x)$ since $d = 1$, and suppose the problem is even, that is, $q(\pi - x) \equiv q(x)$ and $H = h$. Let us define the functions

$$\psi(x, \lambda_n) = \varphi(\pi - x, \lambda_n), \quad n \geq 0. \tag{1.3}$$

Since $q(\pi - x) = q(x)$ the function $\psi(x, \lambda_n)$ is a solution of the differential equation (1):

$$-\psi''(x, \lambda_n) + q(x)\psi(x, \lambda_n) = \lambda_n \psi(x, \lambda_n). \tag{1.4}$$

To prove that $\psi(x, \lambda_n)$ is an eigenfunction of the problem (1)–(3) we must consider the boundary behavior of $\psi(x, \lambda_n)$. By construction,

$$\psi(0, \lambda_n) = \varphi(\pi, \lambda_n), \quad \psi'(0, \lambda_n) = -\varphi'(\pi, \lambda_n)\,.$$

By these last two equations, the assumed symmetry, and because $\varphi(x, \lambda_n)$ is an eigenfunction,

$$\psi'(0, \lambda_n) = -\varphi'(\pi, \lambda_n) = H\varphi(\pi, \lambda_n) = H\psi(0, \lambda_n) = h\psi(0, \lambda_n),$$

and

$$\psi'(\pi, \lambda_n) = -\varphi'(0, \lambda_n) = -h\varphi(0, \lambda_n) = -h\psi(\pi, \lambda_n) = -H\psi(\pi, \lambda_n).$$

Therefore $\psi(x, \lambda_n)$ is an eigenfunction, so the simplicity of the spectrum (see, e.g., [3] for some functional analysis background material) implies that $\psi(x, \lambda_n)$ is a multiple of $\varphi(x, \lambda_n)$: $\psi(x, \lambda_n) = c_n \varphi(x, \lambda_n)$. We can now prove (1.2). Since $\psi(x, \lambda_n)$ and $\varphi(x, \lambda_n)$ have the same norm in L^2 (by construction), and because we consider real scalars only, $c_n = \pm 1$. Thus $\psi(x, \lambda_n) = \pm\varphi(x, \lambda_n)$. We can now put $x = 0$ and use the definition of $\psi(x, \lambda_n)$ to obtain (1.2).

P r o o f o f S u f f i c i e n c y. We will make use of Marchenko's uniqueness theorem here. In addition to (1)–(3), in which $d = 1$ and we write $q(x)$ instead of $Q(x)$, we assume that (1.2) holds. We consider also the "reflected" problem

$$-y'' + q(\pi - x)y = \lambda\, y, \ 0 \leq x \leq \pi, \tag{1*}$$

$$y'(0) - H\, y(0) = 0, \tag{2*}$$

$$y'(\pi) + hy(\pi) = 0. \tag{3*}$$

We observe, by inspection, that the functions $\psi(x, \lambda_n) = \varphi(\pi - x, \lambda_n)$ defined in (1.3) are eigenfunctions of (1*)–(3*). This new problem therefore has the same spectrum as (1)–(3). We wish to show that the problems (1)–(3) and (1*)–(3*) have the same spectral function. Let us renormalize the eigenfunctions $\varphi(x,\lambda_n)$, recalling that each $\varphi(x,\lambda_n)$ has norm 1 in L^2. We define, for each λ, $\tilde\varphi(x,\lambda)$ to be the solution of (1) with initial values $\tilde\varphi(0,\lambda)=1$, $\tilde\varphi'(0,\lambda)=h$. Then $\varphi(x,\lambda_n)=\varphi(0,\lambda_n)\tilde\varphi(x,\lambda_n)$. Hence for all $n=0,1,2\ldots,$ $\varphi(0,\lambda_n)^2\|\tilde\varphi(\bullet,\lambda_n)\|_2^2=1$. Now we can define the *spectral function* $\rho(\lambda)$ of (1)–(3) in terms of the *norming constants* $\alpha_n^2:=\|\tilde\varphi(\bullet,\lambda_n)\|_2^2$, and we can apply the relation $\varphi(0,\lambda_n)^2\|\tilde\varphi(\bullet,\lambda_n)\|_2^2=1$:

$$\rho(\lambda):=\sum_{\lambda_n<\lambda}\frac{1}{\alpha_n^2}=\sum_{\lambda_n<\lambda}|\varphi(0,\lambda_n)|^2.$$

