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Monge–Ampère operators and Jessen functions of holomorphic almost periodic mappings

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For a holomorphic almost periodic mapping f from a tube domain of $\mathbf{C}^n$ into $\mathbf{C}^q$, the properties of its Jessen function, i.e., the mean value of the function $\log|f|^2$, are studied. In particular, certain relations between the Jessen function and behavior of the mapping and its zero set are obtained. To this end certain operators Φ_l on plurisubharmonic functions are introduced in a way that for a smooth function u,

$$(\Phi_l[u])^l\,(dd^c|z|^2)^n = (dd^c u)^l \wedge (dd^c|z|^2)^{n-l}.$$

1. Introduction

In this paper we study relations between behavior of a holomorphic almost periodic mapping and distribution of its zero set. Our considerations are based on pluripotential theory methods and have required some developments in the machinery of the complex Monge–Ampère operators. Some of the results were announced in [10].

Let G be a convex domain in $\mathbf{R}^n$, $T_G = \mathbf{R}^n + iG = \{z = x + iy :\ x \in \mathbf{R}^n,\ y \in G\}$. A holomorphic mapping $f : T_G \to \mathbf{C}^q$, $q \le n$, is said to be *almost periodic in* T_G if for any $\epsilon > 0$ and any subdomain $G' \subset\subset G$ the inequality

$$\|f(z+\tau) - f(z)\|_{G',\infty} < \epsilon$$

holds for all τ from a relatively dense subset X_ϵ of $\mathbf{R}^n$. Here

$$\|g(z)\|_{G',\infty} = \sup\{|g(z)| :\ z \in T_{G'}\}. \tag{1.1}$$

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An example of such a mapping is one whose components f_k have the form

$$\sum_j c_{kj} \exp\{i(z,\lambda_j)\},\ c_{kj} \in \mathbf{C},\ \lambda_j \in \mathbf{R}^n.$$

A classical result of B. Jessen and H. Tornehave [6] for $n = q = 1$ states that for a holomorphic almost periodic function f in a strip $T_{(a,b)} = \{x + iy : -\infty < x < \infty,\ a < y < b\}$, the function $\log|f|$ has the mean value

$$A_f(y) = \lim_{t\to\infty} \frac{1}{2t} \int\limits_{-t}^{t} \log|f(x+iy)|\, dx$$

which is a convex function on (a,b) with the following remarkable property: if the derivative A'_f exists at points a_1 and b_1, $a < a_1 < b_1 < b$, then the set of zeros of the function f in the strip $T_{(a_1,b_1)}$ has the density equal to

$$\frac{1}{2\pi}\left(A'_f(b_1) - A'_f(a_1)\right).$$

This result was extended in [12] to almost periodic holomorphic functions $f : T_G \to \mathbf{C}$. Namely, the function $\log|f(x+iy)|$ was shown to possess its mean value $A_f(y)$ along any plane $\{x + iy :\ x \in \mathbf{R}^n\}$, the function A_f (*the Jessen function*) being convex, and the density of the zero set Z_f of the function f in any subdomain $T_{G'}$, $G' \subset\subset G$, exists and coincides with the Riesz measure $\mu_f(G')$ of the function A_f, provided $\mu_f(\partial G') = 0$.

The case $q > 1$ was considered in [13]. Under certain condition of regularity of a mapping f (see the definition of the class R_q in the next section), a current $\tilde{A}_f^{(q-1)}$ of bidegree $(q-1, q-1)$ was constructed as the mean value of the current $\log|f|^2(dd^c\log|f|^2)^{q-1}$, so the density of Z_f in a domain $T_{G'}$ exists and coincides with the trace measure of the current $dd^c\tilde{A}_f^{(q-1)}$ in $\{\max|x_j| < 1\} + iG'$, $G' \subset\subset G$. The existence of mean values $\tilde{A}_f^{(l)}$ of the currents $\log|f|^2(dd^c\log|f|^2)^l$, $l < q-1$, was proved in [9]. The current $\tilde{A}_f^{(0)}$ is equivalent to a convex function and can be regarded as the Jessen function of the mapping f. It is a much simpler object than the current $dd^c\tilde{A}_f^{(q-1)}$, and, on the other hand, it could give us some information on the distribution of zeros of the mapping. A natural conjecture that $dd^c\tilde{A}_f^{(l-1)} = \left[dd^c\tilde{A}_f^{(0)}\right]^l$ was shown to fail for $l > 1$ [9]. Moreover, an example by Ronkin (also described in [9]) proves that, generally speaking, $\operatorname{supp} dd^c\tilde{A}_f^{(q-1)} \neq \operatorname{supp}\left[dd^c\tilde{A}_f^{(0)}\right]^q$. Nevertheless one can hope for some relations between these sets. Since for a subdomain G' of G, $Z_f \cap T_{G'} = \emptyset$ if and only if $dd^c\tilde{A}_f^{(q-1)}(T_{G'}) = 0$, the question arises: given the function $\tilde{A}_f^{(0)}$, to find tube domains $T_{G'}$ where the mapping f has no zeros.

To answer it we study relations between the Jessen function $\tilde{A}_f^{(0)}$ and the currents $\tilde{A}_f^{(l)}$, $l < q$. We show that for $f \in R_q$, $q \le n$,

$$\operatorname{supp} dd^c \tilde{A}_f^{(q-1)} \subset \operatorname{supp} \left[dd^c \tilde{A}_f^{(0)}\right]^q \tag{1.2}$$

(Corollary 3.1); for the case $q = n$ it was earlier proved in [9]. Moreover, under no assumption on regularity of a mapping f, the condition $\left[dd^c \tilde{A}_f^{(0)}\right]^q = 0$ on a domain $T_{G'}$ implies $Z_f \cap T_{G'} = \emptyset$.

Then we obtain that, similarly,

$$\operatorname{supp} dd^c \tilde{A}_f^{(l-1)} \subset \operatorname{supp} \left[dd^c \tilde{A}_f^{(0)}\right]^l, \quad \forall l < q \tag{1.3}$$

(Corollary 5.5). We also show that the degeneration of the currents $dd^c \tilde{A}_f^{(l-1)}$ for $l < q$ and $f \in R_q$ cannot be of local character: their supports are either T_G or $\emptyset$, and the latter is the case of rank $f \le l$. So if $\left[dd^c \tilde{A}_f^{(0)}\right]^l = 0$ on some $T_{G'}$ then f has no zeros in the whole T_G.

As a consequence it follows that if a convex, piecewise linear function A in $G \subset \mathbf{R}^n$ is the Jessen function of a mapping $f \in R_q(T_G)$ for $q > 1$, then A is linear. It contrasts with the case $q = 1$, for which a description of the (non-empty) set of piecewise linear Jessen functions was obtained in [14]. Moreover, every convex piecewise linear function on an interval (a, b) is the Jessen function of some holomorphic almost periodic function in the strip $T_{(a,b)}$ [6] (note that every holomorphic almost periodic function $f : T_G \to \mathbf{C}$ is regular).

The proof of (1.2) makes use of technique of maximal plurisubharmonic and convex functions, while the situation for $l < q$ in (1.3) is treated in an absolutely different way. The Monge–Ampère operator $(dd^c)^l$ has the property $(dd^c(u + v))^l \ge (dd^c u)^l + (dd^c v)^l$ for plurisubharmonic functions u and v, however it is not possible to relate

$$\left(dd^c \frac{1}{N} \sum_1^N u_j\right)^l$$

to

$$\frac{1}{N} \sum_1^N (dd^c u_j)^l$$

independently of N. So one can hardly hope for good relations concerning mean values of families of plurisubharmonic functions. Besides, as is known the Monge–Ampère operators cannot be defined for all plurisubharmonic functions.

