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Operator theoretic approach to orthogonal polynomials on an arc of the unit circle

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We study the probability measures on the unit circle and the multiplication operators acting on appropriate L^2 spaces. When such a measure does not satisfy the Szegő condition, orthonormal polynomials form an orthonormal basis in this Hilbert space. The multiplication operator can be represented by an upper Hessenberg matrix. The main result concerns certain infinite-dimensional perturbations of the "constant" Hessenberg matrix which have a finite number of eigenvalues off the essential spectrum.

1. Introduction

Let μ be a probability measure on the unit circle $\mathbb{T} = \{|\zeta| = 1\}$ with infinite support, $\operatorname{supp}\mu$. The polynomials $\varphi_n(z) = \varphi_n(z,\mu) = \kappa_n(\mu)z^n + \dots$, orthonormal on the unit circle with respect to μ are uniquely determined by the requirement that $\kappa_n = \kappa_n(\mu) > 0$ and

$$\int\limits_{\mathbb{T}} \varphi_n(\zeta)\overline{\varphi_m(\zeta)}\, d\mu = \delta_{n,m}, \qquad n, m = 0, 1, \dots, \qquad \zeta \in \mathbb{T}.$$

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The monic orthogonal polynomials Φ_n are defined by

$$\Phi_n(z) = \Phi_n(z, \mu) = \kappa_n^{-1} \varphi_n(z) = z^n + \dots, \qquad n = 0, 1, \dots .$$

The numbers $a_n = \Phi_n(0, \mu), \quad n = 1, 2, \dots$, which play a key role throughout the whole paper, are known as *reflection coefficients*. Since all zeros of Φ_n are inside the unit circle (cf. [11, Section 8, p. 9]), we have $|\Phi_n(0, \mu)| < 1, \quad n = 1, 2, \dots$. What is more to the point, given a sequence of complex numbers $\{\gamma_n\}$ with the only restriction $|\gamma_n| < 1$ there is a unique probability measure μ with infinite support such that $\gamma_n = \Phi_n(0, \mu)$ for $n = 1, 2, \dots$. This result is known as Favard's Theorem for the unit circle (cf. [6]).

The orthonormal polynomials φ_n along with another sequence ψ_n of polynomials, which are also orthonormal with respect to some probability measure τ (and known as the *second kind polynomials*), satisfy the (Szegő) recurrence relations

$$\begin{pmatrix} \varphi_n(z) & \psi_n(z) \\ \varphi_n^*(z) & -\psi_n^*(z) \end{pmatrix} = T_n(z) \begin{pmatrix} \varphi_{n-1}(z) & \psi_{n-1}(z) \\ \varphi_{n-1}^*(z) & -\psi_{n-1}^*(z) \end{pmatrix},$$

$$T_n(z) \stackrel{\text{def}}{=} \frac{\kappa_n}{\kappa_{n-1}} \begin{pmatrix} z & a_n \\ z\,\overline{a_n} & 1 \end{pmatrix} \tag{1}$$

for $n = 1, 2, \dots$ with the initial condition

$$\begin{pmatrix} \varphi_0 & \psi_0 \\ \varphi_0^* & -\psi_0^* \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(cf. [15, formula (8), p. 395]). Here the reversed *–polynomial of a polynomial p_n of degree n is defined by $p_n^*(z) \stackrel{\text{def}}{=} z^n\,\overline{p_n}(z^{-1})$. The relation $\psi_n(0) = -\varphi_n(0)$ for $n \geq 1$ follows directly from (1).

Note that the reflection coefficients a_n determine completely the system of orthonormal polynomials by (1), since

$$\frac{\kappa_{n-1}^2}{\kappa_n^2} = 1 - |a_n|^2, \qquad \kappa_n^{-2} = \prod_{k=1}^{n} (1 - |a_k|^2), \quad n \geq 1, \quad \kappa_0 = 1 \tag{2}$$

(cf. [11, p. 7]).

