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On an isometric representation with the maximal set of spectral subspaces*

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It was proved the theorem. Let G be a locally compact noncompact separable Abelian group. Then there exists an isometric representation of the group G in a Banach space X without eigenvectors for which any spectral subspace $L(K) \neq \{0\}$ if K contains a nonempty perfect subset.

Let G be a locally compact separable Abelian group, G^* be its character group, $T = T(g)$, $g \in G$ be a strongly continuous isometric representation of the group G ($\| T(g) \| \equiv 1 \; \forall g \in G$) in a Banach space X. Following [9], the set $\sigma(T)$ of such characters $\chi \in G^*$ for each of which there exists a normalized sequence $\{x_n\}_{n=1}^{\infty} \subset X$ satisfying the condition

$$\lim_{n\to\infty} (T(g)x_n - \chi(g)x_n) = 0\,,$$

for all $g \in G$, is called the spectrum of the representation T. As has been proved in [9] and [10], $\sigma(T)$ is not empty and closed in G^*.

As has been proved in [10] for each compact $Q \subset G^*$ there exists a subspace $L(Q)$ which is called a spectral subspace of T such that

1. $L(Q)$ is invariant with respect of T;
2. the restriction $T(Q) = T|L(Q)$ is a uniformly continuous representation;
3. $\sigma(T(Q)) \subset \sigma(T) \cap Q$;

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4. the interior $\mathrm{Int}(\sigma(T) \cap Q) \subset \sigma(T(Q))$, where Int is taken in the topology with respect to $\sigma(T)$ (but not with respect to the whole of G^*);

5. if a subspace L is invariant with respect of T and $\sigma(T|L) \subset Q$, then $L \subset L(Q)$.

An interesting and difficult problem is the following: what are the conditions on a compact Q in order that $L(Q) \neq \{0\}$. [1] In this article we discuss one of the aspects of this general problem.

If a function $\Phi \in L^1(G)$, then denote by $\hat{\Phi}(\chi)$ its Fourier transform:

$$\hat{\Phi}(\chi) = \int\limits_G \Phi(g)\overline{\chi(g)}dg,$$

and by $\hat{\Phi}_T$ its Fourier transform with respect of a representation T:

$$\hat{\Phi}_T = \int\limits_G \Phi(g)T(-g)dg.$$

Let Q be a compact in G^*. Denote by $F(Q)$ the set of functions $\Phi \in L^1(G)$ for each of which $\hat{\Phi}(\chi) = 1$ in a neighborhood of Q. As has been proved in [10] any spectral subspace $L(Q)$ can be defined in the following way:

$$L(Q) = \{x \in X : \hat{\Phi}_T x = x \quad \forall \Phi \in F(Q)\}. \tag{1}$$

Let a function $a \in L^\infty(G)$. Denote by $\sigma(a)$ its Beurling spectrum. If $x \in X$, $\varphi \in X^*$, then define by $\sigma_{\varphi,x}$ the Beurling spectrum of a function $\varphi(T(g)x) \in L^\infty(G)$. As has been proved in [4] the spectral subspace $L(Q)$ can be defined as follows:

$$L(Q) = \{x \in X : \sigma_{\varphi,x} \subset Q \quad \forall \varphi \in X^*\}. \tag{2}$$

It may be noted that

$$\sigma(T) = \bigcup_{\varphi,x} \sigma_{\varphi,x},$$

where the union is taken by all $\varphi \in X^*$, $x \in X$ [4].

A vector $x \in X$, $x \neq 0$, is called to be an eigenvector of a representation T if there exists a character $\chi \in G^*$ (an eigencharacter of the representation) such that

$$T(g)x = \chi(g)x \quad \forall g \in G.$$

[1] The problem of the description of spectral subspaces is substantal even for one isometric operator in a Banach space (the same for a representation of the group $\mathbb{Z}$). For example, it was proved in [7] that if T is an isometric operator in the space of continuous functions $C[0,1]$ without eigenvectors, then $L(Q) = \{0\}$ if and only if Q is a uniqueness set for trigonometrical series.

Unless otherwise specified in what follows we will assume that a representation T do not have any eigenvectors. A spectral subspace $L(K)$, maybe a trivial one, corresponds for each compact $K \subset G^*$. Our aim is to construct a representation T with the following property: the set of such compacts K for each of which $L(K) \neq \{0\}$ is a maximal one. It should be noticed that if $L(K) \neq \{0\}$, then the compact K can not consist of a countable set of points, since then (see, e.g., [4]) there would be in $L(K)$ an eigenvector of T. It means that if a representation T has no eigenvectors and a spectral subspace $L(K) \neq \{0\}$, then it is necessary that the compact K contain a nonempty perfect subset.

In the sequel we will assume that G is a noncompact one, for otherwise G^* is countable and hense any isometric representation T of G would have an eigenvector. The following theorem takes place.