To avoid such difficulties we introduce operators Φ_l acting on plurisubharmonic functions as follows. Given a smooth plurisubharmonic function u, $\Phi_l[u]$ is a function g_l such that

$$(g_l)^l (dd^c|z|^2)^n = (dd^c u)^l \wedge (dd^c|z|^2)^{n-l}, \tag{1.4}$$

and we extend it to be a measure for an arbitrary plurisubharmonic function u (as was done for $l = n$ by Bedford and Taylor [1] when solving the complex Monge–Ampère equation). These operators are well adapted to functions of the form $u = \log|f|^2$, where f is a holomorphic mapping into $\mathbf{C}^q$. If the zero set of f is a complete intersection then $\Phi_l[u]$ with $l < q$ is absolutely continuous with respect to the Lebesgue measure in $\mathbf{C}^n$ and its density g_l satisfies (1.4). The operators Φ_l have the concavity property

$$\Phi_l[t\,u + (1-t)\,v] \geq t\Phi_l[u] + (1-t)\Phi_l[v]\,,$$

and we hope they could be of independent interest.

The paper is organized as follows. Section 2 contains preliminaries on holomorphic almost periodic mappings. In Section 3 we prove the conclusion (1.2). We introduce and study the operators Φ_l in Section 4, and we obtain relations (1.3) in Section 5.

2. Notations and preliminary results

Throughout the paper G is a convex domain in $\mathbf{R}^n$ and $T_G = \mathbf{R}^n + iG = \{z = x + iy \in \mathbf{C}^n : x \in \mathbf{R}^n, y \in G\}$ is a tube domain in $\mathbf{C}^n$ with the base G. For $t \in \mathbf{R}^n_+$ and $\mathbf{1} = (1,\ldots,1) \in \mathbf{R}^n_+$, $t^{\mathbf{1}} = t_1 \cdot \ldots \cdot t_n$, $m(t) = \min_{1\leq j\leq n} t_j$, $t \cdot x = (t_1x_1,\ldots,t_nx_n) \in \mathbf{R}^n$,

$$\Pi_t(x^0) = \{x \in \mathbf{R}^n : |x_j - x_j^0| < t_j,\ 1 \leq j \leq n\}, \quad \Pi_t = \Pi_t(0).$$

Further, $\mathcal{M}_{\updownarrow}$ is the collection of all ordered samples of the length $l \leq n$ from the set $\{1,\ldots,n\}$; for $I = (i_1,\ldots,i_l) \in \mathcal{M}_l$ and $z \in \mathbf{C}^n$ we set $z_I = (z_{i_1},\ldots,z_{i_l}) \in \mathbf{C}^l$, $z^I = z_{i_1} \cdot \ldots \cdot z_{i_l}$ and $dz^I = dz_{i_1} \wedge \ldots \wedge z_{i_l}$. We denote by $CNV(G)$ the collection of all convex functions on G and by $PSH(\Omega)$ the collection of all plurisubharmonic functions on a domain Ω.

Let $\mathcal{D}_{p,q}(\Omega)$ be the space of smooth and compactly supported (p,q)-differential forms on a domain $\Omega \subset \mathbf{C}^n$, and let $\mathcal{D}'_{p,q}(\Omega)$ be its dual space of currents of bidimension (p,q) (bidegree $(n-p, n-q)$) (see [7]). A current $T \in \mathcal{D}'_{p,p}(\Omega)$ is called positive if $\langle T, \phi\rangle \geq 0$ for every $\phi \in \mathcal{D}_{p,p}(\Omega)$ of the form

$$\phi = (i\lambda_1 \wedge \bar\lambda_1) \wedge (i\lambda_2 \wedge \bar\lambda_2) \wedge \ldots \wedge (i\lambda_p \wedge \bar\lambda_p),$$

where $\lambda_j \in \mathcal{D}_{1,0}(\Omega)$, $1 \le j \le p$. The coefficients of positive currents are known to be measures.

We set $d = \partial + \bar\partial$, $d^c = (\partial - \bar\partial)/4i$, so $\beta_l = \left(dd^c|z|^2\right)^l/l!$ is the standard Euclid volume form in $\mathbf{C}^l$. A positive current $T \in \mathcal{D}'_{0,0}(\Omega)$ has the form $T = *T\beta_n$, where $*T$ is a positive measure (or function). Throughout the paper we will identify such currents T with the corresponding measures (functions) $*T$.

Let F be a holomorphic mapping from a domain $\Omega \subset \mathbf{C}^n$ into $\mathbf{C}^q$, $q \le n$, let $Z_F = F^{-1}(0)$ be its zero set regarded as a holomorphic chain and $|Z_F|$ be its support. We will say that Z_F is a *complete intersection* if $\dim |Z_F| \le n - q$, i.e., $\dim |Z_F| = n - q$ or $Z_F = \emptyset$. As is known [5], for such a mapping the currents

$$a_F^{(l)} := (dd^c \log |F|^2)^l$$

and

$$A_F^{(l)} := \log |F|^2 a_F^{(l)}$$

are correctly defined for $l < q$, their coefficients $a_{IJ}^{(l)}$ and $A_{IJ}^{(l)}$ are locally summable functions, $dd^c A_F^{(l)} = a_F^{(l+1)} \ge 0$, and the current $a_F^{(q)} := dd^c A_F^{(q-1)}$ coincides with the current of integration over the chain Z_F (up to the multiplication constant π^{-q}) . In particular, the $2\,(n-q)$-dimensional volume $V_F(\Omega')$ of Z_F in a domain $\Omega' \subset\subset \Omega$ is equal (up to π^{-q}) to the trace measure of $a_F^{(q)}$ in Ω':

$$V_F(\Omega') = \frac{1}{\pi^q} \int\limits_{\Omega'} a_F^{(q)} \wedge \beta_{n-q}.$$

Let now $f : T_G \to \mathbf{C}^q$, $1 \le q \le n$, be a holomorphic almost periodic mapping in T_G. It means that for any sequence $\{h_j\} \subset \mathbf{R}^n$ one can choose a subsequence $\{h_{j_k}\}$ such that the mappings $f(z + h_{j_k})$ converge as $k \to \infty$ with respect to any norm $\|\cdot\|_{G',\infty}$, $G' \subset\subset G$ (see (1.1)). This is equivalent to the definition given in the introduction.

An almost periodic mapping f is said to be *regular* if Z_F is a complete intersection for each its limit mapping F [13]. The class of all regular holomorphic almost periodic mappings $T_G \to \mathbf{C}^q$ will be denoted by $R_q\,(T_G)$.

For $f \in R_q\,(T_G)$, $t \in \mathbf{R}^n_+$ and $l < q$ we set

$$\begin{aligned} a_{f,t}^{(l)} &= \sum_{I,J} a_{IJ}^{(l)}\,(t\cdot x + iy) dz^I \wedge d\bar z^J, \\ A_{f,t}^{(l)} &= \sum_{I,J} A_{IJ}^{(l)}\,(t\cdot x + iy) dz^I \wedge d\bar z^J, \end{aligned}$$

where $a_{IJ}^{(l)}$ and $A_{IJ}^{(l)}$ are the coefficients of the currents $a_f^{(l)}$ and $A_f^{(l)}$, respectively.