From (2) it follows that $\det T_n(z) = z$. If we set

$$\mathcal{T}_n(z) \stackrel{\text{def}}{=} T_n(z) T_{n-1}(z) \dots T_1(z) \tag{3}$$

then $\det \mathcal{T}_n = z^n$, which implies that $\mathcal{T}_n$ is a fundamental matrix solution to difference equation (1).

G. Szegő and Ja.L. Geronimus developed an important theory for the orthogonal polynomials which satisfy the *Szegő condition*

$$\log \mu' \in L^1(\mathbb{T}) \iff \sum_{n=0}^{\infty} |a_n|^2 < \infty \,, \tag{4}$$

where μ' is the Radon–Nikodym derivative with respect to Lebesgue measure on the unit circle (cf. [11, Chapters 8; 25, Chapters 10–11, p. 274–295]).

In the present paper we study certain classes of measures with reflection coefficients close in a sense to a nonzero complex constant a, $0 < |a| < 1$ and thereby not satisfying the Szegő condition (cf. [15, 16]). The orthonormal polynomial system $\{\varphi_n\}_0^\infty$ forms an orthonormal basis in an appropriate Hilbert space of functions on the unit circle, so that it seems reasonable to invoke the multiplication operator and its matrix representation in this basis. Such approach is well-known in the real line case (cf. [2, 5]). We are led to Jacobi matrices, which contain coefficients of the three-term recurrence relation for the corresponding orthogonal polynomials as their entries. Matrix representation of the multiplication operator in the unit circle case is much more complex, yet the Hilbert space operators prove useful in this setting as well (cf. [15, Section 3; 9]).

Given a probability measure μ on $\mathbb{T}$ with infinite support, let $L^2(\mu, \mathbb{T})$ be the Hilbert space of measurable, square-integrable functions on the unit circle with the inner product and norm

$$(f,g)_\mu = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}\, d\mu, \qquad ||f||_\mu^2 = (f,f)_\mu \,, \tag{5}$$

respectively. The orthonormal polynomial system $\{\varphi_n\}_0^\infty$ forms an orthonormal basis in $L^2(\mu, \mathbb{T})$ if and only if (4) is violated:

$$\log \mu' \notin L^1(\mathbb{T}) \iff \sum_{n=0}^{\infty} |a_n|^2 = \infty \tag{6}$$

(cf. [20; 17, Theorem 3.3(a), p. 49]).

The key role in our investigation is played by the unitary multiplication operator $U(\mu)$ which is defined by

$$[U(\mu)]f(\zeta) = \zeta f(\zeta), \quad \zeta \in \mathbb{T}, \quad f \in L^2(\mu, \mathbb{T}), \tag{7}$$

and its matrix representation in the orthonormal basis $\{\varphi_n\}_0^\infty$. We have (cf. [21, Proposition 4.2, p. 45; 14, formula (6), p. 394])

$$z\varphi_j(z) = \left(\frac{h_{j+1}}{h_j}\right)^{1/2} \varphi_{j+1}(z) - \sum_{k=0}^{j} \left(\frac{h_j}{h_k}\right)^{1/2} a_{j+1}\overline{a_k}\varphi_k(z), \tag{8}$$

where

$$h_j = ||\Phi_j||_\mu^2 = h_0 \prod_{k=1}^{j} (1 - |a_k|^2), \quad h_0 = 1.$$

Therefore

$$U(\mu) = \begin{pmatrix} u_{00} & u_{01} & \cdots \\ u_{10} & u_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad u_{kj} = (U(\mu)\varphi_j, \varphi_k)_\mu \,, \tag{9}$$

where for $j = 0, 1, \ldots$

$$u_{kj} = \begin{cases} -a_{j+1}\overline{a_k} \prod_{p=k+1}^{j} (1 - |a_p|^2)^{1/2}, & \text{for } k = 0, 1, \ldots, j, \\ (1 - |a_{j+1}|^2)^{1/2}, & \text{for } k = j+1, \\ 0, & \text{for } k \geq j+2 \end{cases} \tag{10}$$

(cf. [15, p. 401; 26, p. 409]). The infinite matrices (9)–(10), in which all entries below the subdiagonal vanish are called the *Hessenberg matrices*.

