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Asymptotic behaviour of harmonic 1-forms on Riemannian surfaces of increasing genus

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2-dimensional compact oriented Riemannian manifolds M_ε consisting of one or several copies of some base surface Γ with a large number of thin tubes, endowed with a metric depending on a small parameter ε are considered. The asymptotic behaviour of harmonic 1-forms on M_ε is studied when the number of tubes increases and their thickness vanishes, as $\varepsilon \to 0$. We obtain the homogenized equations on the base surface Γ describing the leading term of the asymptotics.

§ 1. Description of the problem

Let Γ be a 2-dimensional compact manifold without an edge endowed with Riemannian metric tensor $\{g_{\alpha\beta}(x);\ \alpha, \beta = 1, 2\}$. Let, for any small $\varepsilon > 0$, a family of closed pairwise disjoint circles (holes) $F_{\varepsilon i} \subset \Gamma$ $(i = 1, \dots, 2N(\varepsilon))$ is given. We suppose that the total number $2N(\varepsilon)$ of holes tends to infinity and their diameters (relative to the metric on Γ) tend to zero, as $\varepsilon \to 0$.

Denote Γ_ε the domain on Γ

$$\Gamma_\varepsilon = \Gamma \setminus \bigcup_{i=1}^{2N(\varepsilon)} F_{\varepsilon i}$$

and consider the disjoint union of m, $m \geq 1$, copies of Γ_ε: $\{\Gamma_\varepsilon^k,\ k = 1, \dots, m\}$. Everywhere below the upper index (-es) will mean the relevance to corresponding sheet (s).

We suppose that the set of all holes $\{F_{\varepsilon i}^k;\ i = 1, \dots, 2N(\varepsilon),\ k = 1, \dots, m\}$ is partitioned into subsets of two elements $(F_{\varepsilon i}^k, F_{\varepsilon j}^l)$ — linked pairs of holes, and for each linked pair $(F^k_{\varepsilon i}, F^l_{\varepsilon j})$ we are given a 2-dimensional compact manifold $T^{kl}_{\varepsilon ij}$ diffeomorfic to the tube $S_1 \times [0,1]$ (S_1 is the unit circle). It's boundary $\partial T^{kl}_{\varepsilon ij}$ consists of two components

$$\partial T^{kl}_{\varepsilon ij} = S^{1k}_{\varepsilon i} \bigcup S^{1l}_{\varepsilon j}.$$

Suppose as well that for each i, k we are given a diffeomorphism:

$$h^k_{\varepsilon i} : S^{1k}_{\varepsilon i} \leftrightarrow \partial F^k_{\varepsilon i}, \quad i = 1, \ldots, 2N(\varepsilon),\ k = 1, \ldots, m.$$

We construct a new manifold

$$M_\varepsilon = \left(\bigcup_{k=1}^{m} \Gamma^k_\varepsilon \right) \bigcup \left(\bigcup_{l.p.} \overline{T^{kl}_{\varepsilon ij}} \right)$$

by attaching the tubes $T^{kl}_{\varepsilon ij}$ $(i, j = 1, \ldots, 2N(\varepsilon),\ \ k, l = 1, \ldots, m)$ to Γ^k_ε and Γ^l_ε gluing their boundaries to those of the holes $F^k_{\varepsilon i}$, $F^l_{\varepsilon j}$ according to diffeomorphisms $h^k_{\varepsilon i}$. (Here and everywhere below "l.p." denotes linked pairs $([i,k],[j,l])$ of holes.)

We equip M_ε with a differentiable structure and denote by $\{g^\varepsilon_{\alpha\beta}(x); \alpha, \beta = 1, 2\}$ the Riemannian metric tenzor on M_ε. Assume that the metric on M_ε coinsides with the metric of the base surface Γ outside of some sufficiently small neighbourhoods of the tubes $T^{kl}_{\varepsilon ij}$. In addition, M_ε is supposed to be orientable. Thus we obtain the surface of $N(\varepsilon)(m-1)+1$ genus, i.e., being topologically a sphere with $N(\varepsilon)(m-1)+1$ handles attached.

We will study differential 1-forms $v(x) = v_\alpha(x)dx^\alpha$ on M_ε which can be identified with vector fields $v(x) = \{v^\alpha(x),\ \alpha = 1, 2\}$ via the metric: $v^\alpha = g^{\alpha\beta}_\varepsilon v_\beta(x)$. Here and everywhere below $\{g^{\alpha\beta}_\varepsilon(x);\ \alpha, \beta = 1, 2\}$ is the inverse metric tensor, and, as usual, by repeated superscripts and subscripts summation is made.

Let us remind some definitions and facts from the theory of differentiable manifolds. We have the exterior differentiation operator d, which maps r-forms into $(r+1)$-forms $(r = 0, 1)$ according to the formulae

$$d\phi = \frac{\partial \phi}{\partial x^1} dx^1 + \frac{\partial \phi}{\partial x^2} dx^2, \quad d\omega = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) dx^1 \wedge dx^2,$$

where $\phi(x)$ is a function and $\omega(x) = \omega_\alpha(x)dx^\alpha$ is a 1-form on M_ε; $\wedge$ denotes the exterior product of forms.

The star conjugation operator $*$ assigns to any 1-form ω the 1-form

$$*\omega = -\omega_2 dx^1 + \omega_1 dx^2.$$

The dual operator δ (generalized divergence) maps 1-forms into functions. It is defined by

$$(d\phi, \omega) \ = \ (\phi, \delta\omega),$$

where $(\cdot,\cdot)$ denotes scalar product of forms: $(v,u)=\int v\wedge * u$. (It's easy to see that right-hand side of the last equality can be rewrited as follows: $(\phi,\delta\omega)=\int \phi\delta\omega d^2x$, where $d^2x=\sqrt{|g_\varepsilon|}dx^1\wedge dx^2$ is the volume element on M_ε, $|g_\varepsilon|=\det g_\varepsilon$.) In local coordinates the operators δ and $*$ are defined by the following formulae:

$$\delta\omega=-\frac{1}{\sqrt{|g_\epsilon|}}\ \frac{\partial}{\partial x^\alpha}(g_\epsilon^{\alpha\beta}\omega_\beta),$$

$$*\omega=(\epsilon^{12}_{\alpha\gamma}\sqrt{|g_\epsilon|}g_\epsilon^{\gamma\beta})dx^\alpha,$$

where $\epsilon^{12}_{\alpha\beta}$ is the skew-symmetric tensor such that $\epsilon^{12}_{\alpha\beta}=1$.

At last, we have the Beltrami–Laplace operator defined by

$$\Delta=d\delta+\delta d$$

and having the following form in local coordinates:

$$\Delta=\frac{1}{\sqrt{|g_\epsilon|}}\ \sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial x^\alpha}\left(\sqrt{|g_\epsilon|}\ g_\epsilon^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right).$$

A 1-form ω is called closed (irrotational), if $d\omega=0$, coclosed (solenoidal), if $\delta\omega=0$, and harmonic, if it is both closed and coclosed. A 1-form ω is called exact, if there exists a function ϕ such that $\omega=d\phi$.

We will consider 1-dimensional smooth paths (curves) on M_ε, which are parametrized by continuously differentiable functions $x^\alpha(t)$, $t\in[a,b]$, $\alpha=1,2$. Closed curves (for which $x^\alpha(a)=x^\alpha(b)$, $\alpha=1,2$) are called 1-dimensional cycles. Two cycles Z_1,Z_2 are called homologous each to other if there exists a domain $G\subset M_\epsilon$ for which Z_1-Z_2 is the oriented boundary: $Z_1-Z_2=\partial G$. In particular, a cycle Z is homologous to zero, if $Z=\partial G$.

If ω is a closed 1-form and Z is a cycle, then he integral

$$(\omega,Z)=\int\limits_Z\omega$$

is called the period of ω along Z.

Due to Hodge's theorem [1], there exists a unique harmonic form having arbitrary preassigned periods with respect to the cycles, no linear combination of which is homologous to zero.

Our manifold M_ε has the following classes of not homologous to zero cycles: homology classes of A-cycles which go across the tubes and homology classes of B-cycles going along the tubes. The integrals along A-, B-cycles are called A-, B-periods respectively. (More exact description of A-, B-cycles needed for definition of corresponding A-, B-periods will be given in detail in the next paragraph.)

The main goal of this paper is to study the asymptotic behaviour of harmonic 1-forms $u_\varepsilon(x)$ with prescribed A-, B-periods on M_ε, as $\varepsilon \to 0$.

Notice that this paper is related to the theory of homogenization of boundary-value problems in strongly perforated domains. Such problems originate from the need to construct averaged models of physical processes in strongly inhomogeneous media [2, 3]. This branch of mathematical physics has been intensively developing, therefore numerous new tools, approaches, unusual settings of problems arised recently [4–6]. In particular, such objects as Riemannian manifolds and differential forms were applied to investigations by many authors (see, e.g., [7, 8]).

The problem considered here, as many other homogenization problems, can be reduced to a study of Laplace–Beltrami operator on Riemannian manifolds M_ε endowed with a metric depending on a small parameter ε. In [9] the Cauchy problem for the diffusion equation was considered on special Riemannian manifolds with complicated microstructure. Different kinds of homogenized models were obtained in dependence on the topological structure of M_ε and its metric. The asymptotic behaviour of harmonic vector fields on n-dimensional ($n \geq 3$) Riemannian manifolds was studied in [10].

Homogenization of harmonic 1-forms on Riemannian surfaces of more simple structure was studied in [11].

§ 2. Statement of main result

At first let us introduce some notations and state necessary assumptions.

Denote by $x_{\varepsilon i} \in F_{\varepsilon i}$ the center of the hole $F_{\varepsilon i}$, by $a_{\varepsilon i}$ its radius and by $r_{\varepsilon i}$ the distance from $x_{\varepsilon i}$ to the union of the centers of the others holes:

$$F_{\varepsilon i} = \{x \in \Gamma : dist(x, x_{\varepsilon i}) \leq a_{\varepsilon i}\},$$

$$r_{\varepsilon i} = \min_{j \neq i} dist(x_{\varepsilon i}, x_{\varepsilon j}).$$

The distance is measured relatively to the metric $g(x)$ on Γ.

We suppose that when $\varepsilon \to 0$,

$$a_{\varepsilon i} \to 0, \quad r_{\varepsilon i} \to 0, \tag{i}$$

so that

$$|\ln^{-1} a_{\varepsilon i}| \leq C r_{\varepsilon i}^2, \tag{ii}$$

where $C > 0$ doesn't depend on ε and i, i.e., the holes' sizes become exponentialy small in comparison with the distances from them to neighbouring ones.

Let us make an assumption about the distribution (relative positions) of holes on Γ. Let $R_{\varepsilon i}$ be the annulus in R^2, centered at the point $x_{\varepsilon i}$ with inner radius $a_{\varepsilon i}$ and outer radius $r_{\varepsilon i}/2$:

$$R_{\varepsilon i} = \{x \in \Gamma : a_{\varepsilon i} < dist(x, x_{\varepsilon i}) < \frac{r_{\varepsilon i}}{2}\}.$$

Clearly that Γ can be covered by a system of convex non-intersecting polyhedrons $\{\Pi_{\varepsilon i},\ i = 1, \ldots, 2N(\varepsilon)\}$ such that, for any i, $R_{\varepsilon i} \subset \Pi_{\varepsilon i} \subset \Gamma$; $\Gamma \subset \bigcup_i \Pi_{\varepsilon i}$. We assume that

$$d_{\varepsilon i} = diam(\Pi_{\varepsilon i}) \leq C r_{\varepsilon i}, \tag{iii}$$

where $C > 0$ doesn't depend on ε, i.e., covering polyhedrons are not very prolate.

Now let us introduce the quantities $V^{kl}_{\varepsilon ij}$ characterizing the metric on tubes $T^{kl}_{\varepsilon ij}$. For each linked pair of holes $(F^k_{\varepsilon i}, F^l_{\varepsilon j})$ consider the following domain $D^{kl}_{\varepsilon ij} \subset M_\varepsilon$:

$$D^{kl}_{\varepsilon ij} = \overline{T^{kl}_{\varepsilon ij}} \bigcup R^k_{\varepsilon i} \bigcup R^l_{\varepsilon j}.$$

Its boundary consists of two components

$$\partial D^{kl}_{\varepsilon ij} = S^k_{\varepsilon i} \bigcup S^l_{\varepsilon j},$$

where

$$S_{\varepsilon i} = \{x \in \Gamma : dist(x, x_{\varepsilon i}) = \frac{r_{\varepsilon i}}{2}\}.$$

In the domain $D^{kl}_{\varepsilon ij}$ consider the boundary value problem

$$\begin{cases} \Delta v(x) = 0, & x \in D^{kl}_{\varepsilon ij}; \\ v(x) = 0, & x \in S^k_{\varepsilon i}; \\ v(x) = 1, & x \in S^l_{\varepsilon j}. \end{cases} \tag{2.1}$$

There exists a unique solution $v^{kl}_{\varepsilon ij} = v^{kl}_{\varepsilon ij}(x)$ of this problem.