When we do the same for the problem (1*)–(3*), and take into account the assumption (1.2), we find that the spectral function for (1*)–(3*) has the form

$$\rho^*(\lambda):=\sum_{\lambda_n<\lambda}\frac{1}{{\alpha^*}_n^2}=\sum_{\lambda_n<\lambda}|\psi(0,\lambda_n)|^2$$

$$=\sum_{\lambda_n<\lambda}|\varphi(\pi,\lambda_n)|^2=\sum_{\lambda_n<\lambda}|\varphi(0,\lambda_n)|^2=\rho(\lambda).$$

Thus the problems (1)–(3) and (1*)–(3*) have the same spectral functions. By Marchenko's uniqueness theorem, they have the same potentials and the same boundary conditions at $x=0$. That is,

$$q(\pi-x)\equiv q(x)\ \text{ for }\ 0\le x\le\pi,\quad\text{and}\quad H=h\,.$$

From the Sturm oscillation theorem it follows that $\tilde\varphi(\pi,\lambda_n)=(-1)^n$ if $\tilde\varphi(\pi,\lambda_0)=1$. We do not yet have an analogue of this last result in the vector-valued case.

§ 2. The vector-valued case

Theorem 2.1. *The Sturm–Liouville problem (1)–(3) is even if and only if, for each eigenvalue λ, whose multiplicity is $r=r_\lambda$ (where $1\le r\le d$, and where $\varphi_1(x,\lambda),\ldots,\varphi_r(x,\lambda)$ denote orthonormal eigenfunctions belonging to λ), there exists an $r\times r$ matrix $A=(a_{ij})$ (which may depend on λ and on the choice of basis $\{\varphi_i(x,\lambda)\}_{i=1}^r$, but does not depend on x) such that A is orthogonal and symmetric and, for $1\le i\le r$,*

$$\varphi_i(\pi,\lambda)=\sum_{j=1}^r a_{ij}\varphi_j(0,\lambda)\,.$$

R e m a r k. We do not consider Dirichlet boundary conditions in this paper.

P r o o f. Suppose the problem is even, so that $Q(x) \equiv Q(\pi - x)$ and $H = h$. This part of the proof is straightforward. By exploiting reflection, and the hypothesis that $H = h$, we can show that the functions

$$\varphi_1(\pi - x, \lambda), \ldots, \varphi_r(\pi - x, \lambda)$$

are solutions of the same problem. They are also mutually orthogonal and normalized, so there exists an *orthogonal* $r \times r$ matrix A such that, for $1 \le i \le r$, we have

$$\varphi_i(\pi - x, \lambda) = \sum_{j=1}^{r} a_{ij}\varphi_j(x, \lambda), \quad \text{for} \quad 0 \le x \le \pi\,. \tag{2.2}$$

If we now replace x by $\pi - x$ in this equation, and then substitute the original equations into the right-hand side, we have

$$\varphi_i(x, \lambda) = \sum_{j=1}^{r} a_{ij}\varphi_j(\pi - x, \lambda) = \sum_{j=1}^{r} a_{ij} \sum_{k=1}^{r} a_{jk}\varphi_k(x, \lambda)$$

$$= \sum_{k=1}^{r} \left(\sum_{j=1}^{r} a_{ij}a_{jk} \right) \varphi_k(x, \lambda) \tag{2.3}$$

for $1 \le i \le r$. The orthonormality of the functions $\varphi_k(x, \lambda)$ now implies that $\sum_{j=1}^{r} a_{ij}a_{jk} = \delta_{ik}$, so $A^2 = I$. Since A is also orthogonal, we have $A^2 = I = AA^T$, so A is symmetric as well, as was to be shown. Finally, we may put $x = 0$ in the equation (2.2) to obtain

$$\varphi_i(\pi, \lambda) = \sum_{j=1}^{r} a_{ij}\varphi_j(0, \lambda), \tag{2.4}$$

and this completes the proof that the conditions concerning the matrix A are necessary.

Next, we must deduce the symmetry of problem (1)–(3) from the assumptions $\varphi_i(\pi, \lambda) = \sum_{j=1}^{r_\lambda} a_{ij}(\lambda)\varphi_j(0, \lambda)$, valid for each eigenvalue λ, where we assume that $A = A(\lambda) = \left(a_{ij}(\lambda)\right)_{r\times r}$ is orthogonal and symmetric. This part of the proof is not so straightforward. We will use the completeness of the eigenfunctions, the uniform convergence of the Fourier series (with respect to the eigenfunctions) of $C^2[0, \pi]$ functions, and the two following theorems.