Given a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ such that $a_0 = 1$, $|a_n| \leq 1$ for $n \geq 1$, consider the operator U in ℓ^2 space defined by (9)–(10) in the standard orthonormal basis $\{e_n\}_{n=0}^{\infty}$. It can be checked by direct calculation that $UU^* = U^*U = I$, i.e., U is a unitary operator (we call such operators *Hessenberg operators*). The spectral measure $E_U(B)$ of the operator U is defined on Borel sets B of the unit circle. If $|a_n| < 1, \ n \geq 1$, the operator U is unitarily equivalent to multiplication operator (7) in $L^2(\mu, \mathbb{T})$ with $\mu(B) = (E_U(B)e_0, e_0)^*$. In particular, $\operatorname{supp} \mu$ coincides with the spectrum $\sigma(U)$ of the operator U. The set of mass points of the measure μ is exactly the discrete spectrum $\sigma_d(U)$ (i.e., the set of all eigenvalues) of the operator U. Therefore we can shed the light upon the connection between measures and their reflection coefficients by employing the spectral theory of unitary operators in the Hilbert space (cf. [27, p. 60–65]).

Let U be a unitary operator in the Hilbert space H with spectral measure $E_U(B)$. According to the Spectral Theorem for each continuous on $\mathbb{T}$ function F and for any vectors $f, g \in H$ the relation

$$(F(U)f, g) = \int_{\mathbb{T}} F(\zeta)\, d(E_U(\zeta)f, g) \tag{11}$$

holds. For $|\tau_1| = |\tau_2| = 1$ we denote by (τ_1, τ_2) ($[\tau_1, \tau_2]$) the open (closed) arc on $\mathbb{T}$ swept out as τ_1 moves to τ_2 counterclockwise. Let

$$\Gamma_\alpha = (\bar{\tau}, \tau), \quad \tau = e^{i\alpha}, \quad 0 < \alpha < \pi\,. \tag{12}$$

*The operator U is simple with e_0 being a cyclic vector.

The following proposition from spectral theory of unitary operators is of particular importance for our purpose.

Theorem 1. *Let there exist a subspace* $H_N \subset H$ *with* $\operatorname{codim} H_N = N$ *such that for every* $g \in H_N$ *and for some* $0 < \alpha < \pi$

$$(\Re U g, g) \le \left(1 - 2\sin^2 \frac{\alpha}{2}\right) ||g||^2, \quad \Re U = \frac{1}{2}(U + U^*) \tag{13}$$

holds. Then the arc Γ_α (12) *contains no more than* N *points of* $\sigma(U)$.

P r o o f. Assume the contrary. In that case there exists an arc $\Gamma_\beta \subset \Gamma_\alpha$ such that

$$\dim \tilde{H} > N, \quad \tilde{H} = E_U(\Gamma_\beta)H.$$

From (11) we see that for $h \in \tilde{H}$

$$||(U-I)h||^2 = \int_{\Gamma_\beta} |\zeta - 1|^2 \, d(E_U(\zeta)h, h) \le 4\sin^2 \frac{\beta}{2} ||h||^2 < 4\sin^2 \frac{\alpha}{2} ||h||^2. \tag{14}$$