Theorem 2.1 [13, 9]. *Let $f \in R_q(T_G)$, $l < q$. Then*

(1) the currents $a^{(l)}_{f,t}$ and $A^{(l)}_{f,t}$ converge in the weak topology of currents as $m(t) \to \infty$ to some currents $\tilde{a}^{(l)}_f$ and $\tilde{A}^{(l)}_f$, respectively;

(2) the coefficients of $\tilde{a}^{(l)}_f$ and $\tilde{A}^{(l)}_f$ are locally summable functions depending on y only;

(3) $dd^c\tilde{A}^{(l-1)}_f = \tilde{a}^{(l)}_f \geq 0$;

(4) if a measure μ_f on G is defined so that given Borel set $K \subset\subset G$, $\mu_f(K)$ is equal to the Kähler mass of the current $\tilde{a}^{(q)}_f := dd^c\tilde{A}^{(q-1)}_f$ on the set $\Pi_{\mathbf{1}} + iK$, then for any domain $G' \subset\subset G$ with $\mu_f(\partial G') = 0$ there exists

$$\lim_{m(t)\to\infty} \frac{1}{t^{\mathbf{1}}} V_f(\Pi_t + iG') = \pi^{-q}\mu_f(G').$$

The coefficients $\tilde{a}^{(l)}_{IJ}$ and $\tilde{A}^{(l)}_{IJ}$ are L^1_{loc}-limits of the coefficients $a^{(l)}_{IJ}$ and $A^{(l)}_{IJ}$ of the currents $a^{(l)}_f$ and $A^{(l)}_f$, respectively:

$$\begin{aligned}
\tilde{a}^{(l)}_{IJ}(y) &= \lim_{m(t)\to\infty} \fint_{\Pi_t(x^0)} a^{(l)}_{IJ}(x+iy)\,dx, \\
\tilde{A}^{(l)}_{IJ}(y) &= \lim_{m(t)\to\infty} \fint_{\Pi_t(x^0)} A^{(l)}_{IJ}(x+iy)\,dx, \quad \forall x^0 \in \mathbf{R}^n.
\end{aligned}$$

Here

$$\fint_D g\,d\tau = \frac{1}{\tau(D)} \int_D g\,d\tau \tag{2.1}$$

is the mean value of a function g over a set D with respect to a measure τ.

Theorem 2.2 [9]*. *Let $f \in R_q(T_G)$. Then $\tilde{a}^{(q)}_f = 0$ on a domain $T_{G'}$, $G' \subset G$, if and only if $|Z_f| \cap T_{G'} = \emptyset$.*

The current (function) $\tilde{A}^{(0)}_f$ is equivalent to a convex function A_f which is called the *Jessen function* of the mapping f.

Theorem 2.3 [9]**. *Let f be a holomorphic almost periodic mapping from T_G to $\mathbf{C}^q$. Then*

$$\fint_{\Pi_t(x^0)} \log|f(x+iy)|^2\,dx \to A_f(y) \in CNV(G) \text{ as } m(t) \to \infty,$$

*For $n = 1$ see [6], for $n > 1$, $q = 1$ see [14].

**For $q = 1$ see [12].

uniformly in $x^0 \in \mathbf{R}^n$ and $y \in G'$ for each $G' \subset\subset G$. Besides, if $f \in R_q(T_G)$ then the function A_f is equivalent to $\tilde{A}_f^{(0)}$.

The function A_f was shown to be a uniform limit of the mean values $A_{f,\theta}$ of the almost periodic functions $\log|f|_\theta^2$ as $\theta \to 0$, where

$$|f|_\theta := \max\{|f|, \theta\}. \tag{2.2}$$

3. A relation between the Jessen function and distribution of zeros

As was shown in [9], if $(dd^c A_f)^n = 0$ for $f \in R_n(T_G)$ on a domain $T_{G'}$, $G' \subset G$, then $|Z_f| \cap T_{G'} = \emptyset$. Here we extend this result to almost periodic mappings $f : T_G \to \mathbf{C}^q$, $q \le n$. To this end we need to recall some facts concerning the real Monge–Ampère operator and maximal convex functions [11].

The real Monge–Ampère operator $\mathbf{M}$ is defined on smooth functions $v(y)$, $y \in \mathbf{R}^n$, by the equation

$$\mathbf{M}v(y) = \det\left(\frac{\partial^2 v}{\partial y_j \partial y_k}\right)_{j,k=1}^n$$

and extends in a unique way to arbitrary convex functions. Besides, for any function $v(y) \in CNV(D)$ regarded as a function from $PSH(T_D)$,

$$\mathbf{M}v\, dx = c_n (dd^c v)^n.$$

The Dirichlet problem

$$\begin{cases} \mathbf{M}v(y) = 0, & y \in D, \\ v(y) = \phi(y), & y \in \partial D \end{cases}$$

in a bounded strictly convex domain D for any function $\phi \in C(\partial D)$ has a unique solution $v_\phi(y)$ in the class $CNV(D)$ of all convex functions on D. Furthermore, the solution is the upper envelope of the set of convex functions in $\bar{D}$ not exceeding ϕ on ∂D:

$$v_\phi(y) = \sup\{v(y) \in CNV(D) :\ v(y') \le \phi(y'),\ \forall y' \in \partial D\}. \tag{3.1}$$

Theorem 3.1. *Let f be a holomorphic almost periodic mapping from $T_G \subset \mathbf{C}^n$ to $\mathbf{C}^q$, $q \le n$, and $(dd^c A_f)^q = 0$ on some domain T_{G_0}, $G_0 \subset\subset G$. Then $|Z_f| \cap T_{G_0} = \emptyset$.*

An immediate consequence of theorems 2.2 and 3.1 is the following

Corollary 3.1. *Let* $f \in R_q(T_G)$, $q \le n$. *Then*

$$\operatorname{supp} \tilde{a}_f^{(q)} \subset \operatorname{supp} (dd^c A_f)^q .$$

P r o o f of T h e o r e m 3.1. Let, contrary to the statement, $|Z_f| \cap T_{G_0} \ne \emptyset$. Then there exists a point $z^0 \in T_{G_0}$ such that $f(z^0) = 0$ and $\dim |Z_f| \cap \{z \in \mathbf{C}^n : z_I = z_I^0\} = 0$ for some $I \in \mathcal{M}_{n-q}$. For the sake of brevity we can take $z^0 = 0$ and $I = \{q+1, \ldots, n\}$. Points of the space $\mathbf{C}^n$ will be denoted by $z = (z', z'')$, where $z' \in \mathbf{C}^q$ and $z'' \in \mathbf{C}^{n-q}$.

Choose domains $G' \subset \mathbf{R}^q$ and $G'' \subset \mathbf{R}^{n-q}$ such that $0 \in G' \times G'' \subset\subset G_0$ and consider the family of the holomorphic mappings

$$f_s(z') := f(z' + s', s'') : T_{G'} \to \mathbf{C}^q,$$

where $s = (s', s'') \in \mathbf{R}^n$.

Let $B' \subset\subset G'$ be a ball in $\mathbf{R}^q$ with the center at 0. As the zero set of the mapping f_0 is discrete, one can choose a bounded neighborhood U' of 0 in $\mathbf{C}^q$ in such a way that for some $\theta > 0$ the inequality $|f_0| \ge 2\theta$ holds on $\partial U'$. In view of almost periodicity of the mapping f there exists a relatively dense subset $\{h_j\}$ of $\mathbf{R}^n$ such that

$$\|f(z + h_j) - f(z)\|_{G_0,\infty} < \frac{\theta}{4}, \quad j = 1, 2, \ldots, \tag{3.2}$$

and for some bounded neighbourhood V_0 of 0 in $\mathbf{R}^n$

$$\|f(z + s) - f(z)\|_{G_0,\infty} < \frac{\theta}{4}, \quad \forall s \in V_0. \tag{3.3}$$

Set $V_j = V_0 + h_j$ and $V = \bigcup_j V_j$; we can also arrange the domains V_j to be disjoint keeping the set $\{h_j\}$ relatively dense in $\mathbf{R}^n$. By (3.2) and (3.3),

$$\|f(z + s) - f(z)\|_{G_0,\infty} < \frac{\theta}{2}, \quad \forall s \in V,$$

then

$$\|f_s(z') - f_0(z')\|_{G',\infty} < \frac{\theta}{2}, \quad \forall s \in V, \tag{3.4}$$

and, by Rouché's theorem, each mapping $f_s(z')$ has a zero in the domain U' and $|f_s| > \theta$ on $\partial U'$, $\forall s \in V$.