Theorem 1. *Let G be a locally compact noncompact separable Abelian group. Then there exists an isometric representation of the group G in a Banach space X without eigenvectors for which any spectral subspace $L(K) \neq \{0\}$ if K contains a nonempty perfect subset.*

To prove the theorem we need two lemmas. Denote by $CB(G)$ the space of continuous bounded functions on G and by $AP(G)$ the space of almost periodic functions on G. The space $AP(G)$ is a proper subspace of $CB(G)$ in view of noncompactness of the group G.

Consider the left regular representation T of the group G in the space $CB(G)$:

$$T(g)f(h) = f(h+g), \quad f \in CB(G).$$

Since the space $AP(G)$ is invariant with respect of T, the representation T induces the representation $\hat{T}$ in the factor-space $CB(G)/AP(G)$. It is obvious that $\hat{T}$ is isometric too.

Lemma 1. *The representation $\hat{T}$ has no eigenvectors.*

P r o o f. Let $[f] \in CB(G)/AP(G)$, $[f] \neq 0$ is an eigenvector of $\hat{T}$, i.e., for some $\chi \in G^*$

$$\hat{T}(g)[f] = \chi(g)[f] \quad \forall g \in G. \tag{3}$$

Equality (3) means that

$$f(h+g) - \chi(g)f(h) = a(h), \quad g, h \in G, \tag{4}$$

where $a \in AP(G)$, $f \in [f]$. Put $\varphi(h) = \frac{f(h)}{\chi(h)}$. It is follows from (4) that the function φ satisfies the equation

$$\varphi(h+g) - \varphi(h) = \frac{a(h)}{\chi(g)\chi(h)}, \quad g, h \in G. \tag{5}$$

It follows from (5), using the well-known Doss's theorem [2], that $\varphi \in AP(G)$ and therefore $f \in AP(G)$, i.e., $[f] = 0$, contrary to the assumption.

R e m a r k 1. We note that if T is the left regular representation of a group G in the space $L^\infty(G)$ (this representation is not a strongly continuous!), then the representation $\hat{T}$ induced by T in the factor-space $L^\infty(G)/AP(G)$ has no eigenvectors too. To prove we should repeat the proof of Lemma 1 using instead of the Doss's theorem its generalization (see [1, 6]).

On the other hand, it should be noticed that if T is an isometric representation in a Banach space X and X_0 is the subspace of X generated by all eigenvectors of T, then the induced representation $\hat{T}$ in the factor-space X/X_0 generally can has an eigenvector.

Let $\Phi \in L^1(G)$ and $a \in L^\infty(G)$. Denote by

$$(\Phi * a)(h) = \int_G \Phi(g)a(h-g)dg.$$

Lemma 2. *Let K be a compact in G^* . Then the following equality takes place:*

$$\{a \in L^\infty(G) : \Phi * a = a \quad \forall \Phi \in F(K)\} = \{a \in L^\infty(G) : \sigma(a) \subset K\}.$$

P r o o f. Let $\Phi * a = a$ for any $\Phi \in F(K)$. By lemma XI.4.12 in [3] the Beurling spectrum $\sigma(\Phi * a)$ does not contain any points which are interior for the set of zeros of the function $\hat{\Phi}(\chi)$. Let $\chi_0 \notin K$. Choose $\Phi \in F(K)$ in such a way that $\hat{\Phi}(\chi) = 0$ in a neighborhood of the point χ_0. Then $\chi_0 \notin \sigma(\Phi * a)$, i.e., $\chi_0 \notin \sigma(a)$ and hence $\sigma(a) \subset K$.

Let us assume that $\sigma(a) \subset K$. We verify that for any function $\Phi \in F(K)$ is fulfilled

$$\Phi * a = a. \tag{6}$$

Let $\hat{\Phi}(\chi) = 1$ in a neighborhood U of the compact K. We notice that (6) is fulfilled if $a(h) = \chi(h)$, $\chi \in U$. Hence (6) takes place for all functions $a \in L^\infty(G)$ which are contained in the $L^1(G)$ closed linear subspace generated by characters from U. By Theorem XI.4.13 in [2] any function $a \in L^\infty(G)$ such that $\sigma(a) \subset K$ belongs to this subspace and therefore (6) is fulfilled for all such functions.

P r o o f o f T h e o r e m 1. We will check that the representation $\hat{T}$ of the group G in the factor-space $X = CB(G)/AP(G)$ is the required. By Lemma 1 the representation $\hat{T}$ has no eigenvectors. Let K be a compact in G^* and let K_0 be a perfect subset in K. It is known (see, e.g., [6]), that there exists a continuous bounded not almost periodic function $f(h)$ on G such that $\sigma(f) = K_0$. Let $\Phi \in F(K)$. As appears from Lemma 2 that

$$(\Phi * f)(h) = \int_G \Phi(g)T(-g)f(h)dg = f(h).$$

This implies that

$$\hat{\Phi}_{\hat{T}}[f] = \int_G \Phi(g)\hat{T}(-g)[f]dg = [f].$$

Since this equality is correct for any function $\Phi \in F(K)$ then as appears from (1), $[f] \in L(K)$. Since $[f] \neq 0$, Theorem 1 is proved.