Since $\operatorname{codim} H_N = \dim {H_N}^\perp = N$, we can find a vector $h_0 \in \tilde{H}$, which is orthogonal to ${H_N}^\perp$, i.e., $h_0 \in \tilde{H} \cap H_N$. Relation (14) then yields

$$||(U-I)h_0||^2 = 2\{||h_0||^2 - (\Re U h_0, h_0)\} < 4\sin^2 \frac{\alpha}{2} ||h_0||^2$$

or

$$(\Re U h_0, h_0) > \left(1 - 2\sin^2 \frac{\alpha}{2}\right) ||h_0||^2,$$

that contradicts (13). ■

Note that assumption (13) is equivalent to

$$(Tg, g) \le 2\cos^2 \frac{\alpha}{2} ||g||^2, \quad T = \Re U + I. \tag{15}$$

In the present paper we are primarily interested in the set of mass points of the measure μ (in other words, in discrete spectrum of the corresponding operator U) on arc Γ_α (12)*. We proceed as follows: in Sections 2 and 3 we study the discrete spectrum for constant reflection coefficients and its perturbations. In Section 4 we discuss the relation between linear difference equations, orthonormal polynomials and resolvents of the Hessenberg operators (cf. [8, 9]). The main result on finiteness of discrete spectrum is presented in Section 5.

*Concerning the set of mass points on the complement arc see [15, Section 5].

2. Constant reflection coefficients

We begin with the simplest case of the measures with constant reflection coefficients (cf. [13, p. 93–94; 15, p. 402]) and the associated multiplication operators.

Consider operator $U = U_a$ (9)–(10) in ℓ^2 space with $a_0 = 1$, $a_1 = a_2 = \ldots = a$, $0 < |a| < 1$

$$U_a = \begin{pmatrix} -a & -a\rho & -a\rho^2 & \ldots \\ \rho & -|a|^2 & -|a|^2\rho & \ldots \\ 0 & \rho & -|a|^2 & \ldots \\ \vdots & \vdots & \ddots & \end{pmatrix}, \quad \rho^2 = 1 - |a|^2. \tag{16}$$

This operator is unitarily equivalent to multiplication operator (7) in $L^2(\mu_a, \mathbb{T})$ for a certain measure μ_a. If the eigenvector $x = (x_0, x_1, \ldots)'$ corresponds to the eigenvalue $\lambda \in \mathbb{T}$, then

$$-a \sum_{k=0}^{\infty} \rho^k x_k = \lambda x_0, \tag{17}$$

$$\rho x_{n-1} - |a|^2 \sum_{k=n}^{\infty} \rho^{k-n} x_k = \lambda x_n, \quad n = 1, 2, \ldots . \tag{18}$$

Combining the adjacent equations from (18) gives

$$\rho\lambda x_{n+1} - (\lambda + 1)x_n + \rho x_{n-1} = 0, \quad n = 1, 2, \ldots . \tag{19}$$

The characteristic equation $\rho z t^2 - (z+1)t + \rho = 0$ for the three–term recurrence relation (19) has two solutions

$$t_\pm(z) = \frac{r_\pm(z)}{z}, \qquad r_\pm(z) = \frac{z + 1 \pm \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2\rho}, \tag{20}$$

where $\alpha = 2 \arcsin |a|$ and that branch of the square root is chosen for which $r_-(0) = 0$ (cf. [15, p. 398]). For the arc

$$\Gamma_\alpha = \{(\bar{\tau}, \tau),\ \tau = e^{i\alpha}\}, \qquad \alpha = 2 \arcsin |a|, \tag{21}$$

denote $\Delta_\alpha = \mathbb{T} \backslash \Gamma_\alpha$. The function $r_-(z)(r_+(z))$ maps the domain $\mathbb{C} \backslash \Delta_\alpha$ conformally onto the open unit disk $\mathbb{D}$ (the exterior of the closed unit disk). Note that

$$r_+(z) + r_-(z) = \frac{z+1}{\rho}, \qquad r_+(z) r_-(z) = z, \tag{22}$$

$|r_\pm(\zeta)| = 1$ for $\zeta \in \Delta_\alpha$ and $r_+(\zeta) \neq r_-(\zeta)$ for $\zeta \neq e^{\pm i\alpha}$, where

$$r_\pm(\zeta) = \lim_{r \to 1-0} r_\pm(r\zeta).$$

Hence

$$x_n = x_n(\lambda) = a(\lambda)t_+^n(\lambda) + b(\lambda)t_-^n(\lambda), \qquad n = 0, 1, \dots \, . \tag{23}$$