We set

$$V^{kl}_{\varepsilon ij} = \int\limits_{D^{kl}_{\varepsilon ij}} dv^{kl}_{\varepsilon ij} \wedge * dv^{kl}_{\varepsilon ij} \tag{2.2}$$

for linked pairs of holes $(F^k_{\varepsilon i}, F^l_{\varepsilon j})$ and $V^{kl}_{\varepsilon ij} = 0$ otherwise.

A solution $v^{kl}_{\varepsilon ij}(x)$ of the problem (2.1) minimizes the functional (2.2) over the class of functions satisfying the boundary conditions on $S^k_{\varepsilon i}$ and $S^l_{\varepsilon j}$. Therefore using variational methods, it is easy to obtain the inequality

$$V^{kl}_{\varepsilon ij} \leq C |\ln^{-1} \check{a}_{\varepsilon ij}|, \tag{2.3}$$

where $\check{a}_{\varepsilon ij} = \min\{a_{\varepsilon i}, a_{\varepsilon j}\}$, $C > 0$ doesn't depend ε.

The quantities (2.2) are positive and because of $v^{lk}_{\varepsilon ji} = 1 - v^{kl}_{\varepsilon ij}$, possess the symmetry $V^{kl}_{\varepsilon ij} = V^{lk}_{\varepsilon ji}$. They characterize the metric on tubes $T^{kl}_{\varepsilon ij}$. We assume that this metric satisfies an additional condition which provides the inequality

$$V^{kl}_{\varepsilon ij} \geq C |\ln^{-1} \hat{a}_{\varepsilon ij}|, \tag{iv}$$

where $\hat{a}_{\varepsilon ij} = \max\{a_{\varepsilon i}, a_{\varepsilon j}\}$, $C > 0$ doesn't depend on ε. Here $([i,k],[\,j,l\,])$ are linked pairs, that is why the inequality (iv) in view of (2.3) means that radiuses of linked circles are of the same order, as $\epsilon \to 0$.

At last, we should focus our attention on the precise definition of basic A-, B-cycles associated with A-, B-periods of a harmonic 1-form u_ε. We define A-cycle corresponding to the tube $T^{kl}_{\varepsilon ij}$ so that it envelopes the hole $F^k_{\varepsilon i}$ once and oriented in such a way that $F^k_{\varepsilon i}$ remains on the left.

Now we are interested in the description of B-cycles. They consist of components belonging to some tubes $T^{kl}_{\varepsilon ij}$ and situated on some copies of Γ_ε. We reduce the study of B-cycles to the investigation of special unclosed paths $L^{kl}_{\varepsilon ij}$ associated with the tubes. Let us turn to the precise definition of such paths.

Let us fix a point $x_0 \in \Gamma_\varepsilon$ and define oriented curves $L^{kl}_{\varepsilon ij}$ joining the points x^k_0 and x^l_0 and going along the tube $T^{kl}_{\varepsilon ij}$:

$$x^k_0 \to \partial F^k_{\varepsilon i} \to T^{kl}_{\varepsilon ij} \to \partial F^l_{\varepsilon j} \to x^l_0.$$

At first, we suppose that for any point $x \in \Gamma$ there exists a smooth curve $L(x_0, x)$ joining it with base point x_0 and oriented in the direction from x_0 to x. Then we assume that on Γ^k_ε $L^{kl}_{\varepsilon ij}$ goes along $L(x^k_0, x^k_{\varepsilon i})$ from x^k_0 to $\partial F^k_{\varepsilon i}$ bypassing the holes met on Γ^k_ε so that they are situated on the left. (Such a way of bypassings of holes will be ment everywhere below as well.)

The tube $T^{kl}_{\varepsilon ij}$ can be parametrized as $(z,\varphi) \in S_1 \times [0.1]$. We assume that after meeting the hole $F^k_{\varepsilon i}$, the path $L^{kl}_{\varepsilon ij}$ goes along its boundary $\partial F^k_{\varepsilon i}$ up to crossing the curve $\varphi = 0$. On the tube $L^{kl}_{\varepsilon ij}$ goes along $\varphi = 0$ in the direction from $\partial F^k_{\varepsilon i}$ to $\partial F^l_{\varepsilon j}$. Then it goes along $\partial F^l_{\varepsilon j}$. Meeting the path $L(x^l_0, x^l_{\varepsilon j})$, $L^{kl}_{\varepsilon ij}$ goes along it in the direction from $\partial F^l_{\varepsilon j}$ to x^l_0 bypassing the holes met on Γ^l_ε.

It's evident that any cycle on M_ε can be composed by summation of determined above A-cycles and curves $L^{kl}_{\varepsilon ij}$. That is why we consider them basic for our manifold.

We denote $A^{kl}_{\varepsilon ij}$ the integral of harmonic 1-form $u_\varepsilon(x)$ along A-cycle associated with the tube $T^{kl}_{\varepsilon ij}$ and $B^{kl}_{\varepsilon ij}$ the integral of $u_\varepsilon(x)$ along $L^{kl}_{\varepsilon ij}$. We will call the numbers $A^{kl}_{\varepsilon ij}$, $B^{kl}_{\varepsilon ij}$ A-periods and B-quaziperiods respectively. Everywhere below saying "A-periods (B-periods) are given", we will mean that we are given a set of numbers $A^{kl}_{\varepsilon ij}$ ($B^{kl}_{\varepsilon ij}$). Clearly that

$$A^{kl}_{\varepsilon ij} = -A^{lk}_{\varepsilon ji}; \quad B^{kl}_{\varepsilon ij} = -B^{lk}_{\varepsilon ji}.$$

Now we introduce the following generalized functions on $\Gamma \times \Gamma$:

$$V_{\varepsilon kl}(x,y) = \sum_{i,j} V^{kl}_{\varepsilon ij}\delta(x - x_{\varepsilon i})\ \delta(y - x_{\varepsilon j}), \quad k,l = \overline{1,m}; \tag{2.4}$$

$$a_{\varepsilon kl}(x,y) = \sum_{i,j} A^{kl}_{\varepsilon ij}\delta(x - x_{\varepsilon i})\delta(y - x_{\varepsilon j}), \quad k,l = \overline{1,m}; \tag{2.5}$$

$$b_{\varepsilon kl}(x,y) = \sum_{i,j} B^{kl}_{\varepsilon ij}V^{kl}_{\varepsilon ij}\delta(x - x_{\varepsilon i})\delta(y - x_{\varepsilon j}), \quad k,l = \overline{1,m}, \tag{2.6}$$

where $\delta(x)$ is delta-function on Γ.

We suppose that these functions converge in the distribution sense (in $D'(\Gamma\times\Gamma)$) to $V_{kl}(x,y)$, $a_{kl}(x,y)$, $b_{kl}(x,y)$ respectively, as $\varepsilon\to 0$:

$$w - \lim_{\varepsilon\to 0} V_{\varepsilon kl}(x,y) = V_{kl}(x,y), \quad k,l = \overline{1,m}; \tag{v}$$

$$w - \lim_{\varepsilon\to 0} a_{\varepsilon kl}(x,y) = a_{kl}(x,y), \quad k,l = \overline{1,m}; \tag{j}$$

$$w - \lim_{\varepsilon\to 0} b_{\varepsilon kl}(x,y) = b_{kl}(x,y), \quad k,l = \overline{1,m}. \tag{jj}$$

At last, we assume the following inequalities fulfilled:

$$\sum_{l.p.} |A^{kl}_{\varepsilon ij}|^2\, |\ln \hat{a}_{\varepsilon ij}| \le C_1; \tag{jjj}$$

$$\sum_{l.p.} |B^{kl}_{\varepsilon ij}|^2\, |\ln \check{a}_{\varepsilon ij}|^{-1} \le C_2 \tag{jv}$$

with independent on ε positive constants C_1, C_2.

We denote $L_2(M_\varepsilon)$ the Hilbert space of 1-forms on the manifold M_ε with scalar product related to the metric $g_\varepsilon(x)$, $L_2(\Gamma)$ the Hilbert space of 1-forms on Γ with scalar product related to the metric $g(x)$ and $L_2^{0\varepsilon}(\Gamma)$ the subspace in $L_2(\Gamma)$ of 1-forms that equal zero on all circles $F_{\varepsilon i}$ $(i = 1,\dots,2N(\varepsilon))$.

Let us introduce the operators $Q_{\varepsilon k}$, $k = 1,\dots,m$, mapping $L_2(M_\varepsilon)$ into $L_2^{0\varepsilon}(\Gamma)$ by the formula

$$\Big[Q_{\varepsilon k}u_\varepsilon\Big](x) = \begin{cases} u_\varepsilon(x), & x\times\{k\}\in\ \Gamma^k_\varepsilon; \\ 0, & x\in\ \bigcup_i F_{\varepsilon i}. \end{cases}$$

The main result of the paper is follows:

Theorem 1. *Let u_ε be a harmonic 1-form on Riemannian surface M_ε with given A- and B-periods. Suppose that conditions (i)–(v), (j)–(jv) are fulfilled, as $\varepsilon\to 0$.*

Then for any k, 1-form $Q_{\varepsilon k}u_\varepsilon$ converges weakly in $L_2(\Gamma)$ to 1-form $u_k(x)$ which is represented by

$$u_k(x) = *d\varphi_k(x) + d\psi_k(x), \quad k = \overline{1,m}, \tag{2.7}$$

where functions $\varphi_k(x)$, $\psi_k(x)$ satisfy on Γ the following homogenized equations:

$$\Delta\varphi_k(x) = \sum_{l=1}^{m} \int_\Gamma a_{kl}(x,y)\ d^2y, \quad k = \overline{1,m}; \tag{2.8}$$

$$\Delta\psi_k(x) + \sum_{l=1}^{m} \int_\Gamma V_{kl}(x,y)[\psi_l(y) - \psi_k(x)]\ d^2y = \sum_{l=1}^{m} \int_\Gamma b'_{kl}(x,y)\ d^2y; \tag{2.9}$$

$$b'_{kl}(x,y) = b(x,y) + V_{kl}(x,y) \sum_{p,q=1}^{m} \int_\Gamma \int_\Gamma (a_{pq}(\mu,\nu) + V_{pq}(\mu,\nu)[\varphi_p(\mu) - \varphi_q(\nu)])$$
$$\times [w_{xy}^{pql}(\nu) - w_{xy}^{pqk}(\mu)] d^2\mu d^2\nu, \quad k = \overline{1,m},$$

where d, $*$, Δ are exterior derivative, star operator and Laplace operator on Γ relative to the metric $g(x)$, $V_{kl}(x,y)$, $a_{kl}(x,y)$, $b_{kl}(x,y)$ were defined above by (2.4)–(2.6) respectively, $w_{xy}^{pqk}(z)$, $k = 1, \ldots, m$, is a solution of the problem

$$\Delta w_{xy}^{pqk}(z) + \sum_{l=1}^{m} \int V_{kl}(z,\zeta)[w_{xy}^{pql}(\zeta) - w_{xy}^{pqk}(z)]\ d\zeta = 0, \quad z \in \Gamma \setminus L_{xy}^{pqk};$$

$$(w_{xy}^{pqk})^+(z) - (w_{xy}^{pqk})^-(z) = \delta_{pq}^k, \quad z \in L_{xy}^{pqk};$$

$$(*dw_{xy}^{pqk})^+(z) - (*dw_{xy}^{pqk})^-(z) = 0, \quad z \in L_{xy}^{pqk};$$

$$\sum_k \int_\Gamma w_{xy}^{pqk} = 0;$$

$$|w_{xy}^{pqk}| < C \quad \text{on } \Gamma.$$

Here contour

$$L_{xy}^{pqk} = \delta_p^k L^p(x_0,x) + \delta_q^k L^q(y,x_0)$$

consists of one or two components or may be absent on Γ;

$$\delta_{pq}^k = \begin{cases} 1, & k \in \{p,q\}; \\ 0, & otherwise; \end{cases}$$

$(w_{xy}^{pqk})^+(z)$ $((w_{xy}^{pqk})^-(z))$ is the limit value of $w_{xy}^{pqk}(z)$ on the left (right) side of the cut.