Theorem A. Existence of the transmutation (by way of a Goursat problem). *Consider two Sturm–Liouville problems,*

$$-y'' + R_1(x)y = \lambda\, y, \;\; 0 \le x \le \pi,$$

$$y'(0) - h_1\, y(0) = 0,\;\; y'(\pi) + H_1 y(\pi) = 0, \tag{2.5}$$

and

$$-y'' + R_2(x)y = \lambda\, y,\;\; 0 \le x \le \pi,$$
$$y'(0) - h_2\, y(0) = 0,\;\; y'(\pi) + H_2 y(\pi) = 0. \tag{2.6}$$

As usual, we assume that $R_1(x)$ and $R_2(x)$ are continuous, symmetric, $d \times d$ real matrix-valued functions, and we assume that h_1, H_1, h_2, and H_2 are symmetric, $d \times d$ real matrices.

Suppose that

(i) the problems (2.5) *and* (2.6) *have the same eigenvalues, with the same multiplicities,*

and

(ii) there exist systems of eigenfunctions $\{\varphi_{1i}(x,\lambda)\}$ and $\{\varphi_{2i}(x,\lambda)\}$ for the eigenvalues λ, where $1 \le i \le r_\lambda$, such that for each eigenvalue λ, and for $1 \le i \le r_\lambda$,

$$\varphi_{1i}(0,\lambda) = \varphi_{2i}(0,\lambda).$$

Then there exists a transmutation operator, with kernel $K(x,y)$, such that for each eigenvalue λ, and for $1 \le i \le r_\lambda$,

$$\varphi_{2i}(x,\lambda) = \varphi_{1i}(x,\lambda) + \int_0^x K(x,t)\varphi_{1i}(t,\lambda)dt, \tag{2.7}$$

and conversely, if (2.7) *holds, then $\varphi_{1i}(0,\lambda) = \varphi_{2i}(0,\lambda)$ for each eigenvalue λ, and for $1 \le i \le r_\lambda$.*

The proof of Theorem A will be given following the present proof, of Theorem (2.1).

Theorem B. Identification of $\mathcal{F}(x,y)$ in the Gelfand–Levitan equation so that the kernel $K(x,y)$ of the transmutation is its solution.

In addition to the hypotheses of Theorem A, assume that the systems of eigenfunctions $\{\varphi_{1i}(x,\lambda)\}$ and $\{\varphi_{2i}(x,\lambda)\}$ are each orthogonal, though not necessarily normalized. For each eigenvalue λ, and for $1 \le i \le r_\lambda$, let $\alpha_{1i}(\lambda)^2 := \|\varphi_{1i}(\bullet,\lambda)\|_2^2$ and let $\alpha_{2i}(\lambda)^2 := \|\varphi_{2i}(\bullet,\lambda)\|_2^2$.

Suppose that the following series, defining $\mathcal{F}(x,y)$,

$$\mathcal{F}(x,y) := \sum \left(\frac{1}{\alpha_{2i}(\lambda)^2} - \frac{1}{\alpha_{1i}(\lambda)^2}\right) \varphi_{1i}(x,\lambda)\varphi_{1i}(y,\lambda)^T$$

summed over the eigenvalues λ and the i with $1 \le i \le r_\lambda$, converges uniformly in $0 \le y \le x \le \pi$. Then, the kernel $K(x,y)$ of the transmutation operator from

Theorem A is the solution of the Gelfand–Levitan equation

$$K(x,y) + \mathcal{F}(x,y) + \int_0^x K(x,t)\mathcal{F}(t,y)dt = 0.$$

The proof of Theorem B will be given following the proof of Theorem A.

We consider the following (reflected) Sturm–Liouville problem, and seek to show that it coincides with the original one.

$$-y'' + Q(\pi - x)y = \lambda\, y,\ 0 \le x \le \pi, \tag{1*}$$

$$y'(0) - H\, y(0) = 0, \tag{2*}$$

$$y'(\pi) + hy(\pi) = 0. \tag{3*}$$

Because this problem is obtained from the original problem by substituting $\pi - x$ for x, and by reflecting the boundary conditions (i.e., interchanging h and H), (1^*)–(3^*) has the same eigenvalues λ, with the same multiplicities, r_λ, as the original problem, (1)–(3).