As $\sum_{n=0}^{\infty} |x_n|^2 < \infty$, it follows from (20), (23) that $\lambda \in \Gamma_\alpha$ and $a(\lambda) = 0$. For such λ the equality $|t_-(\lambda)| = |r_-(\lambda)\lambda^{-1}| < 1$ holds, so that

$$x = (1, t_-(\lambda), t_-^2(\lambda), \dots)' \in \ell^2. \tag{24}$$

It is easily seen that vector x (24) satisfies (18) for each $\lambda \in \Gamma_\alpha$. Substituting (24) for x into (17) leads to the equality $\lambda = a(\rho t_-(\lambda) - 1)^{-1}$. Combining the latter one with (18) for $n = 1$, we obtain

$$\lambda = \frac{\rho}{t_-(\lambda)(1 - \bar{a})} \, .$$

Finally $t_-(\lambda) = \rho(1 - a)^{-1}$ and

$$\lambda = \frac{1 - a}{1 - \bar{a}} \, . \tag{25}$$

Conversely, let λ be taken from (25) for some a, $0 < |a| < 1$. Define a parameter β by the equality $1 - a = |1 - a| \exp(\frac{i}{2}\beta)$. We have now

$$\lambda = e^{i\beta}, \qquad \cos\frac{\beta}{2} = \frac{1 - \Re a}{|1 - a|} \, . \tag{26}$$

From the elementary inequality

$$\frac{|\Im z|}{|1 - z|} \le |z|, \quad |z| < 1,$$

wherein the equality is attained if and only if $|z - \frac{1}{2}| = \frac{1}{2}$, it follows that

$$|\sin\frac{\beta}{2}| = \frac{|\Im a|}{|1 - \bar{a}|} \le |a| = \sin\frac{\alpha}{2} \, ,$$

i.e., $\lambda \in \Gamma_\alpha$ for $|a - \frac{1}{2}| \ne \frac{1}{2}$, and vector x (24)–(25) satisfies (18). It remains only to make sure that x satisfies (17). We have

$$e^{i\beta}(1 - \rho t_-(e^{i\beta})) + a$$

$$\begin{aligned}
&= \frac{1}{2}(e^{i\beta} - 1 + e^{i\beta/2}\sqrt{2(\cos\beta - \cos\alpha)}) + 1 - |1 - a|e^{i\beta/2} \\
&= \frac{1}{2}e^{i\beta/2}(2i\sin\frac{\beta}{2} + \sqrt{2(\cos\beta - \cos\alpha)}) + e^{i\beta/2}(e^{-i\beta/2} - |1 - a|) \\
&= \frac{1}{2}e^{i\beta/2}\sqrt{2(\cos\beta - \cos\alpha)} + e^{i\beta/2}(\cos\frac{\beta}{2} - |1 - a|) \\
&= \frac{1}{2}e^{i\beta/2}\sqrt{2(\cos\beta - \cos\alpha)} - e^{i\beta/2}\frac{|a - 1/2|^2 - 1/4}{|1 - a|} \, .
\end{aligned}$$

It can be directly derived from (26) that

$$\sqrt{2(\cos\beta - \cos\alpha)} = \frac{2}{|1-a|}\left||a-1/2|^2 - 1/4\right| ,$$

which yields (17) as long as $|a-\frac{1}{2}| > \frac{1}{2}$.

Thus we come to the following conclusion.