§ 3. Representation of harmonic 1-forms on M_ε

First we note that a harmonic 1-form u_ε on M_ε with given A-, B-periods can be represented in the form

$$u_\varepsilon = v_\varepsilon + w_\varepsilon, \tag{3.1}$$

where w_ε is a harmonic 1-form with given A-periods $A^{kl}_{\varepsilon ij}$ and some B-periods, and v_ε is a harmonic 1-form whose A-periods are all equal zero and B-periods are given. We will denote $\hat{B}^{kl}_{\varepsilon ij}$ B-pseudoperiods of w_ε, and $\check{B}^{kl}_{\varepsilon ij} = B^{kl}_{\varepsilon ij} - \hat{B}^{kl}_{\varepsilon ij}$ B-pseudoperiods of v_ε.

Let us make cuts on M_ε along the middle contours $S^{kl}_{\varepsilon ij}$ on each tube $T^{kl}_{\varepsilon ij}$. Thus our manifold is divided on m components M^k_ε. The boundary of the cut manifold is the union

$$\bigcup_{i<j}(S^{kl}_{\varepsilon ij}\bigcup S^{lk}_{\varepsilon ji}),$$

where $S^{kl}_{\varepsilon ij}$ and $S^{lk}_{\varepsilon ji}$ $(i<j)$ denote different banks of the cut $S^{kl}_{\varepsilon ij}$. Besides, it will be useful to denote $T^k_{\varepsilon i}$, $T^l_{\varepsilon j}$ the components of cut tube, i.e.,

$$T^{kl}_{\varepsilon ij} = T^k_{\varepsilon i}\bigcup T^l_{\varepsilon j}.$$

Let us consider a collection of harmonic functions $\psi_{\varepsilon[m]} = \{\psi_{\varepsilon k},\ k = 1,\dots,m\}$ on M^k_ε

$$\Delta\psi_{\varepsilon k} \equiv \delta d\psi_{\varepsilon k} = 0, \quad x \in M^k_\varepsilon, \tag{3.2}$$

which have "jumps" on cuts $S^{kl}_{\varepsilon ij}$ equal to B-pseudoperiods $\check{B}^{kl}_{\varepsilon ij}$ of v_ε:

$$\psi_{\varepsilon k}|_{S^{kl}_{\varepsilon ij}} - \psi_{\varepsilon l}|_{S^{lk}_{\varepsilon ji}} = \check{B}^{kl}_{\varepsilon ij}; \tag{3.3}$$

$$d\psi_{\varepsilon k}|_{S^{kl}_{\varepsilon ij}} - d\psi_{\varepsilon l}|_{S^{lk}_{\varepsilon ji}} = 0. \tag{3.4}$$

Setting

$$\tilde{v}_\varepsilon = d\psi_{\varepsilon k}, \quad x \in M^k_\varepsilon, \quad k = \overline{1,m}, \tag{3.5}$$

we obtain a harmonic 1-form with trivial A-periods and some B-periods. B-pseudoperiods of $\tilde{v}_\varepsilon$ are not equal to the numbers $\check{B}^{kl}_{\varepsilon ij}$. But B-periods of $\tilde{v}_\varepsilon$ and v_ε derived from the corresponding sets of B-pseudoperiods will be the same. Hence $\tilde{v}_\varepsilon = v_\varepsilon$.

It is well known, that a collection of functions $\psi_{\varepsilon[m]} = \{\psi_{\varepsilon k},\ k = 1,\dots,m\}$ exists and minimizes the functional

$$J_{1\varepsilon}[\psi_{\varepsilon[m]}] = \sum_k \int_{M^k_\varepsilon} d\psi_{\varepsilon k} \wedge *d\psi_{\varepsilon k} \tag{3.6}$$

over the class $H^1(M_\varepsilon^k)$ of functions satisfying the conditions (3.3), (3.4) on M_ε^k $(k = 1, \ldots, m)$.

Now let us consider a harmonic 1-form w_ε from (3.1). We will show that it is represented on M_ε^k as follows

$$w_\varepsilon = *d\varphi_{\varepsilon k}, \quad x \in M_\varepsilon^k, \quad k = \overline{1, m}, \tag{3.7}$$

where $\varphi_{\varepsilon k}$ is a harmonic function on M_ε^k,

$$\Delta\varphi_{\varepsilon k} \equiv \delta d\varphi_{\varepsilon k} = 0, \quad x \in M_\varepsilon^k, \tag{3.8}$$

satisfying the following conditions on cuts $S_{\varepsilon ij}^{kl}$:

$$\varphi_{\varepsilon k}|_{S_{\varepsilon ij}^{kl}} - \varphi_{\varepsilon l}|_{S_{\varepsilon ji}^{lk}} = \tilde{B}_{\varepsilon ij}^{kl}; \tag{3.9}$$

$$d\varphi_{\varepsilon k}|_{S_{\varepsilon ij}^{kl}} - d\varphi_{\varepsilon l}|_{S_{\varepsilon ji}^{lk}} = 0; \tag{3.10}$$

$$\int\limits_{S_{\varepsilon ij}^{kl}} *d\varphi_{\varepsilon k} = - \int\limits_{S_{\varepsilon ji}^{lk}} *d\varphi_{\varepsilon l} = A_{\varepsilon ij}^{kl}, \tag{3.11}$$

where $\tilde{B}_{\varepsilon ij}^{kl}$ are arbitrary constants, $A_{\varepsilon ij}^{kl} = -A_{\varepsilon ji}^{lk}$ is A-period of w_ε associated with the tube $T_{\varepsilon ij}^{kl}$.

A collection of functions $\varphi_{\varepsilon[m]} = \{\varphi_{\varepsilon k},\ k = 1, \ldots, m\}$ exists and minimizes the functional

$$J_{2\varepsilon}[\varphi_{\varepsilon[m]}] = \sum_k \int\limits_{M_\varepsilon^k} d\varphi_{\varepsilon k} \wedge *d\varphi_{\varepsilon k} - \sum_{i,k} A_{\varepsilon ij}^{kl} \tilde{B}_{\varepsilon ij}^{kl} \tag{3.12}$$

over the class $H^1(M_\varepsilon^k)$ of functions satisfying the conditions (3.9)–(3.11) on M_ε^k $(k = 1, \ldots, m)$.

Thus we obtain representations of harmonic 1-forms v_ε, w_ε on M_ε that will be used for investigation of the asymptotic behaviour of these forms, as $\varepsilon \to 0$. Notice that existence and uniqueness of these forms follow from above representations if solvabilities of minimization problems (3.6), (3.12) are proved.

Theorem 2. *Let v_ε be a harmonic 1-form with given B-periods and trivial A-periods. Assume that for B-pseudoperiods $\check{B}_{\varepsilon ij}^{kl}$ of v_ε the conditions (i)–(v) and (jj), (jv) are fulfilled, as $\varepsilon \to 0$.*

Then

$$Q_{\varepsilon k} v_\varepsilon \to v_k \quad weakly \quad in \quad L_2(\Gamma),$$

where v_k $(k = 1, \ldots, m)$ *are exact 1-forms on* Γ *such that* $v_k = d\psi_k$, $\{\psi_k(x),\ k = 1, \ldots, m\}$ *is the collection of functions being the solution of the problem*

$$\Delta\psi_k + \sum_{l=1}^{m} \int\limits_{\Gamma} V_{kl}(x,y)[\psi_k(x) - \psi_l(y)]\, d^2y \ = \sum_{l=1}^{m} \int\limits_{\Gamma} \check{b}_{kl}(x,y)\, d^2y, \quad x \in \Gamma,\ k = \overline{1,m}. \tag{3.13}$$

Theorem 3. *Let* w_ε *be a harmonic 1-form with given A-periods* $A^{kl}_{\varepsilon ij}$ *and represented by (3.7)–(3.11). Assume that the conditions (i)–(v) and (j), (jjj) are fulfilled, as* $\varepsilon \to 0$.

Then

$$Q_{\varepsilon k} w_\varepsilon \to w_k \quad \textit{weakly in} \quad L_2(\Gamma),$$

where w_k $(k = 1, \ldots, m)$ *are 1-forms on* Γ *such that* $w_k = *d\varphi_k$, $\{\varphi_k(x),\ k = 1, \ldots, m\}$ *is the collection of functions being the solution of the problem*

$$\Delta\varphi_k = \sum_{l=1}^{m} \int\limits_{\Gamma} a_{kl}(x,y)\, d^2y, \quad x \in \Gamma,\ k = \overline{1,m}. \tag{3.14}$$

With a help of these theorems we obtain the asymptotic behaviour of 1-forms starting from given B-periods of v_ε and given A-periods of w_ε. But we have to study the contribution produced by B-periods of w_ε as well (see § 6). Hence we will obtain the complete asymptotic description (2.7)–(2.9) of harmonic 1-form u_ε .

§ 4. Proof of Theorem 2

Let $\check{B} = \{\check{B}^{kl}_{\varepsilon ij}\}$ be the set of B-pseudoperiods of harmonic 1-form v_ε and let $\psi_{\varepsilon[m]} = \{\psi_{\varepsilon k},\ k = 1, \ldots, m\}$ be the collection of functions of the class $H^1(M^k_\varepsilon, \check{B})$ (defined by (3.2)–(3.4)) minimizing the functional (3.6).

Let us introduce a collection of functions $\psi^0_{\varepsilon[m]} = \{\psi^0_k(x),\ k = 1, \ldots, m\}$ of the class $H^1(M^k_\varepsilon, \check{B})$ and set

$$\phi_\varepsilon(x) = \sum_k (\psi_{\varepsilon k}(x) - \psi^0_{\varepsilon k}(x))\chi_{\varepsilon k}(x); \tag{4.1}$$

$$v^0_\varepsilon = \sum_k d\psi^0_{\varepsilon k}\chi_{\varepsilon k}(x), \tag{4.2}$$

where $\chi_{\varepsilon k}(x)$ is the characteristic function of $\overline{M^k_\varepsilon}$. It follows from properties of $\psi_{\varepsilon[m]}$, $\psi^0_{\varepsilon[m]}$ that ϕ_ε is the function of $H^1(M_\varepsilon)$ minimizing the functional

$$J_\varepsilon(\phi_\varepsilon) = \int\limits_{M_\varepsilon} d\phi_\varepsilon \wedge *d\phi_\varepsilon + 2\sum_k \int\limits_{M^k_\varepsilon} d\psi^0_{\varepsilon k} \wedge *d\phi_\varepsilon. \tag{4.3}$$

It's easy to see that $v_\varepsilon = v_\varepsilon^0 + d\phi_\varepsilon$ on M_ε. We will choose below a collection $\psi^0_{\varepsilon[m]}$ so that, for any k, $Q_{\varepsilon k} v_\varepsilon^0 \to 0$ weakly in $L_2^{(1)}(\Gamma)$, as $\varepsilon \to 0$. Then to prove Theorem 2 we will show that $Q_{\varepsilon k} d\phi_\varepsilon \to d\phi_k$ weakly in $L_2^{(1)}(\Gamma)$, where $\phi_k(x)$ is a function on Γ with square integrable derivatives that minimizes the functional

$$J(\phi) = \sum_k \int_\Gamma |\nabla \phi_k|^2 d^2x$$

$$+\sum_{k,l} \int_\Gamma \int_\Gamma V_{kl}(x,y)[\phi_k(x) - \phi_l(y)]^2 d^2x\, d^2y + 2\sum_{k,l} \int_\Gamma b_{kl}(x,y)\phi_k(x) dx dy. \quad (4.4)$$

Here

$$|\nabla \phi_k|^2 = \sum_{\alpha=1}^{2} \left| \frac{\partial \phi_k}{\partial x^\alpha} \right|^2,$$

$V_{kl}(x,y)$ and $b_{kl}(x,y)$ were defined by (v) and (jj) respectively, integrals are taken with respect to Lebesgue measure on Γ.

At first we describe the abstract scheme of the minimization problems (4.3), (4.4).

Let H_ε be a Hilbert space depending on a parameter $\varepsilon > 0$, with scalar product $(\cdot,\cdot)_\varepsilon$ and norm $||\cdot||_\varepsilon$, F_ε is a continuous linear functional on H_ε uniformly bounded with respect to ε.

Let H be another Hilbert space with scalar product $(\cdot,\cdot)$ and norm $||\cdot||$, F is a continuous linear functional on H.

Let ϕ_ε and ϕ be solutions of the following minimization problems:

$$\inf_{\phi_\varepsilon \in H_\varepsilon} \Big[||\phi_\varepsilon||_\varepsilon^2 + F_\varepsilon(\phi_\varepsilon) \Big], \quad (4.5)$$

$$\inf_{\phi \in H} \Big[||\phi||^2 + F(\phi) \Big] \quad (4.6)$$

respectively.