As proved in [10], if a compact K has an interior point with respect of the spectrum of a representation $\sigma(T)$, then $L(K) \neq \{0\}$. We guess it will be in interest the following result showing that for any noncompact group G there exists the isometric representation T such that this condition of nontriviality of a spectral subspace is necessary.

Proposition 1. *Let G be a locally compact noncompact separable Abelian group. Then there exists an isometric representation T of the group G in a Banach space X without eigenvectors such that any $L(K) \neq \{0\}$ if and only if K has a nonempty interior.*

P r o o f. Put $X = L^1(G)$, then $X^* = L^\infty(G)$ and consider the left regular representation T of G in $L^1(G)$:

$$T(g)f(h) = f(h+g), \;\; f \in L^1(G).$$

It is obvious that T is an isometric representation and in a view of noncompactness of the group G, the representation T has no eigenvectors.

We note that if $\varphi \in L^\infty(G)$ and $f \in L^1(G)$ then

$$\varphi(T(g)f) = \int_G f(h+g)\overline{\varphi(h)}dh$$

and therefore for $\varphi(h) = \chi(h)$ it implies that

$$\sigma_{\varphi,f} = \begin{cases} \chi, & \hat{f}(\chi) \neq 0, \\ \emptyset, & \hat{f}(\chi) = 0. \end{cases} \tag{7}$$

Let a compact K has no interior points and $f \in L(K)$. Choose a character $\chi_0 \notin K$. It follows from (2) and (7) that $\hat{f}(\chi_0) = 0$, i.e., $\hat{f}(\chi) = 0$ on the set $G^* \setminus K$. Since the set $G^* \setminus K$ is everywhere dense and the function $\hat{f}(\chi)$ is continuous, then $\hat{f}(\chi) \equiv 0$, $\chi \in G^*$, i.e., $f = 0$. Proposition 1 is proved.

R e m a r k 2. It is easy to see from the proof of Proposition 1 that any spectral subspace $L(K)$ of the representation T has a form

$$L(K) = \{f \in L^1(G) : \quad \hat{f}(\chi) = 0 \quad \forall \chi \in G^* \setminus K\}. \tag{8}$$

In order to verify it let us denote the right side of the equality (8) by $\hat{L}(K)$. It is proved in Proposition 1 that $L(K) \subset \hat{L}(K)$. Let $f \in \hat{L}(K)$. Since the spectrum $\sigma(f * \psi)$, where $\psi \in L^\infty(G)$, does not contain any interior points of the set of zeros of the function $\hat{f}(\chi)$, then $\sigma(f * \psi) \subset K$. We notice that $\varphi(T(g)f) = f * \psi$, where $\psi(h) = \overline{\varphi(-h)}$. It follows from this that $\sigma_{\varphi,f} \subset K$ for any $\varphi \in L^\infty(G)$, i.e., $f \in L(K)$ in view of (2).

R e m a r k 3. Let a group G be the same as in Proposition 1. Consider the right regular representation T of the group G in the space $L^2(G)$:

$$T(g)f(h) = f(h-g), \quad f \in L^2(G).$$

It is obvious that T is an isometric representation and in view of noncompactness of the group G the representation T has no eigenvectors. Consider the natural isomorphism $L^2(G) \to L^2(G^*)$, namely $f(h) \to \hat{f}(\chi)$. Then the representation T is equivalent to the representation $\hat{T}$ of the group G in the space $L^2(G^*)$:

$$\hat{T}(g)\hat{f}(\chi) = \overline{\chi(g)}\hat{f}(\chi), \quad \hat{f} \in L^2(G^*).$$

It follows from this that for any $\Phi \in L^1(G)$ is fulfilled

$$\hat{\Phi}_{\hat{T}}\hat{f}(\chi) = \hat{\Phi}(\chi)\hat{f}(\chi). \tag{9}$$

Taking into account (1), it follows from (9) that any spectral subspace $L(K)$ of the representation $\hat{T}$, and hence of the representation T too, has the form

$$L(K) = \{f \in L^2(G) : \hat{f}(\chi) \overset{\text{a.e.}}{=} 0 \quad \forall \chi \in G^* \setminus K\}.$$

In particular, $L(K) \neq \{0\}$ if and only if when $m(K) > 0$, where m is the Haar measure on G^*. The set of compacts K for which $L(K) \neq \{0\}$ in this example is narrower then in Theorem 1 but wider then in Proposition 1.

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Об изометрическом представлении с максимальным набором спектральных подпространств

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Доказана теорема. Пусть G — локально компактная некомпактная сепарабельная абелева группа. Тогда существует изометрическое представление группы G в банаховом пространстве X, не имеющее собственных векторов и обладающее тем свойством, что спектральное подпространство $L(K) \neq \{0\}$, если компакт K содержит непустое совершенное подмножество.

Про ізометричне зображення з максимальною множиною спектральних підпросторів

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Доведено теорему. Нехай G — локально компактна некомпактна сепарабельна абелева група. Тоді існує ізометричне зображення групи G у банаховому просторі X, яке не має власних векторів і має властивість, що спектральний підпростір $L(K) \neq \{0\}$, якщо компакт K містить непусту досконалу підмножину.