Let us propose, for eigenfunctions of (1^*)–(3^*), the functions $\psi_i(x,\lambda)$ (where $1 \le i \le r_\lambda$, and λ is an eigenvalue), defined by

$$\psi_i(x,\lambda) := \sum_{j=1}^{r} a_{ij}\varphi_j(\pi - x, \lambda).$$

Then, by substitution and rearrangement, each $\psi_i(x,\lambda)$ is a solution of (1^*), for each original eigenvalue λ, and for $1 \le i \le r_\lambda$.

To examine boundary behavior, we find that, for any $d \times d$ matrix G,

$$\begin{aligned}\psi_i'(x,\lambda) + G\psi_i(x,\lambda) &= -\sum_{j=1}^{r} a_{ij}\varphi_j'(\pi - x,\lambda) + G\sum_{j=1}^{r} a_{ij}\varphi_j(\pi - x,\lambda)\\ &= -\sum_{j=1}^{r} a_{ij}(\varphi_j'(\pi - x,\lambda) - G\varphi_j(\pi - x,\lambda)).\end{aligned}$$

If we choose $x = 0$ and $G = -H$, then for $1 \le j \le r$, $\varphi_j'(\pi - x,\lambda) - G\varphi_j(\pi - x,\lambda) = 0$ by (3), so

$$\psi_i'(0,\lambda) - H\psi_i(0,\lambda) = 0,$$

which is (2^*). If we choose $x = \pi$ and $G = h$, then for $1 \le j \le r$, $\varphi_j'(\pi - x,\lambda) - G\varphi_j(\pi - x,\lambda) = 0$ by (2), so

$$\psi_i'(\pi,\lambda) + h\psi_i(\pi,\lambda) = 0,$$

which is (3*). Thus, each of our defined functions $\psi_i(x,\lambda)$ is a solution of (1*)–(3*).

Now let us note that, by the definition of $\psi_i(x,\lambda)$, and by hypothesis,

$$\psi_i(0,\lambda) = \sum_{j=1}^{r} a_{ij}(\lambda)\varphi_j(\pi,\lambda) = \sum_{j=1}^{r} a_{ij}(\lambda)\sum_{k=1}^{r} a_{jk}(\lambda)\varphi_k(0,\lambda) = \varphi_i(0,\lambda),$$

since A is orthogonal and symmetric. Therefore the hypotheses of Theorem A are satisfied, so there exists a transmutation operator with kernel $K(x,y)$ such that $\psi_i(x,\lambda) = \varphi_i(x,\lambda) + \int_0^x K(x,t)\varphi_i(t,\lambda)dt$, for each eigenvalue λ and each i such that $1 \le i \le r_\lambda$.

We shall next apply Theorem B to show that $K(x,y) \equiv 0$. In Theorem B, the L^2-norms of $\varphi_i(x,\lambda)$ and $\psi_i(x,\lambda)$ can be denoted by $\alpha_i(\lambda)$ and $\beta_i(\lambda)$, respectively. In the present case, the eigenfunctions $\varphi_i(x,\lambda)$ comprise an orthonormal set. Thus

$$\begin{aligned}\beta_i^2(\lambda) &= \|\psi_i(\bullet,\lambda)\|_2^2 = \left\|\sum_{k=1}^{r} a_{ik}\varphi_k(\bullet,\lambda)\right\|_2^2 \\ &= \sum_{k=1}^{r} a_{ik}^2\,\|\varphi_k(\bullet,\lambda)\|_2^2 = \sum_{k=1}^{r} a_{ik}^2 = 1 = \alpha_i(\lambda)^2.\end{aligned}$$

According to Theorem B, the function $\mathcal{F}(x,y) \equiv 0$, so the solution of the corresponding Gelfand–Levitan equation is identically zero. But this is $K(x,y)$, so by Theorem A, $\varphi_i(x,\lambda) \equiv \psi_i(x,\lambda)$ for all eigenvalues λ and for all i, $1 \le i \le r_\lambda$. Therefore, with λ and i as above, we have

$$\begin{aligned}&-\varphi_i''(x,\lambda) + Q(x)\varphi_i(x,\lambda) = \lambda\varphi_i(x,\lambda) = \lambda\psi_i(x,\lambda) \\ &= -\psi_i''(x,\lambda) + Q(\pi-x)\psi_i(x,\lambda),\ 0 \le x \le \pi.\end{aligned}$$

Hence

$$(Q(x) - Q(\pi-x))\varphi_i(x,\lambda) = 0,\ 0 \le x \le \pi.$$

It follows now, from the completeness of the eigenfunctions, that $Q(x) = Q(\pi-x)$, $0 \le x \le \pi$. It remains to show that $H = h$.