The questions are in what sense and under what conditions ϕ_ε converges to ϕ.

Theorem 4. *Assume that we are given a dense subspace $M \subset H$ and, for each $\varepsilon > 0$, linear operators $Q_\varepsilon : H_\varepsilon \to H$, $P_\varepsilon : M \to H_\varepsilon$ that satisfy conditions (a)–(c):*

$$||Q_\varepsilon u_\varepsilon|| \le C||u_\varepsilon||_\varepsilon; \quad (a)$$

for any $u_\varepsilon \in H_\varepsilon$, any $u \in M$ and any $v_\varepsilon \in H_\varepsilon$ such that $Q_\varepsilon v_\varepsilon \to v$ weakly in H, as $\varepsilon \to 0$, we have:

$$Q_\varepsilon P_\varepsilon u \to u \quad \textit{weakly in } H, \quad as \varepsilon \to 0; \quad (b_1)$$

$$\lim_{\varepsilon\to 0} ||P_\varepsilon u||_\varepsilon^2 = ||u||^2; \qquad (b_2)$$

$$\overline{\lim_{\varepsilon\to 0}} |(P_\varepsilon u, v_\varepsilon)_\varepsilon| \le C||u||\,||v||; \qquad (b_3)$$

$$\lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) = F(v). \qquad (c)$$

Then solution ϕ_ε of the minimization problem (4.5) converges to solution ϕ of the minimization problem (4.6) in the following sense:

$$Q_\varepsilon\phi_\varepsilon \to \phi \quad \textit{weakly in } H, \quad \varepsilon\to 0.$$

We will apply this theorem, which was proved in [9]. In our situation $H_\varepsilon = \hat{H}^1(M_\varepsilon)$ is the Hilbert space of local square integrable functions $\phi_\varepsilon(x)$ on M_ε having square integrable derivatives and for which

$$\sum_k \overline{Q_{\varepsilon k}\phi_\varepsilon} = 0.$$

Scalar product in H_ε is defined by

$$(\phi_\varepsilon, \psi_\varepsilon)_\varepsilon = \int\limits_{M_\varepsilon} d\phi_\varepsilon \wedge * \, d\psi_\varepsilon.$$

The Hilbert space $H = \hat{H}^1(\Gamma)$ we define as the space of local integrable vector-functions $\phi(x) = \{\phi_k(x),\ k = 1,\dots,m\}$ on Γ that have square integrable derivatives and for which

$$\sum_k \int\limits_\Gamma \phi_k = 0.$$

We endow it with scalar product

$$(\phi,\psi) = \sum_k \int\limits_\Gamma d\phi_k \wedge * d\psi_k$$

$$+\sum_{k,l}\int\limits_\Gamma\int\limits_\Gamma V_{kl}(x,y)[\phi_k(x)-\phi_l(y)][\psi_k(x)-\psi_l(y)]\,d^2x\,d^2y.$$

Linear functionals F_ε in $\hat{H}^1(M_\varepsilon)$ and F in $\hat{H}^1(\Gamma)$ are defined by the formulae

$$F_\varepsilon[\phi_\varepsilon] = 2\sum_k \int\limits_{M_\varepsilon^k} d\phi_\varepsilon \wedge * \, d\psi^0_{\varepsilon k}, \quad \phi_\varepsilon \in \hat{H}^1(M_\varepsilon); \qquad (4.7)$$

$$F[\phi] = 2\sum_{k,l} \int\limits_\Gamma b_{kl}(x,y)\phi(x)\,dxdy, \quad \phi\in\hat{H}^1(\Gamma). \qquad (4.8)$$

Then minimization problems (4.3), (4.4) can be reformulated as (4.5), (4.6) respectively. To apply Theorem 4 we must define operators Q_ε, P_ε and functions $\psi^0_{\varepsilon k}$ so that conditions (a), $(b_1 - b_3)$, (c) of Theorem 4 are satisfied.

By virtue of (ii) there exists an extension operator $Q'_\varepsilon : H^1(M_\varepsilon) \to H^1(\Gamma)$ such that

$$||Q'_\varepsilon \phi_\varepsilon||_{H^1(\Gamma)} \le C||\phi_\varepsilon||_{H^1(M_\varepsilon)} \tag{4.9}$$

for any $\phi_\varepsilon \in H^1(M_\varepsilon)$ [10]. Here and below we denote $H^1(G)$ the Sobolev space of functions on domain $G \subset R^2$, constant C doesn't depend on ε.

Such a continuation, of course, is not unique. However, we may choose a unique one that minimizes norms in spaces $H(F_{\varepsilon i})$. Keeping in mind this, we define the operator $Q_\varepsilon : \hat{H}^1(M_\varepsilon) \to \hat{H}^1(\Gamma)$ in the following way:

$$\Big[Q_\varepsilon \phi_\varepsilon\Big](x) = Q'_\varepsilon \phi_\varepsilon - \overline{Q'_\varepsilon \phi_\varepsilon}, \quad \phi_\varepsilon \in \hat{H}^1(M_\varepsilon).$$

It obviously follows from (4.9) that Q_ε is a linear operator satisfying the condition (a) of Theorem 4.

Now let us define the operator $P_\varepsilon : M \to \hat{H}^1(M_\varepsilon)$ and functions $\psi^0_{\varepsilon k}(x)$ representing the functional F_ε in (4.7). First let us introduce on M_ε the following functions:

$$\varphi^k_{\varepsilon i}(x) = \begin{cases} 1, & x \in T^k_{\varepsilon i}; \\ \varphi\left(\frac{|x-x_{\varepsilon i}|}{r_{\varepsilon i}}\right), & x \in R^k_{\varepsilon i}; \\ 0, & x \in M^k_\varepsilon \setminus (T^k_{\varepsilon i} \bigcup R^k_{\varepsilon i}); \end{cases}$$

$$\hat{\varphi}^k_{\varepsilon i}(x) = \begin{cases} 1, & x \in T^k_{\varepsilon i}; \\ \varphi\left(\frac{|x-x_{\varepsilon i}|}{4a_{\varepsilon i}}\right), & x \in R^k_{\varepsilon i}; \\ 0, & x \in M^k_\varepsilon \setminus (T^k_{\varepsilon i} \bigcup R^k_{\varepsilon i}), \end{cases}$$

where $\varphi(t) \ge 0$ is a twice continuously differentiable function on real line such that $\varphi(t) = 1$ for $t \le \frac{1}{4}$ and $\varphi(t) = 0$ for $t \ge \frac{1}{2}$, other notations correspond to introduced in §§ 1, 2.

We choose the space $C^2_0(\Gamma)$ of twice continuously differentiable functions with compact support in Γ as a dense subset M in $\hat{H}^1(\Gamma)$ and set

$$\Big[P'_\varepsilon \phi\Big](x) = \phi_k(x)\left(1 - \sum_i \hat{\varphi}^k_{\varepsilon i}(x)\right) + \sum_i \phi_k(x_{\varepsilon i})\hat{\varphi}^k_{\varepsilon i}(x)$$

$$+ \sum_i v^{kl}_{\varepsilon ij}(\phi_l(x_{\varepsilon j}) - \phi_k(x_{\varepsilon i}))\varphi^k_{\varepsilon i}(x),$$

$$P_\varepsilon \phi = P'_\varepsilon \phi - \overline{Q_\varepsilon P'_\varepsilon \phi}.$$

Besides we define the functions $\psi_\varepsilon^0(x)$ as follows:

$$\psi_{\varepsilon k}^0(x) = \sum_i \sum_{j,l} \check{B}_{\varepsilon ij}^{kl} v_{\varepsilon ij}^{kl}(x) \varphi_{\varepsilon i}^k(x), \quad x \in M_\varepsilon^k, \ k = 1, \dots, m. \tag{4.10}$$

Here $v_{\varepsilon ij}^{kl}(x)$ is a solution of the problem (2.1) and $\check{B} = \{\check{B}_{\varepsilon ij}^{kl}\}$ is the set of B-quaziperiods of 1-form under consideration.

Since $v_{\varepsilon ij}^{kl}(x) = 1 - v_{\varepsilon ji}^{lk}(x)$, $P_\varepsilon \phi \in \hat{H}^1(M_\varepsilon)$ and $\psi_{\varepsilon[m]}^0 \in H^1(M_\varepsilon^k, \check{B})$. Hence one can show that conditions (a) and (b_i) of Theorem 4 are fulfilled. It remains to check only the condition (c).

Let $\phi_\varepsilon(x)$ be a function of $\hat{H}^1(M_\varepsilon)$ such that $Q_\varepsilon \phi_\varepsilon$ converges weakly to ϕ in $\hat{H}^1(\Gamma)$, when $\varepsilon \to 0$. Taking into account (4.7), (4.10) and properties of functions $v_{\varepsilon ij}^{kl}(x)$ and $\varphi_{\varepsilon i}^k(x)$, by using Green's formulae we obtain

$$F_\varepsilon[\phi_\varepsilon] = 2 \sum_{i,k} \sum_{j,l} \check{B}_{\varepsilon ij}^{kl} \int\limits_{R_{\varepsilon i}^k} \Delta(v_{\varepsilon ij}^{kl} \varphi_{\varepsilon i}^k) \wedge * \phi_\varepsilon \tag{4.11}$$

and according to (2.2)

$$\int\limits_{R_{\varepsilon i}^k} \Delta(v_{\varepsilon ij}^{kl} \varphi_{\varepsilon i}^k) \wedge *1 = \int\limits_{\partial F_{\varepsilon i}^k} * dv_{\varepsilon ij}^{kl}$$

$$= \int\limits_{S_{\varepsilon j}^l} v_{\varepsilon ij}^{kl} \wedge * dv_{\varepsilon ij}^{kl} = \int\limits_{D_{\varepsilon ij}^{kl}} dv_{\varepsilon ij}^{kl} \wedge * dv_{\varepsilon ij}^{kl} = V_{\varepsilon ij}^{kl}. \tag{4.12}$$

We can triangulate Γ by convex polyhedrons $\Pi_{\varepsilon i}$ containing the sets $R_{\varepsilon i}$ and satisfying the condition (iii). For such polyhedrons the Poincare inequality

$$\int\limits_{\Pi_{\varepsilon i}} \phi^2 \, dxdy \le C \left(r_{\varepsilon i}^2 \bar{\phi}_i^2 + r_{\varepsilon i}^2 \int\limits_{\Pi_{\varepsilon i}} |\nabla \phi|^2 \, dxdy \right) \tag{4.13}$$

holds true for any $\phi \in H^1(\Pi_{\varepsilon i})$ [10]. Here $\bar{\phi}_i$ is the mean value of ϕ on $\Pi_{\varepsilon i}$, constant C does not depend on ε.

Let $\chi_{\varepsilon i}(x)$ be the characteristic function of the polyhedron $\Pi_{\varepsilon i}$ and $\overline{\phi_{\varepsilon k}^i}$ be the mean value of the function $(Q_\varepsilon \phi_\varepsilon)_k$ in $\Pi_{\varepsilon i}$. We set

$$\bar{\chi}_{\varepsilon k}(x) = \sum_i \bar{\phi}_{\varepsilon k}^i \chi_{\varepsilon i}(x)$$

and

$$\overline{B_{\varepsilon k}}(x) = \sum_i \sum_{j,l} \check{B}_{\varepsilon ij}^{kl} V_{\varepsilon ij}^{kl} |\Pi_{\varepsilon i}|^{-1} \chi_{\varepsilon i}(x).$$

Then according to (4.11), (4.12) and properties of functions $v^{kl}_{\varepsilon ij}, \varphi^{k}_{\varepsilon i}$ we get

$$F_\varepsilon[\phi_\varepsilon] = 2\sum_k \int \overline{B_{\varepsilon k}}(x)\bar{\chi}_{\varepsilon k}(x)\, d^2x + E_\varepsilon[\phi_\varepsilon], \tag{4.14}$$

where

$$E_\varepsilon[\phi_\varepsilon] = 2\sum_{i,k} \check{B}^{kl}_{\varepsilon ij} \int \Delta(v^{kl}_{\varepsilon ij}\varphi^{k}_{\varepsilon i})\,[(Q_\varepsilon\phi_\varepsilon)_k - \bar{\chi}_{\varepsilon k}]\, d^2x. \tag{4.15}$$

Since $(Q_\varepsilon\phi_\varepsilon)_k$ weakly converges to ϕ_k, as $\varepsilon \to 0$, it remains uniformly bounded in $H^1(\Gamma)$, therefore converges strongly in $L_2(\Gamma)$. Then from the Poincare inequality and condition (i) we obtain that $\bar{\chi}_{\varepsilon k}(x)$ also converges to ϕ_k in $L_2(\Gamma)$.