Set

$$\begin{aligned} W_j &= \{(z', s) \in U' \times V_j : |f_s(z')| < \theta\}, \\ W &= \bigcup_j W_j \subset T_{B'} \times V, \end{aligned}$$

and define the function

$$v^{(\theta)}(z',s)=\begin{cases}\log|f_s(z')|^2, & \text{when} \quad (z',s)\in(T_{G'}\times\mathbf{R}^n)\setminus W,\\ 2\log\theta, & \text{when} \quad (z',s)\in W.\end{cases}$$

For each fixed $s\in\mathbf{R}^n$, $v^{(\theta)}(z',s)$ is plurisubharmonic in $T_{G'}$, coincides with $\log|f_s(z')|^2$ in $T_{G'}\setminus T_{B'}$ and

$$\log|f_s(z')|^2\le v^{(\theta)}(z',s)\le\log|f_s(z')|^2_\theta$$

everywhere in $T_{G'}$ (where $|\cdot|_\theta$ is defined by (2.2)).

Set, further,

$$\tilde{v}^{(\theta)}(z')=\limsup_{\zeta'\to z'}\limsup_{m(t)\to\infty}\fint_{\Pi_t} v^{(\theta)}(\zeta',s)\,ds.$$

It is a plurisubharmonic function in $T_{G'}$ which satisfies the inequalities

$$A_f(y',0)\le\tilde{v}^{(\theta)}(z')\le A_{f,\theta}(y',0)$$

in $T_{G'}$ and coincides with $A_f(y',0)$ in $T_{G'}\setminus T_{B'}$ (see the last paragraph in the previous section).

The function

$$w^{(\theta)}(y'):=\sup\{\tilde{v}^{(\theta)}(x'+iy'):\ x'\in\mathbf{R}^q\}$$

is convex in G' and equal to $A_f(y',0)$ in $G'\setminus B'$. We are going to show that

$$w^{(\theta)}(0)>A_f(0). \tag{3.5}$$

By (3.4), $|f_s(0)|<\frac{\theta}{2}$, $\forall s\in V$, therefore

$$\begin{aligned} w^{(\theta)}(0)-A_f(0) &\ge \liminf_{m(t)\to\infty}\fint_{\Pi_t}\left[v^{(\theta)}(0,s)-\log|f(s)|^2\right]ds\\ &\ge \liminf_{m(t)\to\infty}\frac{1}{2^n t^{\mathbf{1}}}\int_{\Pi_t\cap V}\left[v^{(\theta)}(0,s)-\log|f(s)|^2\right]ds\\ &\ge \liminf_{m(t)\to\infty}\frac{1}{2^n t^{\mathbf{1}}}\int_{\Pi_t\cap V}\left[2\log\theta-2\log\frac{\theta}{2}\right]ds\\ &= \log 4\ \mathrm{mes}_n V_0>0, \end{aligned}$$

that proves (3.5).

So, the function $A_f(y',0)$ is not maximal in B' and thus, by (3.1), $\mathbf{M}A(y',0)\neq 0$ on B'. Hence,

$$[dd^cA_f(y',0)]^q\neq 0 \text{ on } T_{G_0}. \tag{3.6}$$

It implies that $(dd^cA_f)^q\neq 0$ on T_{G_0}, too. Indeed, let $A_{(\epsilon)}(y)$ be a family of smooth convex functions in G_0 decreasing to $A_f(y)$ as $\epsilon\to 0$. By (3.6) there is a nonnegative function $\phi(z')\in\mathcal{D}(T_{B'})$ such that

$$\int\limits_{T_{G'}}[dd^cA_f(y',0)]^q\phi(z')>0. \tag{3.7}$$

Take a function $\psi(z'')\in\mathcal{D}(T_{G''})$, $\psi\geq 0$, $\psi(0)=1$. Then by the monotone convergence theorem for the Monge–Ampère currents [2],

$$\begin{aligned}\int\limits_{T_{G'}}[dd^cA_f(y',0)]^q\phi(z') &= \lim_{\epsilon\to 0}\int\limits_{T_{G'}}[dd^cA_{(\epsilon)}(y',0)]^q\phi(z')\\ &= \lim_{\epsilon\to 0}\int\limits_{T_{G_0}}[dd^cA_{(\epsilon)}(y',y'')]^q\phi(z')\wedge[dd^c\log|z''|^2]^{n-q}\psi(z'')\\ &= \int\limits_{T_{G_0}}[dd^cA_f(y)]^q\wedge[dd^c\log|z''|^2]^{n-q}\phi(z')\psi(z'').\end{aligned}$$

By (3.7), the latter value is strictly positive, therefore $(dd^cA_f)^q\neq 0$ on T_{G_0}. So, we come to the contradiction and the theorem is proved.

4. Operators $\Phi_l[u]$

In order to study properties of the currents $(dd^cA_f)^l$ with $l<q$ we are going to introduce some operators Φ_l acting on plurisubharmonic functions. In case of a smooth function u, $\Phi_l[u]$ will coincide with $g^{1/l}\beta_n$, where g is the density of the absolutely continuous measure $(dd^cu)^l\wedge\beta_{n-l}$. The operator $\Phi_n[u]$ was introduced by Bedford and Taylor in [1] by means of a measure theoretic construction from [4]. The definition and most of the properties of $\Phi_l[u]$ for $l<n$ are similar to ones of $\Phi_n[u]$, and we will refer for their proofs to [1].

Let C_+ denote the cone of positive semi-definite $n\times n$ matrices imbedded in $\mathbf{C}^{n^2}$. Let $A\in C_+$ and let $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$ be its eigenvalues. Consider the function

$$\Psi_l(A)=\left[\sum_{J\in\mathcal{M}_l}\lambda^J\right]^{\frac{1}{l}},\quad l\leq n$$

(as usual, $\lambda^J = \lambda_{j_1} \cdot \lambda_{j_2} \cdot \ldots \cdot \lambda_{j_l}$ for $J = (j_1, j_2, \ldots, j_l)$).

It is clear that Ψ_l is a continuous nonnegative and positively homogeneous of degree 1 function on C_+. The crucial point is that it is concave [8]:

$$\Psi_l(\tau A + (1-\tau)B) \geq \tau\Psi_l(A) + (1-\tau)\Psi_l(B), \quad 0 < \tau < 1, \quad A, B \in C_+.$$

So the construction from [4] can be applied to extend the functions Ψ_l to matrix valued measures.

Let $M_+(\Omega)$ be the set of all vector valued Borel measures μ on a domain Ω with values in the cone C_+, i.e., $\mu = (\mu_{ij})_{i,j=1}^n$ and $\mu(E) \in C_+$ for each Borel set $E \subset\subset \Omega$. For $\mu \in M_+$ choose a nonnegative scalar Borel measure λ on Ω such that μ is absolutely continuous with respect to λ (e.g., $\lambda = \sum_{1 \leq i,j \leq n} |\mu_{ij}|$, where $|\mu_{ij}|$ is the total variation measure of μ_{ij}). By the Radon–Nikodym theorem,

$$d\mu = h\, d\lambda,$$

where $h : \Omega \to C_+$.

Set

$$\Psi_l(\mu) = \Psi_l(h)\lambda.$$

It defines a nonnegative Borel measure $\Psi_l(\mu) \in M_+(\Omega)$ which does not depend on the choice of the measure λ (see [4]).