The identities $\varphi_i(x,\lambda) = \psi_i(x,\lambda) = \sum_{j=1}^{r} a_{ij}\varphi_j(\pi-x,\lambda)$ yield

$$\varphi_i'(x,\lambda) = -\sum_{j=1}^{r} a_{ij}\varphi_j'(\pi-x,\lambda) = \psi_i'(x,\lambda)$$

so that, when we put $x = 0$, and use (2) and (3),

$$h\varphi_i(0,\lambda) = \varphi_i'(0,\lambda) = -\sum_{j=1}^{r} a_{ij}\varphi_j'(\pi,\lambda)$$

$$= -\sum_{j=1}^{r} a_{ij}\left(-H\varphi_j(\pi,\lambda)\right) = H\psi_i(0,\lambda) = H\varphi_i(0,\lambda).$$

We thus have

$$(h-H)\varphi_i(0,\lambda) = 0 \tag{2.8}$$

for all eigenvalues λ and for $1 \le i \le r_\lambda$. What we want, however, is to know that, for an arbitrary vector v in $\mathbb{R}^d$, $(h-H)v = 0$. Thus, suppose not – so that there exists $v \in \mathbb{R}^d$ such that $(h-H)v \ne 0$.

We will define a certain C^2 function $f(x)$, depending on v, such that $f(x)$ is in the domain of the operator determined by problem (1)–(3), and then we will use the uniform convergence of the Fourier series of f with respect to the eigenfunctions $\varphi_i(x,\lambda)$, and the boundary conditions (2) and (3), to conclude that, contrary to assumption, $(h-H)v = 0$.

In fact, $f(x)$ will have the form $M(x)v$, where $M(x)$ is a $d \times d$ matrix for each real x, chosen so that f is in the domain of the operator determined by problem (1)–(3). Our example is the polynomial

$$f(x) := \left[I + xh + \pi((x/\pi)^2 - (x/\pi)^3)(h + H + \pi Hh)\right] v, \quad \text{for} \quad 0 \le x \le \pi.$$

Then

$$f'(x) = \left[h + (2x/\pi - 3x^2/\pi^2)(h + H + \pi Hh)\right] v, \quad \text{for} \quad 0 \le x \le \pi.$$

By inspection, $f(0) = v$, $f'(0) = hv$, so (2) holds. To show that (3) holds, we set $x = \pi$ in the equations for $f(x)$ and $f'(x)$, obtaining $f(\pi) = (I + \pi h)v$, $f'(\pi) = (h - (h + H + \pi Hh))v = -H(I + \pi h)v = -Hf(\pi)$, so (3) holds, and thus f is in the domain of the operator determined by problem (1)–(3).

To write the Fourier series of f with respect to the eigenfunctions $\varphi_i(x,\lambda)$, let us denote by Λ the *set* of all the eigenvalues, λ, of the problem (1)–(3). For each $\lambda \in \Lambda$, we let r_λ denote the multiplicity of the eigenvalue λ. Then

$$f(x) = \sum_{\lambda\in\Lambda}\sum_{i=1}^{r_\lambda}\left(\int_0^\pi f(t)^T\varphi_i(t,\lambda)\,dt\right)\varphi_i(x,\lambda),$$

the sum being taken over all eigenvalues λ and, for each eigenvalue, over all i with $1 \le i \le r_\lambda$. The series converges to $f(x)$ everywhere in $[0,\pi]$, in particular at 0, to $f(0) = v = \sum_{\lambda\in\Lambda}\sum_{i=1}^{r_\lambda}\left(\int_0^\pi f(t)^T\varphi_i(t,\lambda)dt\right)\varphi_i(0,\lambda)$. But then

$$(h-H)v = \sum_{\lambda\in\Lambda}\sum_{i=1}^{r_\lambda}\left[\int_0^\pi f(t)^T\varphi_i(t,\lambda)\,dt\right](h-H)\varphi_i(0,\lambda) = 0,$$

by (2.8). This gives the desired contradiction, and completes the proof of sufficiency.

R e m a r k. The last argument shows that the set $\{\varphi_i(0,\lambda) : \lambda$ an eigenvalue and $1 \le i \le r_\lambda\}$ spans $\mathbb{R}^d$.

§ 3. The proof of Theorem A

The necessity is immediate. If equation (2.7) holds, we may set $x = 0$ and use the definition of a transmutation operator.