Then, taking into account (jj), (jv) and (2.3), it is easy to obtain that

$$\lim_{\varepsilon\to 0}\sum_k \int \overline{B_{\varepsilon k}}(x)\bar{\chi}_{\varepsilon k}(x)\, d^2x = \sum_{k,l}\int b_{kl}(x,y)\phi_k(x)\, dxdy. \tag{4.16}$$

With the help of the estimates

$$\left|D^\alpha v^{kl}_{\varepsilon ij}(x)\right| \le \begin{cases} C|ln a_{\varepsilon i}|^{-1}|ln\rho_{\varepsilon i}|, & |\alpha| = 0; \\ C|ln a_{\varepsilon i}|^{-1}\rho_{\varepsilon i}^{-|\alpha|}, & |\alpha| = 1,2; \end{cases}$$

$$\rho_{\varepsilon i} = dist(x, x_{\varepsilon i}); \quad x \in R^k_{\varepsilon i}, \tag{4.17}$$

for the solution $v^{kl}_{\varepsilon ij}$ of the problem (2.1), we get the following estimate for $E_\varepsilon[\phi_\varepsilon]$ in (4.15):

$$|E_\varepsilon[\phi_\varepsilon]| \le C\Big\{\sum_i \frac{1}{r^2_{\varepsilon i}|ln a_{\varepsilon i}|^2}\Big\}^{1/2}\sum_k ||(Q_\varepsilon\phi_\varepsilon)_k - \bar{\chi}_{\varepsilon k}||_{L_2(\Gamma)}.$$

Then in view of (i), (ii) and the convergence of $(Q_\varepsilon\phi_\varepsilon)_k$ and $\bar{\chi}_{\varepsilon k}$ to ϕ_k in $L_2(\Gamma)$

$$\lim_{\varepsilon\to 0} E_\varepsilon[\phi_\varepsilon] = 0. \tag{4.18}$$

It follows from (4.14), (4.16), (4.18) and (4.8) that condition (c) of Theorem 4 is fulfilled.

Applying Theorem 4, we conclude that $Q_\varepsilon\phi_\varepsilon$ converges weakly to $\phi = (\phi_1 \ldots \phi_m)$ in $H^1(\Gamma)$, where ϕ_ε and ϕ are solutions of the minimization problems (4.3), (4.4) respectively. It means that 1-form $d(Q_\varepsilon\phi_\varepsilon)_k$ converges to 1-form $d\phi_k$ weakly in $L_2(\Gamma)$, as $\varepsilon \to 0$. We have

$$Q_{\varepsilon k}[d\phi_\varepsilon] = d(Q_\varepsilon\phi_\varepsilon)_k - d(Q_\varepsilon\phi_\varepsilon)_k \wedge \chi_\varepsilon$$

with χ_ε the characteristic function of the union of all circles $F_{\varepsilon i}$. In view of conditions (i), (ii) we have

$$\lim_{\varepsilon\to 0}\int \chi_\varepsilon = 0.$$

Therefore $d(Q_\varepsilon\phi_\varepsilon)\wedge\chi_\varepsilon$ converges weakly to zero in $L_2(\Gamma)$, so $Q_{\varepsilon k}[d\phi_\varepsilon]$ converges to $d\phi_k$. Finally, it follows from (4.2), (4.10) that $Q_{\varepsilon k}[v_\varepsilon^0]$ converges weakly to zero in $L_2(\Gamma)$, when $\varepsilon\to 0$, what can be shown in the same manner, using estimates (4.17) and (jv).

Thus a harmonic 1-form $v_\varepsilon = v_\varepsilon^0 + d\phi_\varepsilon$ converges to $d\psi_k$ in the sense defined above, and this proves Theorem 2.

§ 5. Proof of Theorem 3

Let $\varphi_{\varepsilon[m]} = \{\varphi_{\varepsilon k}(x),\ k,\dots,m\}\in H^1(M_\varepsilon^c)$ be a collection of functions minimizing the functional (3.9). The function

$$\phi_\varepsilon(x) = \sum_k (\varphi_{\varepsilon k}(x) - \psi^0_{\varepsilon k}(x))\chi_{\varepsilon k}(x)$$

and a collection of arbitrary constants $\tilde B_\varepsilon = \{\tilde B^{kl}_{\varepsilon ij}\}$ (considered as independent variables) minimize the functional

$$J[\phi_\varepsilon,\tilde B_\varepsilon] = \int\limits_{M_\varepsilon} d\phi_\varepsilon\wedge * d\phi_\varepsilon + \frac{1}{2}\sum_{i,k}\hat V^{kl}_{\varepsilon ij}(\tilde B^{kl}_{\varepsilon ij})^2$$

$$+2\sum_{i,k}\tilde B^{kl}_{\varepsilon ij}\int\limits_{R^k_{\varepsilon i}} d(v^{kl}_{\varepsilon ij}\varphi^k_{\varepsilon i})\wedge * d\varphi_\varepsilon - \sum_{i,k} A^{kl}_{\varepsilon ij}\tilde B^{kl}_{\varepsilon ij}, \tag{5.1}$$

where

$$\hat V^{kl}_{\varepsilon ij} = \int\limits_{R^k_{\varepsilon i}} d(v^{kl}_{\varepsilon ij}\varphi^k_{\varepsilon i})\wedge * d(v^{kl}_{\varepsilon ij}\varphi^k_{\varepsilon i}) + \int\limits_{R^l_{\varepsilon j}} d(v^{lk}_{\varepsilon ji}\varphi^l_{\varepsilon j})\wedge * d(v^{lk}_{\varepsilon ji}\varphi^l_{\varepsilon j}).$$

Taking into account (2.2), properties of functions $v^{kl}_{\varepsilon ij}(x)$, $\varphi^k_{\varepsilon i}(x)$ and conditions $(i), (ii), (iv)$, we obtain

$$\hat V^{kl}_{\varepsilon ij} = V^{kl}_{\varepsilon ij}\,(1+o(1)) \qquad (\varepsilon\to 0). \tag{5.2}$$

The third term in the right-hand side of (5.1) we transform with a help of the Green's formula and represent $J(\phi_\varepsilon,\tilde B_\varepsilon)$ in the form

$$J(\phi_\varepsilon,\tilde B_\varepsilon) = J_0(\phi_\varepsilon,\tilde B_\varepsilon) - \sum_{i,k} A^{kl}_{\varepsilon ij}\tilde B^{kl}_{\varepsilon ij} \tag{5.3}$$

with

$$J_0(\phi_\varepsilon,\tilde B_\varepsilon) = \int\limits_{M_\varepsilon} d\phi_\varepsilon\wedge * d\phi_\varepsilon + \frac{1}{2}\sum_{i,k}\hat V^{kl}_{\varepsilon ij}(\tilde B^{kl}_{\varepsilon ij})^2$$

$$+2\sum_{i,k}\tilde{B}^{kl}_{\varepsilon ij}\int\limits_{R^k_{\varepsilon i}}\Delta(v^{kl}_{\varepsilon ij}\varphi^k_{\varepsilon i})\wedge *\phi_\varepsilon. \tag{5.4}$$

Using the estimates (4.17), the Poincare inequality (4.13), properties of functions $v^{kl}_{\varepsilon ij}(x)$, $\varphi^k_{\varepsilon i}(x)$ and conditions (i), (ii), (iv), and (2.2), one can prove (see [10]) that there exist positive constants C_0 and ε_0 such that for $\varepsilon < \varepsilon_0$

$$J_0(\phi_\varepsilon,\tilde{B}_\varepsilon)\geq C_0\left[\int\limits_{M_\varepsilon}d\phi_\varepsilon\wedge *d\phi_\varepsilon+\sum_{i,k}|\ln a_{\varepsilon i}|^{-1}(\tilde{B}^{kl}_{\varepsilon ij})^2\right]. \tag{5.5}$$

Since $J(\phi_\varepsilon,\tilde{B}_\varepsilon)\leq J(0,0)=0$, from (5.3), (5.5), and (jjj) we get

$$\int\limits_{M_\varepsilon}d\phi_\varepsilon\wedge *d\phi_\varepsilon+\sum_i|\ln a_{\varepsilon i}|^{-1}(\tilde{B}^{kl}_{\varepsilon ij})^2\leq C\sum_i(A^{kl}_{\varepsilon ij})^2|\ln a_{\varepsilon i}|<C', \tag{5.6}$$

where C, C' are constants independent of ε.

Hence the sequence of functions $\{Q_\varepsilon\phi_\varepsilon,\ \varepsilon>0\}$ is weakly compact in $\hat{H}^1(\Gamma)$ and we can select a weakly converging subsequence $\{Q_{\varepsilon_\nu}\phi_{\varepsilon_\nu},\ \varepsilon_\nu\to 0,\}$: $Q_{\varepsilon_\nu}\phi_{\varepsilon_\nu}\to\phi$ weakly in $H^1(\Gamma)$. So, by the compact embedding theorem this subsequence in fact converges in $L_2(\Gamma)$. Without loss of generality we may also consider condition (v) fulfilled, as $\varepsilon=\varepsilon_\nu\to 0$.

Since derivatives of the functional (5.3) in respect of the variables $\tilde{B}^{kl}_{\varepsilon ij}$ is equal to zero at the point of minimum, we have

$$\hat{V}^{kl}_{\varepsilon ij}\tilde{B}^{kl}_{\varepsilon ij}=A^{kl}_{\varepsilon ij}+\int\limits_{R^k_{\varepsilon i}}\Delta(v^{kl}_{\varepsilon ij}\varphi^k_{\varepsilon i})\wedge *\phi_\varepsilon+\int\limits_{R^l_{\varepsilon j}}\Delta(v^{lk}_{\varepsilon ji}\varphi^l_{\varepsilon j})\wedge *\phi_\varepsilon. \tag{5.7}$$

Using the cutoff function, in the same way as in proof of Theorem 2 we get

$$V^{kl}_{\varepsilon ij}\tilde{B}^{kl}_{\varepsilon ij}=\theta^{kl}_{\varepsilon ij}\left[A^{kl}_{\varepsilon ij}+V^{kl}_{\varepsilon ij}(\bar{\phi}^i_{\varepsilon k}-\bar{\phi}^j_{\varepsilon l})\right]+E^{kl}_{\varepsilon ij}, \tag{5.8}$$

where $\bar{\phi}^i_{\varepsilon k}$ is the mean value of the function $(Q_\varepsilon\varphi_\varepsilon)_k$ in the polyhedron $\Pi_{\varepsilon i}$, $\theta^{kl}_{\varepsilon ij}\to 1$ uniformly with respect to i, j, when $\varepsilon=\varepsilon_\nu\to 0$, and $E^{kl}_{\varepsilon ij}$ satisfies the estimate

$$|E^{kl}_{\varepsilon ij}|\leq C\frac{1}{r_{\varepsilon i}|\ln a_{\varepsilon i}|}\left(\int\limits_{\Pi_{\varepsilon i}}|(Q_\varepsilon\varphi_\varepsilon)_k-\bar{\phi}^i_{\varepsilon k}|^2\,d^nx\right)^{1/2}$$

$$+C\frac{1}{r_{\varepsilon j}|\ln a_{\varepsilon j}|}\left(\int\limits_{\Pi_{\varepsilon j}}|(Q_\varepsilon\varphi_\varepsilon)_l-\bar{\phi}^j_{\varepsilon l}|^2\,d^nx\right)^{1/2}. \tag{5.9}$$

Taking into account (5.8), (5.9), (i)–(iii), (v), (j), and the convergence of subsequences $\{Q_{\varepsilon_\nu}\phi_{\varepsilon_\nu}\}$ and χ_{ε_ν} to ϕ, we conclude that

$$\tilde{b}_{\varepsilon kl}(x,y) = \sum_{i,j} V_{\varepsilon ij}^{kl}\ \tilde{B}_{\varepsilon ij}^{kl}\delta(x - x_{\varepsilon i})\delta(y - x_{\varepsilon j})$$

converges, as $\varepsilon = \varepsilon_\nu \to 0$, weakly in $D'(R^2)$ to

$$\tilde{b}_{kl}(x,y) = a_{kl}(x,y) + V_{kl}(x,y)(\phi_k(x) - \phi_l(y)). \tag{5.10}$$

It follows from (5.6) that the collection of constants $\{\tilde{B}_{\varepsilon ij}^{kl}\}$ also satisfies the condition (jv), so all conditions of Theorem 2 for 1-form w_ε with pseudoperiods $\tilde{B}_{\varepsilon ij}^{kl}$ are fulfilled, when $\varepsilon = \varepsilon_\nu \to 0$. Applying Theorem 2 in view of (5.10), we see that the limit ϕ corresponding to the subsequence ε_ν is a solution of the problem (3.14). Since this problem has a unique solution, Theorem 3 is proved.