Theorem 2. *The discrete spectrum* $\sigma_d(U_a)$ *is empty for* $|a-\frac{1}{2}| \le \frac{1}{2}$. *For* $|a-\frac{1}{2}| > \frac{1}{2}$ *the operator* U_a *has the only eigenvalue* λ (25), *which lies on the arc* Γ_α (21) *and corresponds to the eigenvector*

$$x_a = (1, t_-, t_-^2, \ldots)', \quad t_- = \frac{(1-|a|^2)^{1/2}}{1-a}. \tag{27}$$

R e m a r k 3. Let $(U_a + zI)(U_a - zI)^{-1} = \{w_{kj}(z)\}_{k,j=0}^{\infty}$. A straightforward calculation yields the formula for $w_{00}(z)$

$$w_{00}(z) = \int\limits_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d(E_{U_a}(\zeta)e_0, e_0) = \frac{1 - \bar{a}z - \rho r_-(z)}{1 + \bar{a}z - \rho r_-(z)} \tag{28}$$

(the first equality stems from the Spectral Theorem). This relation enables one to find precise expression for the measure μ_a. As it turns out, the measure μ_a is absolutely continuous with respect to the Lebesgue measure on the arc Δ_α, and $\operatorname{supp}\mu_a$ on the arc Γ_α is either empty or consists of one point λ (25) (cf. [13, formulas (XI. 26) and (XI. 27), p. 94]).

3. Perturbation of discrete spectrum

Let μ and μ_a be positive Borel measures on the unit circle with reflection coefficients $\{a_n\}$ and $\{a\}$, U and U_a be corresponding multiplication operators (7) which act on the *different* Hilbert spaces $L^2(\mu, \mathbb{T})$ and $L^2(\mu_a, \mathbb{T})$, respectively. To study both these operators simultaneously it is convenient and instructive to move them on to the *same* Hilbert space ℓ^2 by means of their matrix representations (10) and (16). In our concept the Geronimus operator U_a is the unperturbed object, while the operator U is the perturbed one.

Of particular importance is the difference (or perturbation) $V : \ell^2 \to \ell^2$

$$V = U - U_a = \{v_{k,j}\}_{k,j=0}^{\infty}, \qquad v_{k,j} = u_{k,j} - u_{k,j}^a . \tag{29}$$

An intimate relation between closeness of the sequence $\{a_n\}$ to the constant sequence $\{a\}$ and the properties of perturbation V (29) can be revealed. For instance, the convergence $\lim_{n\to\infty} a_n = a$ provides compactness of the operator V (cf. [15, Theorem 3])*. We continue studying such phenomenon in this section.

To begin with, consider the finite perturbation of the constant sequence $\{a\}$ (cf. [13, formula (XI. 25), p. 94]).

Theorem 4. *Let $a_N = a_{N+1} = \ldots = a, \quad 0 < |a| < 1$. Then the arc Γ_α (21) contains no more than $N+1$ points of* $\operatorname{supp}\mu$.

P r o o f. It is actually not hard to see that under the assumption of Theorem 4 perturbation V (29) is of finite rank. In fact, for $k \ge N$ and $\rho_k^2 \overset{\text{def}}{=} 1 - |a_k|^2$ we have

$$v_{k,k-1} = \rho_k - \rho = 0, \quad v_{k,j} = -a_{j+1}\overline{a_k} \prod_{p=k+1}^{j} \rho_p + |a|^2 \rho^{j-k} = 0, \quad j \ge k,$$

and hence

$$Vh = \sum_{k=0}^{N-1} (h, v_k) e_k, \quad v_k = (\bar{v}_{k0}, \bar{v}_{k1}, \ldots)' \in \ell^2, \quad k = 0, \ldots, N-1.$$