To prove that the suppositions (i) and (ii) are sufficient to assure the existence of a transmutation operator, we make use of the Goursat problem

$$\frac{\partial^2 K}{\partial x^2} - R_2(x)K = \frac{\partial^2 K}{\partial y^2} - KR_1(y), \quad 0 < y < x < \pi;$$

$$2\frac{d}{dx}(K(x,x)) = R_2(x) - R_1(x), \quad 0 \le x \le \pi, \quad \text{and} \quad K(0,0) = h_2 - h_1;$$

$$\left(\frac{\partial K}{\partial y} - Kh_1\right)\bigg|_{y=0} = 0.$$

An argument in a book of Levitan and Sargsjan (see [7, Ch. 6, § 2]) can be adapted straightforwardly (and lengthily!) to solve this Goursat problem in the vector-valued case. Also, see the argument for the existence of the Riemann function in V.A. Marchenko's book [5, Ch. 1, § 1].

Let $\varphi_1(x,\lambda)$ denote any solution of (2.5), in other words, an eigenfunction belonging to the eigenvalue λ. Let $\varphi_2(x,\lambda)$ denote a solution of the differential equation in (2.6), that satisfies the boundary condition $\varphi_2'(0,\lambda) = h_2\varphi_2(0,\lambda)$ at 0, and such that $\varphi_2(0,\lambda) = \varphi_1(0,\lambda)$. Thus $\varphi_2(x,\lambda)$ *is not necessarily an eigenfunction of the problem (2.6).* Straightforward calculation exploiting the fact that $K(x,y)$ is a solution of the Goursat problem shows that the function

$$y(x) := \varphi_1(x,\lambda) + \int_0^x K(x,t)\varphi_1(t,\lambda)\,dt$$

is a solution of the differential equation in (2.6), and that $y(x)$ satisfies the initial conditions

$$\begin{aligned} y(0) &= \varphi_1(0,\lambda) = \varphi_2(0,\lambda), \\ y'(0) &= \varphi_1'(0,\lambda) + K(0,0)\varphi_1(0,\lambda) \\ &= h_1\varphi_1(0,\lambda) + (h_2 - h_1)\varphi_1(0,\lambda) = h_2\varphi_1(0,\lambda) = h_2\varphi_2(0,\lambda). \quad (3.1) \end{aligned}$$

The uniqueness of the solution of an initial value problem implies that

$$\varphi_2(x,\lambda) = y(x) = \varphi_1(x,\lambda) + \int_0^x K(x,t)\varphi_1(t,\lambda)\,dt.$$

Now, by (i) and (ii), there are r_λ eigenfunctions $\varphi_{1i}(x,\lambda)$, $1 \le i \le r_\lambda$ of (2.5), and likewise r_λ eigenfunctions $\varphi_{2i}(x,\lambda)$, $1 \le i \le r_\lambda$ of (2.6), such that

$$\varphi_{1i}(0,\lambda) = \varphi_{2i}(0,\lambda).$$

The calculations that gave (3.1) can be applied to each of these pairs of eigenfunctions, $\varphi_{1i}(x,\lambda)$ and $\varphi_{2i}(x,\lambda)$, so that for each eigenvalue λ,

$$\varphi_{2i}(x,\lambda) = \varphi_{1i}(x,\lambda) + \int_0^x K(x,t)\varphi_{1i}(t,\lambda)\,dt, \quad 1 \le i \le r_\lambda.$$

This is the desired transmutation — interchanging the problems (2.5) and (2.6) shows the invertibility needed for a transmutation operator.

§ 4. The proof of Theorem B

We will restate Theorem B, using a slightly different notation for the eigenvalues and eigenfunctions, in which the eigenvalues are denoted λ_n, with repeated values according to multiplicity.

In addition to the hypotheses of Theorem A, assume that the systems of eigenfunctions $\{\varphi_{1,n}(x)\}$ *and* $\{\varphi_{2,n}(x)\}$ *are each orthogonal, though not necessarily normalized. For each eigenfunction* $\varphi_{1,n}$, *let* $\alpha_{1,n}^2 := \|\varphi_{1,n}\|_2^2$ *and similarly let* $\alpha_{2,n}^2 := \|\varphi_{2,n}\|_2^2$. *Suppose that the following series, defining a function* $\mathcal{F}(x,y)$,

$$\mathcal{F}(x,y) := \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{2,n}^2} - \frac{1}{\alpha_{1,n}^2} \right) \varphi_{1,n}(x)\varphi_{1,n}(y)^T,$$

converges uniformly in $0 \le y \le x \le \pi$. *Then, the kernel* $K(x,y)$ *of the transmutation operator from Theorem A is the solution of the Gelfand–Levitan equation*

$$K(x,y) + \mathcal{F}(x,y) + \int_0^x K(x,t)\mathcal{F}(t,y)dt = 0.$$

P r o o f. From the orthogonality and completeness of the eigenfunctions we can write