§ 6. B-periods of harmonic 1-form w_ε

Here we will study B-periods of harmonic 1-form $w_\varepsilon = *d\varphi_\varepsilon$ defined by (3.7)–(3.11). Taking into account their asymptotics, we will be able to obtain the asymptotic description of harmonic 1-form u_ε and thereby to complete the proof of Theorem 1.

Let us fix a tube $T_{\varepsilon st}^{pq}$ and make cut on it along the corresponding contour $L_{\varepsilon st}^{pq}$ (see § 2) .

At first let us consider the function $w_{\varepsilon st}^{pq}(z)$ being a solution of the following problem:

$$\Delta w_{\varepsilon st}^{pq} \ = \ 0, \qquad z \in M_{\varepsilon st}^{c} = M_\varepsilon \setminus L_{\varepsilon st}^{pq}; \tag{6.1}$$

$$(w_{\varepsilon st}^{pq})^{+} - (w_{\varepsilon st}^{pq})^{-} = 1, \qquad z \in L_{\varepsilon st}^{pq}; \tag{6.2}$$

$$(dw_{\varepsilon st}^{pq})^{+} - (dw_{\varepsilon st}^{pq})^{-} = 0, \qquad z \in L_{\varepsilon st}^{pq} \tag{6.3}$$

and such that

$$\sum_k \overline{Q_{\varepsilon k} w_{\varepsilon st}^{pq}} = 0; \tag{6.4}$$

$$|w_{\varepsilon st}^{pq}| < C. \tag{6.5}$$

Notice that a solution of this problem is unique. Really, let w_1, w_2 be solutions of the problem (6.1)–(6.5), $w_1 \neq w_2$. Their difference will be a harmonic continuousely differentiable function everywhere in M_ε exept points $x_0^p \in \Gamma^p$, $x_0^q \in \Gamma^q$ (the ends of the contour $L_{\varepsilon st}^{pq}$). Then according to the theorem of removal of isolate singularity, we obviously obtain $w_1 \equiv w_2$.

Let us construct the solution of the problem (6.1)–(6.5). At first let us consider the complex surface $\mathbf{C}$ with Euclidean metric and make the cut along a curve joining the points a, $b \in \mathbf{C}$. In some neighbourhood $U(a)$ the bounded harmonic function $u_a = \frac{1}{2\pi} arg(z-a)$ has jump equal 1 on the cut and continuous derivative. In a similar manner we can construct a bounded harmonic function u_b in some neighbourhood $U(b)$ with the needed properties.

Now consider our Riemannian surface Γ with some fixed local chart on it. Then we can find the functions $u_0(z)$, $u_s(z)$ in some neighbourhoods $U(x_0^p)$, $U(x_{\varepsilon s}^p)$ of the ($U(x_0^p)$, $U(x_{\varepsilon s}^p) \subset \Gamma^p \setminus L_{\varepsilon s}^p$) corresponding explicitly constructed above u_a, u_b. (Here $L_{\varepsilon s}^p = L(x_0^p, x_{\varepsilon s}^p)$ is a curve joining the points x_0^p, $x_{\varepsilon s}^p$ being the component of $L_{\varepsilon st}^{pq}$ situated on Γ^p.) Both of them are harmonic in corresponding neighbourhoods with cuts, having jumps equal 1 and continuous derivatives on the cut.

Let us choose the finite covering

$$\bigcup_{i=1}^{k} U_i, \quad U_i \subset \Gamma^p \setminus L_{\varepsilon s}^p$$

of $L_{\varepsilon s}^p$ containing chosen above neighbourhoods $U(x_0^p)$, $U(x_{\varepsilon s}^p)$ ($U_1 = U(x_0^p)$, $U_k = U(x_{\varepsilon s}^p)$) and such that the points x_0, $x_{\varepsilon s}$ covered only by $U(x_0)$, $U(x_{\varepsilon s})$ respectively.

Let $v_i(z)$, $z \in U_i$, $i = 1, \ldots, k$, be the functions having jumps equal 1 on $L_{\varepsilon s}^p$, moreover, $v_1(z)$, $v_k(z)$ equal $u_0(z)$, $u_s(z)$ constructed above respectively. Let $\varphi_i(z) \in C^\infty$, $i = 1, \ldots, k$, be a partition of unity, i.e.,

$$\sum_i \varphi_i = 1; \quad \varphi_i(z) = 0, \ z \in \Gamma \setminus U_i.$$

Then the function

$$w_{0s}^p(z) = \sum_{i=1}^{k} v_i \varphi_i$$

will be a solution of the following problem:

$$\Delta w_{0s}^p = f^p, \quad z \in \Gamma^p \setminus L_{\varepsilon s}^p;$$

$$(w_{0s}^p)^+ - (w_{0s}^p)^- = 1, \quad (dw_{0s}^p)^+ - (dw_{0s}^p)^- = 0, \quad z \in L_{\varepsilon s}^p,$$

where f^p is some function that equals zero in neighbourhoods of holes and such that $\int_\Gamma f^p = 0$.

We can construct by analogy a function $w_{0t}^q(z)$ being a solution of the following problem:

$$\Delta w_{0t}^q = f^q, \quad z \in \Gamma^q \setminus L_{\varepsilon t}^q;$$

$$(w_{0t}^q)^+ - (w_{0t}^q)^- = 1, \quad (dw_{0t}^q)^+ - (dw_{0t}^q)^- = 0, \quad z \in L_{\varepsilon t}^q,$$

where $L^q_{\varepsilon t} = -L(x^q_0, x^q_{\varepsilon t})$ is a curve joining the points $x_{\varepsilon t}$, x_0 being the component of $L^{pq}_{\varepsilon st}$ situated on Γ^q; f^q is some function that equals zero in neighbourhoods of holes and such that $\int_\Gamma f^q = 0$.

Now let us construct the function $w^T_0(z)$ being a solution of the problem

$$\Delta w^T_0 = 0, \quad z \in \hat{T}^{pq}_{\varepsilon st} \setminus (L^{pq}_{\varepsilon st})^T; \tag{6.6}$$

$$(w^T_0)^+ - (w^T_0)^- = 1, \quad (dw^T_0)^+ - (dw^T_0)^- = 0, \quad z \in (L^{pq}_{\varepsilon st})^T. \tag{6.7}$$

Here $(L^{pq}_{\varepsilon st})^T$ is the component of $L^{pq}_{\varepsilon st}$ belonging to the tube,

$$\hat{T}^{pq}_{\varepsilon st} = \overline{T^{pq}_{\varepsilon st}} \bigcup \tilde{R}^p_{\varepsilon s} \bigcup \tilde{R}^q_{\varepsilon t};$$

$$\tilde{R}^k_{\varepsilon i} = \{z \in \Gamma : a_{\varepsilon i} < dist(z, x_{\varepsilon i}) < \frac{r_{\varepsilon i}}{4}\}.$$

After parametrization we can consider $\hat{T}^{pq}_{\varepsilon st} \setminus (L^{pq}_{\varepsilon st})^T$ as a rectangle. It's easy to construct a function $w^T_0(z)$ being harmonic in our rectangle and satisfying the following boundary conditions. We set it equals $u_s(z), u_t(z)$ on corresponding opposite edges of a rectangle. On the another pair of opposite edges it is to be equal to 0 and 1 respectively and is to have equal derivatives on both of them. Thereby we obtain the function $w^T_0(z)$ being the solution of the problem (6.6)–(6.7).

At last we construct on M_ε the function $w_{0\varepsilon}(z)$ in the following way:

$$w_{0\varepsilon}(z) = w_0(z) + \left(w^T_0(z) - w_0(z)\right) \varphi^{pq}_{\varepsilon st}(z)$$
$$+ \sum_{\underline{l.p.}} v^{kl}_{\varepsilon ij}(z) \varphi^k_{\varepsilon i}(z) (w^l_0(x_{\varepsilon j}) - w^k_0(x_{\varepsilon i})),$$

where $w_0(z) = \{w^k_0(z), k = 1, \ldots, m\}$ is a vector-function of the following form:

$$w_0(z) = \begin{cases} w^p_{0s}(z), & k = p; \\ w^q_{0t}(z), & k = q; \\ w^k_0(z), & otherwise, \end{cases}$$

where w^k_0 is a harmonic function on Γ^k, $k \neq p, q$,

$$\varphi^{pq}_{\varepsilon st}(z) = \begin{cases} 1, & z \in \hat{T}^{pq}_{\varepsilon st}; \\ \varphi\left(\frac{|z - x_{\varepsilon s}|}{r_{\varepsilon s}}\right), & z \in \hat{R}^p_{\varepsilon s}; \\ \varphi\left(\frac{|z - x_{\varepsilon t}|}{r_{\varepsilon t}}\right), & z \in \hat{R}^q_{\varepsilon t}; \\ 0, & z \in M_\varepsilon \setminus (\hat{T}^{pq}_{\varepsilon st} \bigcup \hat{R}^p_{\varepsilon s} \bigcup \hat{R}^q_{\varepsilon t}); \end{cases}$$

$$\hat{R}^k_{\varepsilon i} = \{z \in \Gamma : \frac{r_{\varepsilon i}}{4} < dist(z, x_{\varepsilon i}) < \frac{r_{\varepsilon i}}{2}\};$$

$$\varphi_{\varepsilon i}^{k}(z)=\begin{cases}1, & z\in T_{\varepsilon ij}^{kl};\\ \varphi\left(\frac{|z-x_{\varepsilon i}|}{r_{\varepsilon i}}\right), & z\in R_{\varepsilon i}^{k};\\ 0, & z\in M_\varepsilon\setminus(T_{\varepsilon ij}^{kl}\bigcup R_{\varepsilon i}^{k}).\end{cases}$$

Here $\varphi(t)\geq 0$ is a twice continuously differentiable function on R such that $\varphi(t)=1$ for $t\leq\frac{1}{4}$ and $\varphi(t)=0$ for $t\geq\frac{1}{2}$; $\overline{l.p.}$ denotes the set of all linked pairs except $([s,p],[t,q])$.

Thus we have the function $w_{0\varepsilon}(z)$ being a solution of the following problem:

$$\Delta w_{0\varepsilon}(z)=f+f_\varepsilon,\quad z\in M_\varepsilon\setminus L_{\varepsilon st}^{pq};$$

$$(w_{0\varepsilon}(z))^+-(w_{0\varepsilon}(z))^-=1,\quad (dw_{0\varepsilon}(z))^+-(dw_{0\varepsilon}(z))^-=0,\quad z\in L_{\varepsilon st}^{pq}. \tag{6.8}$$

Here f equals zero on Γ^k, $k\neq p,q$, and on all tubes, and equals f^p, f^q on Γ^p, Γ^q respectively; f_ε appears because of bypassings of the holes on Γ_ε.

Clearly, that

$$Q_{\varepsilon k}w_{0\varepsilon}\to w_0^k,\quad k=1,\ldots,m, \tag{6.9}$$

as $\varepsilon\to 0$, by the construction.

Let us take into consideration a smooth function $w_{1\varepsilon}$ satisfying

$$\Delta w_{1\varepsilon}=-f-f_\varepsilon,\quad z\in M_\varepsilon. \tag{6.10}$$

Then

$$w_{\varepsilon st}^{pq}(z)=w_{0\varepsilon}(z)+w_{1\varepsilon}(z)$$

will be the solution of the problem (6.1)–(6.5).