Proposition 4.1. *Let $\mu, \nu \in M_+(\Omega)$. Then*

(1) $\Psi_l(\alpha\mu) = \alpha\Psi_l(\mu)$ for $\alpha \geq 0$;

(2) if μ and ν are mutually singular, then $\Psi_l(\mu + \nu) = \Psi_l(\mu) + \Psi_l(\nu)$;

(3) $\Psi_l(\mu)$ is absolutely continuous with respect to μ;

(4) $\Psi_l(\tau\mu + (1-\tau)\nu) \geq \tau\Psi_l(\mu) + (1-\tau)\Psi_l(\nu), \quad 0 < \tau < 1$;

(5) if $\chi \geq 0$ is a continuous function with compact support, then

$$\Psi_l(\mu * \chi) \geq \Psi_l(\mu) * \chi$$

on any open set Ω' with $\Omega' + \operatorname{supp}\chi \subset \Omega$;

(6) for any Borel subset E of Ω

$$\Psi_l(\mu)(E) = \inf \sum_{j=1}^{\infty} \Psi_l(\mu(E_j)),$$

where the infimum is taken over all disjoint coverings $\{E_j\}$ of the set E;

(7) for any sequence of Borel measures $\mu^j \in M_+(\Omega)$ which converges weakly to μ,

$$\Psi_l(\mu) \geq \varlimsup_{j\to\infty} \Psi_l(\mu^j)$$

provided the measures $\Psi_l(\mu^j)$ converge weakly;

(8) $\Psi_{l+1}(\mu) \le C_l\,\Psi_l(\mu)$, $1 \le l < n$, *where the constants* C_l *depend on* l *and* n *only.*

P r o o f. The properties (1)–(4) are immediate consequence of the definition of Ψ_l. The proofs of (5)–(7) are the same as of Proposition 5.4, Lemma 5.5, and Proposition 5.6 of [1], respectively. Relations (8) follow from Maclaurin's inequality for elementary symmetric functions.

Now let u be a plurisubharmonic function in a domain $\Omega \subset \mathbf{C}^n$ and let

$$H = \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)_{i,j=1}^{n}$$

be its complex Hessian. As is known, $H \in M_+(\Omega)$.

We set

$$\Phi_l[u] = (l\,!)^{1/l}\Psi_l(H), \quad l \le n.$$

Proposition 4.1 implies the following properties of the operator Φ_l.

Proposition 4.2. *Let* $u, v, u_j \in PSH(\Omega)$. *Then*

(1) $\Phi_:[\alpha u] = \alpha\Phi_l[u]$ *for* $\alpha \ge 0$;

(2) $\Phi_l[\tau u + (1-\tau)v] \ge \tau\Phi_l[u] + (1-\tau)\Phi_l[v], \quad 0 < \tau < 1$;

(3) if $\chi \ge 0$ *is a continuous function with compact support, then*

$$\Phi_l[u * \chi] \ge \Phi_l[u] * \chi$$

on any open set Ω' *with* $\Omega' + \operatorname{supp}\chi \subset \Omega$;

(4) if $u_j \to u$ *in* $L^1_{loc}(\Omega)$ *(or, that is the same, in* $\mathcal{D}'(\Omega)$*), then*

$$\Phi_l[u] \ge \lim_{j\to\infty} \Phi_l[u_j]$$

provided the sequence $\Phi_l[u_j]$ *converges weakly;*

(5)

$$\lim_{\epsilon\to 0} \Phi_l[u * \chi_\epsilon] = \Phi_l[u],$$

where $\chi_\epsilon(z) = \epsilon^{-2n}\chi\left(\frac{z}{\epsilon}\right)$ *is a standard smoothing kernel;*

(6) $\Phi_{l+1}[u] \le C_l\,\Phi_l[u]$, $1 \le l < n$, *where* C_l *depend on* l *and* n *only.*

Note that $\Phi_1[u] = dd^c u \wedge \beta_{n-1}$ for any plurisubharmonic function u, and so these properties for $l = 1$ follow immediatly from linearity of the operator dd^c and its continuity in the distribution topology (and of course, with the equalities instead of the inequalities in (2)–(4)).

A relation between $\Phi_l[u]$ and $(dd^c u)^l$ in case of a smooth plurisubharmonic function u is given by the following statement.

Proposition 4.3. *If $u \in PSH(\Omega) \cap C^2(\Omega)$ then*

$$(g_l)^l \beta_n = (dd^c u)^l \wedge \beta_{n-l}, \tag{4.1}$$

where g_l is the density of the measure $\Phi_l[u]$ with respect to the Lebesgue measure in $\mathbf{C}^n$.

P r o o f. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the complex Hessian H of the function u are solutions of the equation $\det(H - \lambda I) = 0$, where I is the unit $n \times n$ matrix. This equation can also be written in the form

$$\left(dd^c u - \lambda dd^c |z|^2\right)^n = 0.$$

But

$$\begin{aligned} \left(dd^c u - \lambda dd^c |z|^2\right)^n &= \sum_{l=0}^{n} \lambda^{n-l} (-1)^{n-l} \binom{n}{l} (dd^c u)^l \wedge \left(dd^c |z|^2\right)^{n-l} \\ &= (-1) \sum_{l=0}^{n} \lambda^{n-l} \frac{(-1)^l n!}{l!} (dd^c u)^l \wedge \beta_{n-l}. \end{aligned}$$

Thus by the Viete theorem we have the desired equations (4.1).

In contrast to the operators $\Phi_l[u]$, the complex Monge–Ampère operators $(dd^c u)^l$, $l > 1$, cannot be defined (at least as positive currents) for all plurisubharmonic functions u. We obtain the following relations between $\Phi_l[u]$ and $(dd^c u)^l$ provided the latter is well-defined.

Theorem 4.1. *Let $u \in PSH(\Omega)$ be such that for some $l > 1$ its regularizations $u_\epsilon = u * \chi_\epsilon$ generate the family of Borel measures*

$$\{(dd^c u_\epsilon)^l \wedge \beta_{n-l}\}_\epsilon$$

which is bounded on each compact subset of Ω. Then

(1) $\Phi_l[u]$ is absolutely continuous with respect to the Lebesgue measure dm_n on $\mathbf{C}^n$; moreover, if $\Phi_l[u] = g_l\, dm_n$, then $g_l \in L^l_{loc}(\Omega)$;

(2) if

$$(dd^c u)^l \wedge \beta_{n-l} = G_l\, dm_n + d\nu$$

is the Lebesgue decomposition of the measure $(dd^c u)^l \wedge \beta_{n-l}$ into its absolutely continuous and singular parts, then

$$g_l \le G_l^{1/l};$$

(3) if

$$H = \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = \tilde{H}\, dm_n + d\nu$$

is the Lebesgue decomposition of the measure H, then

$$g_l = (l!)^{1/l}\Psi_l(\tilde{H}).$$

P r o o f. The statements for $l = n$ are proved in [1, Theorem 5.8]; for the case $l < n$ we only need to use the above Proposition 4.3 and repeat the arguments of the proof of that theorem.

R e m a r k. The question about absolute continuity of the measure $\Phi_n[u]$ for an arbitrary plurisubharmonic function u was posed in [1]. It follows from Theorem 4.1 that all the measures $\Phi_l[u]$, $l \geq 2$, are absolutely continuous if the family $\{(dd^c u_\epsilon)^2 \wedge \beta_{n-l}\}_\epsilon$ is bounded (roughly speaking, if $(dd^c u)^2$ is correctly defined).

We can get more information on $\Phi_l[u]$ in case $u = \log|F|^2$ (that will be used in our study of almost periodic mappings).

Theorem 4.2. *Let F be a holomorphic mapping from $\Omega \subset \mathbf{C}^n$ into $\mathbf{C}^q$, $q \leq n$, such that its zero set Z_f is a complete intersection, and let $u = \log|F|^2$. Then, in the notations of Theorem 4.1,*

(1) $\Phi_l[u] = 0,\ q \leq l \leq n;$

(2) $g_l^l\, dm_n = l!\,[\Psi_l(H)]^l\, dm_n = G_l\, dm_n = (dd^c u)^l \wedge \beta_{n-l} \quad \forall l < q;$

(3) $\Phi_l[u] = 0$ *on a domain* $\omega \subset \Omega$ *for some* $l < q$, *then* $\Phi_l[u] = 0$ *on the whole domain* Ω; *moreover, in this case* $\operatorname{rank} F \leq l$ *and* $Z_F = \emptyset$.