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_{1,n}^2} \varphi_{1,n}(x)\varphi_{1,n}(y)^T = I\delta(x-y) \tag{4.1}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x)\varphi_{2,n}(y)^T = I\delta(x-y), \tag{4.2}$$

where I is the $d \times d$ identity matrix and $\delta(x)$ is the Dirac delta-function. We claim that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x)\varphi_{1,n}(y)^T = 0, \quad 0 \le y < x. \tag{4.3}$$

To show this, we write $\varphi_{1,n}(y)$ in terms of $\varphi_{2,n}(y)$, by using the "inverse" version of (2.7):

$$\varphi_{1,n}(y) = \varphi_{2,n}(y) + \int_0^y \mathcal{L}(y,t)\varphi_{2,n}(t)dt. \tag{4.4}$$

Now we substitute this expression for $\varphi_{1,n}(y)$ into the left-hand side of (4.3), and so find that, if $0 \le y < x$, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x)\varphi_{1,n}(y)^T &= \sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x) \left(\varphi_{2,n}(y) + \int_0^y \mathcal{L}(y,t)\varphi_{2,n}(t)\,dt \right)^T \\
&= \sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x)\varphi_{2,n}^T(y) \\
&\quad + \int_0^y \left(\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x)\varphi_{2,n}^T(t) \right) \mathcal{L}^T(y,t)dt \\
&= I\delta(x-y) + \int_0^y \mathcal{L}(y,t)\delta(x-t)\,dt = 0
\end{aligned}$$

because $0 \le y < x$.

Having shown that, in a distributional sense, (4.3) holds, namely, that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x)\varphi_{1,n}(y)^T = 0, \quad \text{when} \quad 0 \le y < x,$$

let us now express each $\varphi_{2,n}(x)$ in the sum as a transmutation of $\varphi_{1,n}(x)$. That is,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x)\varphi_{1,n}(y)^T \\
= &\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \left(\varphi_{1,n}(x) + \int_0^x K(x,t)\varphi_{1,n}(t)dt \right) \varphi_{1,n}(y)^T
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{1,n}(x)\varphi_{1,n}(y)^T \\
&+ \int_0^x K(x,t) \left(\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{1,n}(t)\varphi_{1,n}(y)^T \right) dt \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{2,n}^2} - \frac{1}{\alpha_{1,n}^2} \right) \varphi_{1,n}(x)\varphi_{1,n}(y)^T \\
&+ \sum_{n=1}^{\infty} \frac{1}{\alpha_{1,n}^2} \varphi_{1,n}(x)\varphi_{1,n}(y)^T \\
&+ \int_0^x K(x,t) \left(\sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{1,n}(t)\varphi_{1,n}(y)^T \right) dt \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{2,n}^2} - \frac{1}{\alpha_{1,n}^2} \right) \varphi_{1,n}(x)\varphi_{1,n}(y)^T \\
&+ \sum_{n=1}^{\infty} \frac{1}{\alpha_{1,n}^2} \varphi_{1,n}(x)\varphi_{1,n}(y)^T \\
&+ \int_0^x K(x,t) \left[\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{2,n}^2} - \frac{1}{\alpha_{1,n}^2} \right) \varphi_{1,n}(t)\varphi_{1,n}(y)^T \right] dt \\
&+ \int_0^x K(x,t) \left(\sum_{n=1}^{\infty} \frac{1}{\alpha_{1,n}^2} \varphi_{1,n}(t)\varphi_{1,n}(y)^T \right) dt.
\end{aligned}
$$