Then

$$U - I = U_a - I + V = U_a - I + \sum_{k=0}^{N-1} (\cdot, v_k) e_k \,.$$

For $|a - \frac{1}{2}| \le \frac{1}{2}$ arc Γ_α (21) is free from $\sigma(U_a)$. By the Spectral Theorem for every $h \in \ell^2$

$$||(U_a - I)h||^2 = \int_{\Delta_\alpha} |\zeta - 1|^2 \, d(E_{U_a}(\zeta)h, h) \ge 4 \sin^2 \frac{\alpha}{2} \, ||h||^2, \quad \Delta_\alpha = \mathbb{T} \backslash \Gamma_\alpha \,.$$

Let $H(v_0, \ldots, v_{N-1})$ be the subspace generated by the vectors $v_0, \ldots, v_{N-1}$ and $H_N = \{H(v_0, \ldots, v_{N-1})\}^\perp$. Then $\operatorname{codim} H_N \le N$ and for every $h \in H_N$ $(U - I)h = (U_a - I)h$. Therefore $||(U - I)h|| = ||(U_a - I)h||$ and

$$||(U - I)h||^2 \ge 4 \sin^2 \frac{\alpha}{2} \, ||h||^2, \quad h \in H_N \,.$$

The rest is immediate from Theorem 1.

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*As a matter of fact, a somewhat weaker assumption known as the *Lopez condition*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1, \quad \lim_{n\to\infty} |a_n| = |a|, \quad 0 < |a| < 1\,, \tag{30}$$

already provides compactness of the perturbation.

For $|a-\frac{1}{2}|>\frac{1}{2}$ the operator U_a has the only eigenvalue $\lambda\in\Gamma_\alpha$, corresponding to eigenvector x_a (29). It remains only to replace in the foregoing argument the subspace $H(v_0,\dots,v_{N-1})$ by the subspace $H(x_a,v_0,\dots,v_{N-1})$. ∎

Further in Section 5 we make use of the result which is actually established in the proof of Theorem 4.

Theorem 5. *Let $U(A)$ and $U(B)$ be the unitary Hessenberg operators, associated with the sequences $A=\{a_n\}$ and $B=\{b_n\}$, respectively. If $a_n=b_n$ for $n\ge N$ then the difference $V=U(A)-U(B)$ is of finite rank at most N.*

In our next statement the sequences $\{a_n\}$ and $\{a\}$ are assumed to be close enough in ℓ^∞-norm. It turnes out that the perturbation V has now small enough operator norm.

Theorem 6. *Suppose that*

$$\gamma\overset{\text{def}}{=}\sup_{j\ge1}|a_j-a|\le\left\{\frac{|a|\rho(1-\rho)}{2}\right\}^2, \qquad \rho^2=1-|a|^2, \tag{31}$$

and define the value β by

$$\cos^2\frac{\beta}{2}\overset{\text{def}}{=}\cos^2\frac{\alpha}{2}+\frac{3\gamma}{\rho(1-\rho)(|a|-\gamma)^2}, \qquad \alpha=2\arcsin|a|. \tag{32}$$

Then for $|a-\frac{1}{2}|\le\frac{1}{2}$ the arc

$$\Gamma_\beta=\{(\bar\tau,\tau),\ \tau=e^{i\beta}\} \tag{33}$$

is free from $\operatorname{supp}\mu$. *For $|a-\frac{1}{2}|>\frac{1}{2}$ the arc Γ_β contains no more than one point of* $\operatorname{supp}\mu$.

P r o o f. We proceed in three steps.

Step 1. It can be easily shown that relation (32) actually defines a certain value β. Indeed, under (31) it is clear that

$$\gamma\le\frac{|a|^2}{64}<\frac{|a|}{64}, \qquad |a|-\gamma>\frac{63|a|}{64}$$

and

$$\cos^2\frac{\alpha}{2}+\frac{3\gamma}{\rho(1-\rho)(|a|-\gamma)^2}\le\rho^2+\frac{3}{4}|a|^2\rho(1-\rho)\left(\frac{64}{63|a|}\right)^2<\rho^2+\rho(1-\rho)=\rho<1.$$