P r o o f. The statement (1) for $l = q$ follows directly from Theorem 4.1,(1) and the fact that $(dd^c u)^q$ is the current of integration over the chain Z_F. In its turn it implies $\Phi_l[u] = 0$ for $l > q$ by Proposition 4.2,(6).

For $l = 1$ the statement (2) is evident since $\Phi_1[u] = dd^c u \wedge \beta_{n-1}$, where the current $dd^c u$ has locally summable coefficients.

To prove (2) for $l > 1$ we observe, first, that the function $u = \log|f|^2$ satisfies the condition on the family

$$\{(dd^c u_\epsilon)^l \wedge \beta_{n-l}\}_\epsilon, \quad 1 < l < q,$$

of Theorem 4.1 (see [3]).

Further, let

$$H_\epsilon = \left(\frac{\partial^2 u_\epsilon}{\partial z_i \partial \bar{z}_j}\right)$$

and

$$(dd^c u_\epsilon)^l \wedge \beta_{n-l} = G_{l,\epsilon}\, dm_n.$$

By Proposition 4.3,

$$G_{l,\epsilon} = l![\Psi_l(H_\epsilon)]^l.$$

On the other hand, the complex Hessian H of the function u has the form

$$H = |F|^{-2} h,$$

where h is a smooth matrix function. Therefore for $l < q$ the coefficients of $(dd^c u_\epsilon)^l$ converge to the coefficients of $(dd^c u)^l$ in $L_{loc}^{(2q-1)/2l}$, and so $G_{l,\epsilon} \to G_l$ as $\epsilon \to 0$. By the same arguments, $[\Psi_l(H_\epsilon)]^l \to [\Psi_l(H)]^l$ in $L_{loc}^{(2q-1)/2l}$, and we come to the desired equations (2).

Now let $g_l = 0$ on a domain $\omega \subset \Omega$ for some $l < q$. Then by the just proved statement (2), $G_l = 0$ on ω and so, $(dd^c u)^l = 0$ on ω. It implies, in particular, that $(dd^c u)^q = 0$ on ω, hence $Z_F \cap \omega = \emptyset$ and $u \in C^\infty(\omega)$.

The Hessian H of the function u has the form

$$H = |F|^{-4} \left(|F|^2 \sum_{k=1}^n \frac{\partial F_k}{\partial z_i} \overline{\frac{\partial F_k}{\partial z_j}} - \sum_{k=1}^n \frac{\partial F_k}{\partial z_i} \bar{F}_k \cdot \overline{\sum_{k=1}^n \frac{\partial F_k}{\partial z_j} \bar{F}_k} \right)_{i,j=1}^n .$$

Let

$$J = \left(\frac{\partial F_j}{\partial z_i} \right)_{1 \le i \le n,\, 1 \le j \le q}$$

be the Jacobi $n \times q$ matrix of the mapping F. Taking F as a row vector and denoting by A^* the matrix adjoint to a matrix A, we have by direct calculation that

$$\begin{aligned} H &= |F|^{-4}(FF^*JJ^* - JF^*FJ^*) \\ &= |F|^{-4}(JFF^*J^* - JF^*FJ^*) \\ &= |F|^{-4} J (FF^* I_q - F^*F) J^*, \end{aligned}$$

where I_q is the unit $q \times q$ matrix. Observe that the matrix F^*F is of rank 1, hence

$$F^*F = U \begin{pmatrix} |F|^2 & 0 \\ 0 & 0 \end{pmatrix} U^*$$

with some unitary $q \times q$ matrix U. Therefore

$$FF^* I_q - F^*F = |F|^2 U \begin{pmatrix} 0 & 0 \\ 0 & I_{q-1} \end{pmatrix} U^*.$$

Denoting

$$B = |F|^{-1} J U \begin{pmatrix} 0 & 0 \\ 0 & I_{q-1} \end{pmatrix},$$

we have $H = BB^*$. Besides, $\operatorname{rank} B \ge \operatorname{rank} J - 1$.

On the other hand, the condition $(dd^c u)^l = 0$ implies that $\operatorname{rank} H \le l - 1$ on ω. Thus $\operatorname{rank} B \le l - 1$, too, and $\operatorname{rank} J \le l$ on ω.

Since every minor of order $l+1$ of the matrix J is equal to zero identically on the domain ω, then it is zero on the whole domain Ω. So, $\operatorname{rank} F = \operatorname{rank} J \le l$ on Ω, and Z_F can be a complete intersection only if $Z_F = \emptyset$.

The matrix

$$U \begin{pmatrix} 0 & 0 \\ 0 & I_{q-1} \end{pmatrix}$$

has the form $[0, u^{(1)}, \dots, u^{(q-1)}]$, and the columns $u^{(j)}$ satisfy the equations $F^*F\, u^{(j)} = 0,\ 1 \le j \le q-1$ (they are the eigenvectors of the operator F^*F corresponding to the zero eigenvalue). Define the column vectors $g^{(j)}$ by $g_j^{(j)} = f_{j+1}$, $g_{j+1}^{(j)} = -f_j$, $g_i^{(j)} = 0$, $\forall i \in \{1,\dots,q\} \setminus \{j, j+1\}$, $1 \le j \le q-1$. The vectors $\{g^{(j)}\}$ form a linearly independent system in the subspace generated by $u^{(1)}, \dots, u^{(q-1)}$ for every $z \in \Omega \setminus \bigcup_j Z_{f_j}$. Therefore $\operatorname{rank} B = \operatorname{rank} JG$ in $\Omega \setminus \bigcup_j Z_{f_j}$, where $G = [g^{(1)}, \dots, g^{(q-1)}]$. Since JG is a holomorphic matrix function in Ω and $\operatorname{rank} JG < l$ on ω, then $\operatorname{rank} JG < l$ everywhere on Ω, and the same is true for the matrix H. Thus $\Phi_l[u] \equiv 0$ in Ω.

The theorem is proved.

The last statement of the theorem can be reformulated in the following way (which might be known, however we have no corresponding references).

Corollary 4.1. *Let F be a holomorphic mapping from a domain $\Omega \subset \mathbf{C}^n$ to $\mathbf{C}^q$, $q \le n$, such that Z_F is a complete intersection. If*

$$(dd^c \log |F|^2)^l = 0 \tag{4.2}$$

on a domain $\omega \subset \Omega$ for some $l < q$, then (4.2) holds identically on Ω, $\operatorname{rank} F \le l$ *and $Z_F = \emptyset$.*

In what follows, we will need a result on limit transitions for the operators $\Phi_l[u]$. The general situation is described by Proposition 4.2,(4). However for our special case we have the following, more precise, relation.

Theorem 4.3. *If a sequence of holomorphic mappings $F_j : \Omega \to \mathbf{C}^q$, $q \le n$, converges uniformly on compact subsets of Ω to a mapping F such that Z_F is a complete intersection, then the sequence $\Phi_l[\log |F_j|^2]$ converges to $\Phi_l[\log |F|^2]$ in $L^l_{loc}(\Omega)$.*

P r o o f. Let according to Theorem 4.1,(1)

$$\Phi_l[\log |F_j|^2] = g_{l,j}\, dm_n, \quad \Phi_l[\log |F|^2] = g_l\, dm_n, \quad l < q.$$

For any compact set $E \subset \Omega$,

$$\int\limits_E |g_{l,j} - g_l|^l\, dm_n = \int\limits_{E \setminus U} |g_{l,j} - g_l|^l\, dm_n + \int\limits_{E \cap U} |g_{l,j} - g_l|^l\, dm_n,$$

where U is some neighbourhood of $|Z_F|$. Since $g_{l,j} \to g_l$ uniformly on $E \setminus U$ then the first term converges to zero as $j \to \infty$. The second term is not greater than

$$2^{l-1} \int\limits_{E \cap U} \left(|g_{l,j}|^l + |g_l|^l \right) \, dm_n$$

which by [9, Theorem 1] can be made however small by the appropriate choice of the set U. Therefore $g_{l,j} \to g_l$ in L^l_{loc}.