Lemma 1. *The solution of the problem (6.1)–(6.5) converges in the sense*

$$Q_{\varepsilon k}w_{\varepsilon st}^{pq}\to w_{st}^{pqk}\ \in L_2(\Gamma\setminus L_{st}^{pqk}),\quad k=\overline{1,m},$$

as $\varepsilon\to 0$, to the solution $w_{st}^{pqk}(z)$ of the problem

$$\Delta w_{st}^{pqk}(z)+\sum_{l=1}^{m}\int V_{kl}(z,\zeta)[w_{st}^{pql}(\zeta)-w_{st}^{pqk}(z)]\,d\zeta=0,\quad z\in\Gamma\setminus L_{st}^{pqk}; \tag{6.11}$$

$$(w_{st}^{pqk})^+(z)-(w_{st}^{pqk})^-(z)=\delta_{pq}^{k},\quad z\in L_{st}^{pqk}; \tag{6.12}$$

$$(dw_{st}^{pqk})^+(z)-(dw_{st}^{pqk})^-(z)=0,\quad z\in L_{st}^{pqk}; \tag{6.13}$$

$$\sum_k\int_\Gamma w_{st}^{pqk}=0; \tag{6.14}$$

$$|w_{st}^{pqk}(z)|<C,\quad z\in\Gamma, \tag{6.15}$$

where the contour

$$L_{st}^{pqk} = \delta_p^k L^p(x_0, s) + \delta_q^k L^q(t, x_0)$$

consists of one or two components or is absent on Γ*;*

$$\delta_{pq}^k = \begin{cases} 1, & k \in \{p, q\}; \\ 0, & otherwise. \end{cases}$$

P r o o f. At first let us study the asymptotic behaviour of f_ε. As it was mentioned above, f_ε appears because of bypassings of holes on Γ, so it equals zero on tubes. Let us show that

$$f_\varepsilon \to \hat{f}^k, \quad weakly\ in\ L_2(\Gamma),\ k = 1, \ldots, m, \tag{6.16}$$

as $\varepsilon \to 0$, where

$$\hat{f}^k(z) = \sum_{l=1}^m \int V_{kl}(z, \zeta)(w_0^l(\zeta) - w_0^k(z))d\zeta, \quad z \in \Gamma.$$

To obtain the sought convergence of f_ε we will show that

$$\int f_\varepsilon^2 dz < C, \tag{6.17}$$

where C doesn't depend on ε, and

$$\int g f_\varepsilon^k dz \to \int g(z) \left(\sum_{l=1}^m \int V_{kl}(z, \zeta)(w_0^l(\zeta) - w_0^k(z))d\zeta \right) dz,\ z \in \Gamma,\ k = 1, \ldots, m, \tag{6.18}$$

as $\varepsilon \to 0$, for any $g \in C^2(\Gamma)$.

Let us consider

$$\int f_\varepsilon^2 dz = \int [\Delta(w_{0\varepsilon} - w_0)]^2$$

$$= \int\limits_{\hat{R}_{\varepsilon s}^p \bigcup \hat{R}_{\varepsilon t}^q} [\Delta \left(w_0^T(z) - w_0(z) \right) \varphi_{\varepsilon st}^{pq}(z)]^2$$

$$+ \sum_{\overline{l.p.}} \int_{R_{\varepsilon i}^k} [(w_0^l(x_{\varepsilon j}) - w_0^k(x_{\varepsilon i}))\Delta(v_{\varepsilon ij}^{kl}(z)\varphi_{\varepsilon i}^k(z))]^2$$

$$= I_1 + I_2.$$

Because of boundness of functions w_0^T, w_0, and properties of $\varphi_{\varepsilon st}^{pq}(z)$, the first term of the right-hand side of the last equality is bounded:

$$|I_1| \leq C_1,$$

where C_1 doesn't depend on ε. Taking into account properties of functions $v^{kl}_{\varepsilon ij}(z)$, $\varphi^k_{\varepsilon i}(z)$, we can estimate the second one as follows:

$$|I_2| \le C_2 \sum_i \frac{1}{r^2_{\varepsilon i}|lna_{\varepsilon i}|^2},$$

with C_2 not depending on ε. Hence the estimate (6.17) holds.

Now let us take a function $g \in C^2(\Gamma)$ and consider

$$\int g f^k_\varepsilon dz = \sum_i \int g(x_{\varepsilon i}) f^k_\varepsilon dz + \sum_i \int (g(z) - g(x_{\varepsilon i})) f^k_\varepsilon dz$$

$$= \sum_i \int g(x_{\varepsilon i}) \sum_{i,j,l} (w^l_0(x_{\varepsilon j}) - w^k_0(x_{\varepsilon i})) \Delta(v^{kl}_{\varepsilon ij}(z)\varphi^k_{\varepsilon i}(z)) dz + I_3 + I_4 + I_5, \ k = 1, \dots, m, \tag{6.19}$$

where

$$I_3 = \sum_i \int g(x_{\varepsilon i}) \Delta \left(w^T_0(z) - w_0(z) \right) \varphi^{pq}_{\varepsilon st}(z) dz;$$

$$I_4 = \sum_i \int (g(z) - g(x_{\varepsilon i})) \Delta \left(w^T_0(z) - w_0(z) \right) \varphi^{pq}_{\varepsilon st}(z) dz;$$

$$I_5 = \sum_i \int (g(z) - g(x_{\varepsilon i})) \sum_{i,j,l} (w^l_0(x_{\varepsilon j}) - w^k_0(x_{\varepsilon i})) \Delta(v^{kl}_{\varepsilon ij}(z)\varphi^k_{\varepsilon i}(z)).$$

In view of properties of functions w^T_0, w_0, $v^{kl}_{\varepsilon ij}(z)$, $\varphi^{pq}_{\varepsilon st}(z)$, $\varphi^k_{\varepsilon i}(z)$, and the estimate (4.17) we obtain:

$$|I_3| \le C_3 \Big\{ \sum_i r^2_{\varepsilon i} \Big\}^{1/2}; \tag{6.20}$$

$$|I_4| \le C_4 \Big\{ \sum_i r^4_{\varepsilon i} \Big\}^{1/2}; \tag{6.21}$$

$$|I_5| \le C_5 \Big\{ \sum_i \frac{1}{|\ln a_{\varepsilon i}|^2} \Big\}^{1/2}, \tag{6.22}$$

where constants C_3, C_4, C_5 do not depend on ε.

The first term of the right-hand side of (6.19) can be expressed in the following way:

$$\sum_i \int g(x_{\varepsilon i}) \sum_{i,j,l} (w^l_0(x_{\varepsilon j}) - w^k_0(x_{\varepsilon i})) \Delta(v^{kl}_{\varepsilon ij}(z)\varphi^k_{\varepsilon i}(z))\, dz$$

$$= \sum_i g(x_{\varepsilon i}) \sum_{i,j,l} (w^l_0(x_{\varepsilon j}) - w^k_0(x_{\varepsilon i})) \int\limits_{R^k_{\varepsilon i}} \Delta(v^{kl}_{\varepsilon ij}(z)\varphi^k_{\varepsilon i}(z))$$

$$= \sum_i g(x_{\varepsilon i}) \sum_{i,j,l} (w_0^l(x_{\varepsilon j}) - w_0^k(x_{\varepsilon i})) \int_{\partial F_{\varepsilon i}^k} *dv_{\varepsilon ij}^{kl}$$

$$= \sum_i g(x_{\varepsilon i}) \sum_{i,j,l} (w_0^l(x_{\varepsilon j}) - w_0^k(x_{\varepsilon i})) V_{\varepsilon ij}^{kl}, \quad k = 1, \ldots, m. \tag{6.23}$$

Hence from (6.19), (6.23), and the estimates (6.20)–(6.22) we obtain the convergence (6.18), thereby (6.16) is proved.

Taking into account this result, in a similar manner as in [9] we obtain that

$$Q_{\varepsilon k} w_{1\varepsilon} \to w_1^k \quad in\ L_2(\Gamma), \quad k = 1, \ldots, m, \tag{6.24}$$

as $\varepsilon \to 0$, where $w_1^k(z)$ satisfies the equation

$$\Delta w_1^k + \sum_{l=1}^m \int V_{kl}(z,\zeta)(w_1^l(\zeta) - w_1^k(z))d\zeta = -f^k - \sum_{l=1}^m \int V_{kl}(z,\zeta)(w_0^l(\zeta) - w_0^k(z))d\zeta. \tag{6.25}$$

Then in view of (6.19), (6.24), (6.25), $w_{\varepsilon st}^{pq}(z)$ converges in the sense

$$Q_{\varepsilon k} w_{\varepsilon st}^{pq} \to w_{st}^{pqk} \quad in\ L_2(\Gamma), \quad k = 1, \ldots, m,$$

to the solution of the problem (6.11)–(6.15), so Lemma 1 is proved.

Lemma 2. *Let $w_{\varepsilon st}^{pq}$ be a solution the problem (6.1)–(6.5). Then the following asymptotic formula holds true:*

$$\int_{S_{\varepsilon ij}^{kl}} *dw_{\varepsilon st}^{pq} = V_{\varepsilon ij}^{kl}((\bar{w}_{\varepsilon st}^{pq})^{ik} - (\bar{w}_{\varepsilon st}^{pq})^{jl}\ ) + \hat{E}_{\varepsilon ij}^{kl}, \tag{6.26}$$

where

$$|\hat{E}_{\varepsilon ij}^{kl}| \le C \left(\frac{1}{r_{\varepsilon i} |\ln a_{\varepsilon i}|^2} + \frac{1}{r_{\varepsilon j} |\ln a_{\varepsilon j}|^2} \right) \tag{6.27}$$

with constant C not depending on ε.

P r o o f. Let us write down Green's formulae for functions $w_{\varepsilon st}^{pq}$ and $v_{\varepsilon ij}^{kl}$:

$$\int_{\partial D_{\varepsilon ij}^{kl}} *dw_{\varepsilon st}^{pq} \wedge v_{\varepsilon ij}^{kl} = \int_{D_{\varepsilon ij}^{kl}} dv_{\varepsilon ij}^{kl} \wedge *dw_{\varepsilon st}^{pq} + \int_{D_{\varepsilon ij}^{kl}} \Delta w_{\varepsilon st}^{pq} v_{\varepsilon ij}^{kl};$$

$$\int_{\partial D_{\varepsilon ij}^{kl}} *dv_{\varepsilon ij}^{kl} \wedge w_{\varepsilon st}^{pq} = \int_{D_{\varepsilon ij}^{kl}} dv_{\varepsilon ij}^{kl} \wedge *dw_{\varepsilon st}^{pq} + \int_{D_{\varepsilon ij}^{kl}} \Delta v_{\varepsilon ij}^{kl} w_{\varepsilon st}^{pq}.$$

Since $v^{kl}_{\varepsilon ij}$, $w^{pq}_{\varepsilon st}$ are harmonic and by the definition of $v^{kl}_{\varepsilon ij}$ (see § 2) we have

$$\int\limits_{\partial D^{kl}_{\varepsilon ij}} *dw^{pq}_{\varepsilon st}\wedge v^{kl}_{\varepsilon ij} = \int\limits_{S^{lk}_{\varepsilon ji}} *dw^{pq}_{\varepsilon st}.$$

Consequently,

$$\int\limits_{S^{lk}_{\varepsilon ji}} *dw^{pq}_{\varepsilon st} = \int\limits_{\partial D^{kl}_{\varepsilon ij}} *dv^{kl}_{\varepsilon ij}\wedge w^{pq}_{\varepsilon st}. \tag{6.28}$$

Set

$$(\bar{w}^{pq}_{\varepsilon st})^{ik} = \frac{1}{|\hat{R}^k_{\varepsilon i}|}\int\limits_{\hat{R}^k_{\varepsilon i}} w^{pq}_{\varepsilon st},$$

where $|\hat{R}^k_{\varepsilon i}|$ is the volume of the ring $\hat{R}^k_{\varepsilon i}$. We can represent the right-hand side of (6.28) in the following form:

$$\int\limits_{\partial D^{kl}_{\varepsilon ij}} *dv^{kl}_{\varepsilon ij}\wedge w^{pq}_{\varepsilon st} = \int\limits_{\partial (D^{kl}_{\varepsilon ij})^r} *dv^r_\varepsilon\wedge w^{pq}_{\varepsilon st} + \int\limits_{\partial (D^{kl}_{\varepsilon ij}\setminus (D^{kl}_{\varepsilon ij})^r)} *dv^r_\varepsilon\wedge w^{pq}_{\varepsilon st}, \tag{6.29}$$

where $r\in \hat{R}^k_{\varepsilon i}(\hat{R}^l_{\varepsilon j})$, v^r_ε is a solution of the problem (2.1) on

$$(D^{kl}_{\varepsilon ij})^r = \overline{T}^{kl}_{\varepsilon ij}\bigcup (R^k_{\varepsilon i})^r\bigcup (R^l_{\varepsilon j})^r;$$

$$(R^k_{\varepsilon i})^r = \{x\in\Gamma : a_{\varepsilon i} < dist(x, x_{\varepsilon i}) < r\};$$

$$\partial (D^{kl}_{\varepsilon ij})^r = (S^k_{\varepsilon i})^r\bigcup (S^l_{\varepsilon j})^r;$$

$$(S^k_{\varepsilon i})^r = \{x\in\Gamma : dist(x, x_{\varepsilon i}) = r\}.$$

Let us consider

$$\int\limits_{\partial (D^{kl}_{\varepsilon ij})^r} *dv^r_\varepsilon\wedge w^{pq}_{\varepsilon st} = \int\limits_{(S^k_{\varepsilon i})^r} *dv^r_\varepsilon\wedge (\bar{w}^{pq}_{\varepsilon st})^{ik} + \int\limits_{(S^l_{\varepsilon j})^r} *dv^r_\varepsilon\wedge (\bar{w}^{pq}_{\varepsilon st})^{jl}$$

$$+\int\limits_{(S^k_{\varepsilon i})^r} *dv^r_\varepsilon\wedge (w^{pq}_{\varepsilon st} - (\bar{w}^{pq}_{\varepsilon st})^{ik} + \int\limits_{(S^l_{\varepsilon j})^r} *dv^r_\varepsilon\wedge (w^{pq}_{\varepsilon st} - (\bar{w}^{pq}_{\varepsilon st})^{jl}). \tag{6.30}$$

Using the Poincare inequality and (4.17), we obtain that

$$\int\limits_{(S^k_{\varepsilon i})^r} *dv^r_\varepsilon\wedge (w^{pq}_{\varepsilon st} - (\bar{w}^{pq}_{\varepsilon st})^{ik}) \le \frac{C}{|\ln a_{\varepsilon i}|}\left(\int\limits_{\hat{R}^k_{\varepsilon i}} |\nabla w|^2\right)^{1/2},$$

where constant C doesn't depend on ε. It's obvious that the analogous estimate holds for $\int_{(S^l_{\varepsilon j})^r} *dv^r_\varepsilon \wedge (w^{pq}_{\varepsilon st} - (\bar{w}^{pq}_{\varepsilon st})^{jl})$.