Then, by definitions, and by (4.1) and (4.3),

$$
\begin{aligned}
0 &= \sum_{n=1}^{\infty} \frac{1}{\alpha_{2,n}^2} \varphi_{2,n}(x)\varphi_{1,n}(y)^T \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{2,n}^2} - \frac{1}{\alpha_{1,n}^2} \right) \varphi_{1,n}(x)\varphi_{1,n}(y)^T \\
&+ \sum_{n=1}^{\infty} \frac{1}{\alpha_{1,n}^2} \varphi_{1,n}(x)\varphi_{1,n}(y)^T \\
&+ \int_0^x K(x,t) \left[\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{2,n}^2} - \frac{1}{\alpha_{1,n}^2} \right) \varphi_{1,n}(t)\varphi_{1,n}(y)^T \right] dt \\
&+ \int_0^x K(x,t) \left(\sum_{n=1}^{\infty} \frac{1}{\alpha_{1,n}^2} \varphi_{1,n}(t)\varphi_{1,n}(y)^T \right) dt \\
&= \mathcal{F}(x,y) + I\delta(x-y)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^x K(x,t)\mathcal{F}(t,y)\,dt + \int_0^x K(x,t)\delta(t-y)dt \\
&= \mathcal{F}(x,y) + \int_0^x K(x,t)\mathcal{F}(t,y)\,dt + K(x,y),
\end{aligned}
$$

because $0 \le t \le y < x$. We have shown that, in a certain distribution sense,

$$
0 = \mathcal{F}(x,y) + \int_0^x K(x,t)\mathcal{F}(t,y)\,dt + K(x,y), \quad \text{for} \quad 0 \le y \le x \le \pi .
$$

Since the expression on the right is a continuous function, the desired Gelfand–Levitan equation holds. This completes the proof.

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Характеристика четных векторных задач Штурма–Лиувилля

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Назовем "четной" задачу Штурма–Лиувилля

$$-y'' + Q(x)y = \lambda\, y,\ \ 0 \le x \le \pi, \tag{1}$$

$$y'(0) - h\, y(0) = 0, \tag{2}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{3}$$

если $H = h$ и $Q(\pi - x) \equiv Q(x)$ на промежутке $[0, \pi]$. Рассмотрим векторно-значный случай, где потенциал $Q(x)$ есть вещественная симметричная $d \times d$ матрица при каждом $x \in [0, \pi]$, и элементы матрицы Q и их первые производные (в смысле распределений) все принадлежат $L^2[0, \pi]$. Положим, что h и H есть вещественные симметричные $d \times d$ матрицы.

Доказано, что векторно-значная задача Штурма–Лиувилля (1)–(3) является четной, если и только если для каждого собственного значения λ, кратность которого есть $r = r_\lambda$ (где $1 \le r \le d$ и через $\varphi_1(x, \lambda), \ldots, \varphi_r(x, \lambda)$ обозначены ортонормированные собственные функции, отвечающие λ), существует $r \times r$ матрица $A = (a_{ij})$ (которая может зависеть от λ и выбора базиса $\{\varphi_i(x, \lambda)\}_{i=1}^{r}$, но не зависит от x) такая, что

1) A есть ортогональная и симметричная,

2) при $1 \le i \le r$ $\varphi_i(\pi, \lambda) = \sum_{j=1}^{r} a_{ij}\varphi_j(0, \lambda)$.

В некотором смысле наша теорема может рассматриваться как обобщение результата Н. Левинсона [2].

Характеристика парних векторних задач Штурма–Ліувілля

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Назвемо "парною" задачу Штурма–Ліувілля

$$-y'' + Q(x)y = \lambda\, y,\ \ 0 \le x \le \pi, \tag{1}$$

$$y'(0) - h\, y(0) = 0, \tag{2}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{3}$$

якщо $H = h$ та $Q(\pi - x) \equiv Q(x)$ на проміжку $[0,\, \pi]$. Розглянемо векторно-значний випадок, де потенціал $Q(x)$ є дійсна симетрична $d \times d$ матриця при кожному $x \in [0, \pi]$, і елементи матриці Q та їх перші похідні (у сенсі розподілів) усі належать $L^2[0, \pi]$. Припустимо, що h і H дійсні симетричні $d \times d$ матриці.

Доведено, що векторно-значна задача Штурма–Ліувілля (1)–(3) є парною тоді і тільки тоді, коли для кожного власного значення λ, кратність якого є $r = r_\lambda$ (тут $1 \le r \le d$ і через $\varphi_1(x,\lambda), \ldots, \varphi_r(x,\lambda)$ позначено відповідні ортонормовані власні функції), існує $r \times r$ матриця $A = (a_{ij})$ (яка може залежати від λ та вибору базиса $\{\varphi_i(x,\lambda)\}_{i=1}^r$, але не залежить від x) така, що

1) A є ортогональна та симетрична,

2) при $1 \le i \le r$ $\varphi_i(\pi, \lambda) = \sum_{j=1}^r a_{ij}\varphi_j(0, \lambda)$.

В деякому сенсі наша теорема може розглядатися як узагальнення результату Н. Левінсона [2].