5. Degeneration of the Jessen function

Now we apply the results from the previous section to study the currents $(dd^c A_f)^l$, $l < q$.

Let $f \in R_q(T_G)$, $u = \log |f|^2 \in PSH(T_G)$, $\Phi_f^{(l)} := g_l$, where the functions g_l are the densities of the absolutely continuous measures $\Phi_l[u]$ (see Theorem 4.1,(1)). For $f \in R_q(T_G)$ and $t \in \mathbf{R}^n_+$ we define the functions

$$\Phi_{f,t}^{(l)}(x + iy) = \Phi_f^{(l)}(t \cdot x + iy), \quad l < q.$$

Theorem 5.1. *Let $f \in R_q(T_G)$ and $l < q$. Then the functions $\Phi_{f,t}^{(l)}$ converge weakly to some function $\tilde{\Phi}_f^{(l)}(y) \geq 0$ as $m(t) \to \infty$. Moreover,*

$$\tilde{\Phi}_f^{(l)}(y) = \lim_{m(t) \to \infty} \fint\limits_{\Pi_t} \Phi_f^{(l)}(s + x + iy) \, ds$$

in $L^l_{loc}(G)$ uniformly in $x \in \mathbf{R}^n$.

Sketch of the proof. The way of the proof is similar to one of [13, Theorem 1], and we repeat it in brief. First, let

$$E_\theta(x^0, G') = \{ z \in T_{G'} : \, x \in \Pi_{\mathbf{1}}(x^0), \, |f(z)| < \theta \}.$$

By means of Theorem 4.3 one can prove that for any $\epsilon > 0$ and any $G' \subset\subset G$ there exists $\theta > 0$ such that

$$\int\limits_{E_\theta(x^0, G')} \left[\Phi_f^{(l)} \right]^l \, dx \, dy < \epsilon, \quad \forall x^0 \in \mathbf{R}^n \tag{5.1}$$

(cf. [13, Lemma 1]).

Further, set

$$\Phi_{f,\theta}^{(l)} = \left(\frac{|f|}{|f|_\theta} \right)^2 \Phi_f^{(l)}$$

(as above, $|f|_\theta = \max\{|f|, \theta\}$). It is a uniformly almost periodic function in T_G. Therefore there exists a function $\tilde{\Phi}^{(l)}_{f,\theta}(y)$ such that for any $\epsilon > 0$ and any $G' \subset\subset G$

$$\left| \quad \fint_{\Pi_{\mathbf{1}}(x)} \Phi^{(l)}_{f,\theta}(t\cdot s + iy)\, ds - \tilde{\Phi}^{(l)}_{f,\theta}(y) \right| < \epsilon, \quad \forall x \in \mathbf{R}^n, \tag{5.2}$$

provided $m(t) > m_0 = m_0(\epsilon, G')$.

Denote

$$\|g\|_{G',l} = \left[\int_{G'} |g(s)|^l \, ds \right]^{1/l}.$$

By (5.1) and (5.2), there exists $\tilde{\theta} > 0$ such that for any $\theta' < \theta$ and $\theta'' < \theta$, $\forall x \in \mathbf{R}^n$,

$$\begin{aligned} \| \quad & \tilde{\Phi}^{(l)}_{f,\theta'} - \tilde{\Phi}^{(l)}_{f,\theta''} \|_{G',l} \le \left\| \tilde{\Phi}^{(l)}_{f,\theta'} - \fint_{\Pi_{\mathbf{1}}(x)} \Phi^{(l)}_{f,\theta',t}\, ds \right\|_{G',l} \\ + & \left\| \tilde{\Phi}^{(l)}_{f,\theta''} - \fint_{\Pi_{\mathbf{1}}(x)} \Phi^{(l)}_{f,\theta'',t}\, ds \right\|_{G',l} + \left\| \fint_{\Pi_{\mathbf{1}}(x)} \left[\Phi^{(l)}_{f,\theta',t} - \Phi^{(l)}_{f,\theta'',t} \right] ds \right\|_{G',l} \\ < \quad & 2\epsilon(1 + \mathrm{mes}_n G'). \end{aligned}$$

Therefore the functions $\tilde{\Phi}^{(l)}_{f,\theta}$ converge to a function $\tilde{\Phi}^{(l)}_{f}$ in $L^l_{loc}(G)$.

For any $\epsilon > 0$ and $G' \subset\subset G$, there exist $\hat{\theta} > 0$ and $\hat{m} > 0$ such that

$$\left\| \fint_{\Pi_{\mathbf{1}}(x)} \Phi^{(l)}_{f,\theta,t}\, ds - \tilde{\Phi}^{(l)}_{f} \right\|_{G',l} < \epsilon, \quad \forall x \in \mathbf{R}^n,$$

provided $\theta < \hat{\theta}$ and $m(t) > \hat{m}$. It implies that for $m(t) > \hat{m}$

$$\left\| \fint_{\Pi_{\mathbf{1}}(x)} \Phi^{(l)}_{f,t}\, ds - \tilde{\Phi}^{(l)}_{f} \right\|_{G',l} < \epsilon, \quad \forall x \in \mathbf{R}^n.$$

Thus

$$\tilde{\Phi}^{(l)}_{f} = \lim_{m(t)\to\infty} \fint_{\Pi_{\mathbf{1}}(x)} \Phi^{(l)}_{f,t}\, ds = \lim_{m(t)\to\infty} \fint_{\Pi_{\mathbf{1}}(x)} \Phi^{(l)}_{f}(s + x + iy)\, ds$$

in $L^l_{loc}(G)$ uniformly in $x \in \mathbf{R}^n$.

It also follows that the functions $\Phi^{(l)}_{f,t}$ converge weakly to the function $\tilde\Phi^{(l)}_f$ as $m(t)\to\infty$ (cf. [13, Theorem 1] and [12, Theorem 1]). That completes the proof.

Note that $\Phi^{(1)}_f[u]=dd^cu\wedge\beta_{n-1}$ implies $\tilde\Phi^{(1)}_f=dd^cA_f\wedge\beta_{n-1}$, and $\tilde\Phi^{(q)}_f=0$ because $\Phi^{(q)}_f=0$ (Theorem 4.2,(1)).

Now we are going to relate these limit functions to the currents $\tilde a^{(l)}_f$ from Theorem 2.1.

Theorem 5.2 *Let $f\in R_q(T_G)$ and $l<q$. Then*

$$\left[\tilde\Phi^{(l)}_f\right]^l\beta_n\le\tilde a^{(l)}_f\wedge\beta_{n-l}.$$

P r o o f. Set $u=\log|f|^2$. By the previous theorem and the Hölder inequality, for any nonnegative function $\phi\in\mathcal{D}(T_G)$

$$\begin{aligned}\int\left[\tilde\Phi^{(l)}_f(y)\right]^l\phi\,\beta_n &= \int\lim_{m(t)\to\infty}\left[\;\fint_{\Pi_t}\Phi_l[u(z+s)]\,ds\right]^l\phi\,\beta_n\\ &\le \int\lim_{m(t)\to\infty}\fint_{\Pi_t}\{\Phi_l[u(z+s)]\}^l\,ds\phi\,\beta_n\\ &= \int\lim_{m(t)\to\infty}\fint_{\Pi_t}[dd^cu(z+s)]^l\,ds\wedge\phi\,\beta_n\\ &= \int\phi\tilde a^{(l)}_f\wedge\beta_{n-l},\end{aligned}$$

and the theorem is proved.

We can also estimate $\tilde\Phi^{(l)}_f$ by $\tilde a^{(l)}_f$ from below, but only on domains where f has no zeros.