The other two terms of the right-hand side of (6.30) we rewrite as follows:

$$\int\limits_{(S^k_{\varepsilon i})^r} *dv^r_\varepsilon \wedge (\bar{w}^{pq}_{\varepsilon st})^{ik} + \int\limits_{(S^l_{\varepsilon j})^r} *dv^r_\varepsilon \wedge (\bar{w}^{pq}_{\varepsilon st})^{jl} = (V^{kl}_{\varepsilon ij})^r ((\bar{w}^{pq}_{\varepsilon st})^{jl} - (\bar{w}^{pq}_{\varepsilon st})^{ik}),$$

where

$$(V^{kl}_{\varepsilon ij})^r = \int\limits_{(D^{kl}_{\varepsilon ij})^r} dv^r_\varepsilon \wedge *dv^r_\varepsilon.$$

By using variational methods we obtain the following estimates:

$$(V^{kl}_{\varepsilon ij})^r = V^{kl}_{\varepsilon ij} + (\hat{E}^{kl}_{\varepsilon ij})^r \quad in\ (D^{kl}_{\varepsilon ij})^r, \tag{6.31}$$

where

$$|(\hat{E}^{kl}_{\varepsilon ij})^r| \le C_1 \left(\frac{1}{r_{\varepsilon i} |\ln a_{\varepsilon i}|^2} + \frac{1}{r_{\varepsilon j} |\ln a_{\varepsilon j}|^2} \right), \tag{6.32}$$

where C_1 doesn't depend on ε.

Moreover, it turns out that

$$|(V^{kl}_{\varepsilon ij})^r| \le C_2 \left(\frac{1}{r_{\varepsilon i} |\ln a_{\varepsilon i}|^2} + \frac{1}{r_{\varepsilon j} |\ln a_{\varepsilon j}|^2} \right) \quad in\ D^{kl}_{\varepsilon ij} \setminus (D^{kl}_{\varepsilon ij})^r, \tag{6.33}$$

where C_2 doesn't depend on ε.

Then from (6.28), using (6.29), (6.31)–(6.33), we come to the asyptotic formulae (6.26), (6.27). Lemma 2 is proved.

Lemma 3. *Let $w_\varepsilon = *d\varphi_\varepsilon$ be the harmonic 1-form defined by (3.1), (3.6)–(3.10). Then its B-pseudoperiods are expressed by the formula*

$$\hat{B}^{pq}_{\varepsilon st} = \int\limits_{L^{pq}_{\varepsilon st}} *d\varphi_\varepsilon = \sum_{l.p.} \tilde{B}^{kl}_{\varepsilon ij} \int\limits_{S^{kl}_{\varepsilon ij}} *dw^{pq}_{\varepsilon st}, \tag{6.34}$$

where $\tilde{B}^{kl}_{\varepsilon ij}$ were defined above in (3.9) as "jumps" of φ_ε on $S^{kl}_{\varepsilon ij}$.

P r o o f. Let us make cuts $S^{kl}_{\varepsilon ij}, L^{kl}_{\varepsilon ij}$ along contours accociated with A- and B-cycles respectively on M_ε. Taking into account the orientation, we denote $(S^{kl}_{\varepsilon ij})^+, (S^{kl}_{\varepsilon ij})^-$, $(L^{kl}_{\varepsilon ij})^+, (L^{kl}_{\varepsilon ij})^-$ different banks of them. We obtain m domains $\tilde{M}^k_\varepsilon$ $(k = 1, \ldots, m)$ with piecewise smooth boundaries. Functions φ_ε from (3.7)–(3.11) and the solution of the problem (6.1)–(6.5) $w^{pq}_{\varepsilon st}$ are twice continuously differentiable on $\tilde{M}_\varepsilon^k$ $(k = 1, \ldots, m)$. Writing down Green's formula for φ_ε and $w_{\varepsilon st}^{pq}$ on $\tilde{M}_\varepsilon^k$ $(k = 1, \ldots, m)$ and summing by k, we get

$$0 = \sum_k \int\limits_{\tilde{M}_\varepsilon^k} (\Delta\varphi_\varepsilon w_{\varepsilon st}^{pq} - \Delta w_{\varepsilon st}^{pq} \varphi_\varepsilon)$$

$$= \sum_{l.p.} \left(\int\limits_{(S_{\varepsilon ij}^{kl})^+} *d\varphi_\varepsilon w_{\varepsilon st}^{pq} + \int\limits_{(S_{\varepsilon ij}^{kl})^-} *d\varphi_\varepsilon w_{\varepsilon st}^{pq} + \int\limits_{(L_{\varepsilon ij}^{kl})^+} *d\varphi_\varepsilon w_{\varepsilon st}^{pq} + \int\limits_{(L_{\varepsilon ij}^{kl})^-} *d\varphi_\varepsilon w_{\varepsilon st}^{pq} \right)$$

$$- \sum_{l.p.} \left(\int\limits_{(S_{\varepsilon ij}^{kl})^+} *dw_{\varepsilon st}^{pq} \varphi_\varepsilon + \int\limits_{(S_{\varepsilon ij}^{kl})^-} *dw_{\varepsilon st}^{pq} \varphi_\varepsilon + \int\limits_{(L_{\varepsilon ij}^{kl})^+} *dw_{\varepsilon st}^{pq} \varphi_\varepsilon + \int\limits_{(L_{\varepsilon ij}^{kl})^-} *dw_{\varepsilon st}^{pq} \varphi_\varepsilon \right).$$

According to properties of functions φ_ε, $w_{\varepsilon st}^{pq}$, we have:

$$\int\limits_{(S_{\varepsilon ij}^{kl})^+} *d\varphi_\varepsilon w_{\varepsilon st}^{pq} + \int\limits_{(S_{\varepsilon ij}^{kl})^-} *d\varphi_\varepsilon w_{\varepsilon st}^{pq} = 0;$$

$$\int\limits_{(L_{\varepsilon ij}^{kl})^+} *dw_{\varepsilon st}^{pq} \varphi_\varepsilon + \int\limits_{(L_{\varepsilon ij}^{kl})^-} *dw_{\varepsilon st}^{pq} \varphi_\varepsilon = 0;$$

$$\sum \int\limits_{(L_{\varepsilon ij}^{kl})^+} *d\varphi_\varepsilon w_{\varepsilon st}^{pq} + \int\limits_{(L_{\varepsilon ij}^{kl})^-} *d\varphi_\varepsilon w_{\varepsilon st}^{pq} = \int\limits_{L_{\varepsilon st}^{pq}} *d\varphi.$$

From this we obviousely obtain

$$\hat{B}_{\varepsilon st}^{pq} = \int\limits_{L_{\varepsilon st}^{pq}} *d\varphi = \sum_{l.p.} \tilde{B}_{\varepsilon ij}^{kl} \int\limits_{S_{\varepsilon ij}^{kl}} *dw_{\varepsilon st}^{pq}.$$

Thereby Lemma 3 is proved.

R e m a r k. The function $w_{\varepsilon st}^{pq}$ doesn't belong to the class $W_2^1(\tilde{M}_\varepsilon^k)$ $(k = 1, \ldots, m)$, because it's derivatives grow in neighbourhoods of points x_0^k $(k = 1, \ldots, m)$. That's why applying of Green's formula needs an additional justification, that is omitted here.

It was proved above (see § 5) that

$$\tilde{B}_{\varepsilon ij}^{kl} = \theta_{\varepsilon ij}^{kl} \left[(V_{\varepsilon ij}^{kl})^{-1} A_{\varepsilon ij}^{kl} + (\bar{\varphi}_{\varepsilon k}^i - \bar{\varphi}_{\varepsilon l}^j) \right] + (V_{\varepsilon ij}^{kl})^{-1} E_{\varepsilon ij}^{kl} \tag{6.35}$$

with $E^{kl}_{\varepsilon ij}$ satisfying the estimate (5.9), $\theta^{kl}_{\varepsilon ij} \to 1$ uniformly with respect to i, j, as $\varepsilon \to 0$. Thus we have the asymptotic representation (6.34), (6.26), (6.35) of B-pseudoperiods of the harmonic 1-form w_ε.

To apply Theorem 2 we should check the conditions (jj), (jv) for the found B-pseudoperiods $\hat{B}^{pq}_{\varepsilon st}$ of w_ε . It's easy to see that, due to the conditions (iv), (jjj), and the estimates (2.3), (5.9), (6.27), the condition (jv) holds true. To check the condition (jj) let us consider the following generalized function on $(\Gamma)^2$:

$$\hat{b}_{\varepsilon pq} = \sum_{s,t} V^{pq}_{\varepsilon st} \hat{B}^{pq}_{\varepsilon st} \delta(x - x_{\varepsilon s})\ \delta(y - x_{\varepsilon t}), \tag{6.36}$$

$$\hat{B}^{pq}_{\varepsilon st} = \sum_{k,l} \sum_{i,j} \left[A^{kl}_{\varepsilon ij} + V^{kl}_{\varepsilon ij} (\bar{\varphi}^i_{\varepsilon k} - \bar{\varphi}^j_{\varepsilon l}) \right] ((\bar{w}^{pq}_{\varepsilon st})^{ik} - (\bar{w}^{pq}_{\varepsilon st})^{jl}).$$

Taking into consideration the conditions (i), (ii), (v), and (j), asymptotic formulae (6.26), (6.35), the estimates (5.9) and (6.27), the convergence of $(Q_\varepsilon \varphi_\varepsilon)_k$ to φ_k and the convergence of $Q_{\varepsilon k} w^{pq}_{\varepsilon st}$ to $(w^{pq}_{st})_k$, as $\varepsilon \to 0$, we conclude that the function (6.36) converges weakly in the sense of distributions to

$$\hat{b}_{pq}(x, y) = V_{pq}(x, y) \sum_{k,l=1}^{m} \int_\Gamma \int_\Gamma [a_{kl}(\mu, \nu) + V_{kl}(\mu, \nu)(\varphi_k(\mu) - \varphi_l(\nu))]$$

$$\times (w^{pqk}_{xy}(\mu) - w^{pql}_{xy}(\nu))\, d^2\mu d^2\nu.$$

Hence, applying Theorem 2, we finish the proof of Theorem 1.

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Асимптотическое поведение гармонических 1-форм на римановых многообразиях возрастающего рода

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Рассматриваются двумерные компактные ориентируемые римановы многообразия M_ε, состоящие из одного или нескольких экземпляров базовой поверхности Г с большим числом тонких трубок и наделенные метрикой, которая зависит от малого параметра $\varepsilon > 0$. Изучается асимптотическое поведение гармонических 1-форм на M_ε при $\varepsilon \to 0$, когда число трубок растет, а их толщина уменьшается. Получены усредненные уравнения на базовой поверхности Г, описывающие главный член асимптотик.

Асимптотична поведінка гармонічних 1-форм на ріманових многостатностях зростаючого роду

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Розглядаються двовимірні компактні орієнтовані ріманові многостатності M_ε, що складаються з одного або кількох екземплярів базової поверхні Г з великим числом тонких трубок і наділені метрикою, яка залежна від малого параметра $\varepsilon > 0$. Вивчається асимптотична поведінка гармонічних 1-форм на M_ε при $\varepsilon \to 0$, коли число трубок зростає, а їх товщина зменьшується. Одержано усереднені рівняння на базовій поверхні Г, що описують головний член асимптотик.