Theorem 5.3. *Let $f\in R_q(T_G)$, $l<q$,*

$$|Z_f|\cap T_{G_0}=\emptyset \tag{5.3}$$

for some domain $G_0\subset G$, and

$$(dd^c\log|f|^2)^l\not\equiv0. \tag{5.4}$$

Then

$$\tilde\Phi^{(l)}_f\,\beta_n\ge c\,\tilde a^{(l)}_f\wedge\beta_{n-l}$$

on each domain $T_{G'}$, $G'\subset\subset G_0$, where the constant c depends on f and the domain G'.

P r o o f. Observe, first, that $u := \log|f|^2 \in C^\infty(T_{G_0})$, and due to Corollary 4.1 the condition (5.4) implies $(dd^c \log|f|^2)^l \not\equiv 0$ on any domain $G' \subset G$.

For $z \in T_{G'}$ and $G' \subset\subset G_0$ we have

$$\begin{aligned}
\tilde{\Phi}_f^{(l)}(y)\,\beta_n &= \lim_{m(t)\to\infty} \fint_{\Pi_t} \Phi_f^{(l)}(z+s)\,ds\,\beta_n \\
&\geq \|\Phi_l[u]\|_{G',\infty}^{-(l-1)} \lim_{m(t)\to\infty} \fint_{\Pi_t} \left[\Phi_f^{(l)}(z+s)\right]^l ds\,\beta_n \\
&= \left\|(dd^c u)^l\right\|_{G',\infty}^{\frac{1}{l}-1} \lim_{m(t)\to\infty} \fint_{\Pi_t} [dd^c u(z+s)]^l \wedge \beta_{n-l}\,ds \\
&= \left\|(dd^c u)^l\right\|_{G',\infty}^{\frac{1}{l}-1} \tilde{a}_f^{(l)} \wedge \beta_{n-l}.
\end{aligned}$$

The proof is complete.

Now we will obtain a relation between $\tilde{\Phi}_f^{(l)}$ and the corresponding Jessen function A_f. We would like to stress that it is the point where the concavity properties of the operators Φ_l play the key role.

Theorem 5.4. $\Phi_l[A_f] \geq \tilde{\Phi}_f^{(l)}\,dm_n$.

P r o o f. Let

$$v_t(z) = \fint_{\Pi_t} \log|f(z+s)|^2\,ds.$$

By Proposition 4.2,(4),

$$\Phi_l[A_f] \geq \lim_{m(t)\to\infty} \Phi_l[v_t]. \tag{5.5}$$

Further, by concavity of Φ_l (Proposition 4.2,(2)),

$$\Phi_l[v_t] \geq \fint_{\Pi_t} \Phi_l[u(z+s)]\,ds, \tag{5.6}$$

where $u = \log|f|^2$. Finally, by (5.5) and (5.6),

$$\Phi_l[A_f] \geq \lim_{m(t)\to\infty} \fint_{\Pi_t} \Phi_l[u(z+s)]\,ds = \tilde{\Phi}_f^{(l)}\,dm_n.$$

Corollary 5.1. $(dd^c A_f)^l \wedge \beta_{n-l} \geq \left[\tilde{\Phi}_f^{(l)}\right]^l \beta_n$.

P r o o f. The relation follows immediatly from Theorem 5.4 and Theorem 4.1,(2).

Theorem 5.5. *Let $f \in R_q$, $|Z_f| \cap T_{G_0} = \emptyset$ for a domain $G_0 \subset G$ and $\tilde{\Phi}_f^{(l)} = 0$ on T_{G_0} for some $l < q$. Then $\tilde{a}_f^{(l)} = 0$ on the whole T_G,* $\operatorname{rank} f \le l$ *and $Z_f = \emptyset$.*

P r o o f. Since $|Z_f| \cap T_{G_0} = \emptyset$, the function $u := \log |f|^2$ is smooth and uniformly almost periodic in T_{G_0}, and so is $\Phi_l[u] \equiv \Phi_f^{(l)}$. However,

$$\lim_{m(t)\to\infty} \fint_{\Pi_t} \Phi_f^{(l)}(z+s)\, ds = \tilde{\Phi}_f^{(l)}\, dm_n = 0$$

on G_0. Thus, by standard properties of almost periodic functions, $\Phi_f^{(l)} \equiv 0$ on T_{G_0}. Now Theorem 4.2 implies that $\Phi_f^{(l)} \equiv 0$, $\tilde{a}_f^{(l)} = 0$, $Z_f = \emptyset$, and $\operatorname{rank} f \le l$.

Corollary 5.2. *Let $f \in R_q(T_G)$ and $(dd^c A_f)^l = 0$ on T_{G_0}, $G_0 \subset G$, $l < q$. Then the conclusion of Theorem 5.5 holds.*

P r o o f. Since $(dd^c A_f)^l = 0$ on T_{G_0}, $(dd^c A_f)^q = 0$ on T_{G_0}, too, and thus by Theorem 3.1, $|Z_f| \cap T_{G_0} = \emptyset$. So Corollary 5.1 and Theorem 5.5 imply the desired result.

Corollary 5.3. *If the Jessen function A_f of a mapping $f \in R_q(T_G)$, $q > 1$, is linear on a subdomain of G, then A_f is linear on G.*

P r o o f. The statement follows directly from Corollary 5.2 with $l = 1$, since $dd^c A_f = \tilde{a}_f^{(1)}$.

Corollary 5.4. *Let $f \in R_q(T_G)$ be such that $\tilde{a}_f^{(l)} = 0$ on T_{G_0}, $G_0 \subset G$, $l < q$. Then the conclusion of Theorem 5.5 holds.*

P r o o f. By [9, Theorem 3],

$$\operatorname{supp} \tilde{a}_f^{(q)} \subset \operatorname{supp} \tilde{a}_f^{(l)},$$

so $\tilde{a}_f^{(q)} = 0$ on T_{G_0} and, by Theorem 2.2, $|Z_f| \cap T_{G_0} = \emptyset$. Now we can either refer to Theorem 5.2 and Theorem 5.5 or repeat directly the arguments of the proof of Theorem 5.5.

The combination of the above results gives us

Corollary 5.5. *If $f \in R_q(T_G)$ then*

$$\operatorname{supp} (dd^c A_f)^l \supset \operatorname{supp} \tilde{\Phi}_f^{(l)} = \operatorname{supp} \tilde{a}_f^{(l)}, \quad \forall l < q.$$

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Операторы Монжа–Ампера и функции Йессена голоморфных почти периодических отображений

А. Рашковский

Исследуются свойства функции Йессена, т.е. среднего значения функции $\log|f|^2$, голоморфного почти периодического отображения f трубчатой области пространства $\mathbf{C}^n$ в $\mathbf{C}^q$. В частности, установлены связи между функцией Йессена и поведением самого отображения и его нулевого множества. С этой целью вводятся операторы Φ_l, действующие на плюрисубгармонические функции, которые в случае гладкой функции u имеют вид

$$(\Phi_l[u])^l\,(dd^c|z|^2)^n = (dd^c u)^l \wedge (dd^c|z|^2)^{n-l}.$$

Оператори Монжа–Ампера та функції Йессена голоморфних майже періодичних відображень

О. Рашковський

Досліджуються властивості функції Йессена, тобто середнього значення функції $\log|f|^2$, голоморфного майже періодичного відображення f трубчастої області простору $\mathbf{C}^n$ в $\mathbf{C}^q$. Зокрема, встановлено зв'язки між функцією Йессена і поведінкою самого відображення та його нульової множини. З цією метою вводяться оператори Φ_l, які діють на плюрисубгармонічні функції, що у разі гладкої функції u мають вигляд

$$(\Phi_l[u])^l\,(dd^c|z|^2)^n = (dd^c u)^l \wedge (dd^c|z|^2)^{n-l}.$$