Step 2. We are aiming at proving (15) (with α replaced by β). To this end let us estimate the norm of operator V (29). We can write V as

$$V = S^* D_{-1} + \sum_{k=0}^{\infty} D_k S^k, \tag{34}$$

where S is the shift operator given by the matrix representation

$$S = \begin{pmatrix} 0 & 1 & 0 & 0\dots & \\ 0 & 0 & 1 & 0\dots & \\ 0 & 0 & 0 & 1\dots & \\ 0 & 0 & 0 & 0\dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and D_k are diagonal operators with the elements

$$\operatorname{diag} D_k = \begin{cases} (v_{0,k}, v_{1,k+1}, \dots), & \text{for } k = 0, 1, \dots\,, \\ (\rho_1 - \rho, \rho_2 - \rho, \dots), & \text{for } k = -1, \end{cases} \tag{35}$$

$\rho_k = 1 - |a_k|^2$ (cf. [15, p. 403]). To prove that infinite series in (24) converges in the operator norm, set

$$r_0 = 1, \quad r_m = \sup_{j\geq 0} \rho_{j+1}\rho_{j+2}\dots\rho_{j+m}, \quad m = 1, 2, \dots\,.$$

Under assumption (31) $|a_j| \geq |a| - \gamma > 0$ and $\rho_j^2 \leq \delta^2 \stackrel{\text{def}}{=} 1 - (|a| - \gamma)^2$, so that

$$r_m \leq \delta^m, \quad \sum_{m=0}^{\infty} r_m \leq (1-\delta)^{-1} = \frac{1+\delta}{1-\delta^2} < \frac{2}{(|a|-\gamma)^2}\,. \tag{36}$$

Our further consideration is based on the explicit expression for the matrix elements u_{kj} and u_{kj}^a (see (10) and (16)). We have

$$|v_{0,k}| = |u_{0,k} - u_{0,k}^a| = |a_{k+1} \prod_{p=1}^{k} \rho_p - a\rho^k| \leq |a_{k+1} - a|\rho^k + |a_{k+1}||\prod_{p=1}^{k} \rho_p - \rho^k|,$$

$$\begin{aligned} |v_{j,k+j}| &= |u_{j,k+j} - u_{j,k+j}^a| = |a_{k+j+1}\overline{a_j} \prod_{p=j+1}^{j+k} \rho_p - |a|^2\rho^k| \\ &\leq |a_{k+j+1}\overline{a_j} - |a|^2|\rho^k + |a_{k+j+1}a_j|| \prod_{p=j+1}^{j+k} \rho_p - \rho^k|, \quad k \geq 1. \end{aligned}$$

To evaluate the second terms in the right-hand sides we make use of the identity

$$\prod_{p=j+1}^{j+k} x_p - \prod_{p=j+1}^{j+k} y_p = \sum_{p=j+1}^{j+k} x_{j+1}x_{j+2}\dots x_{p-1}(x_p - y_p)y_{p+1}\dots y_{j+k}\,.$$

Hence

$$|v_{0,k}| \le |a_{k+1} - a|\rho_k + \sum_{p=1}^{k} \rho_1 \dots \rho_{p-1}|\rho_p - \rho|\rho^{k-p},$$

$$\begin{aligned} |v_{j,k+j}| &\le (|a_{k+j+1} - a| + |a_j - a|)\rho^k + \sum_{p=j+1}^{j+k} \rho_{j+1}\dots\rho_{p-1}|\rho_p - \rho|\rho^{j+k-p} \\ &\le (|a_{k+j+1} - a| + |a_j - a|)\rho^k + \sum_{l=1}^{k} \rho_{j+1}\dots\rho_{l+j-1}|\rho_{l+j} - \rho|\rho^{k-l} \end{aligned} \tag{37}$$

for $k, j = 0, 1, \dots$. Next

$$|\rho_n - \rho| = \frac{\big||a_n|^2 - |a|^2\big|}{\rho_n + \rho} \le \frac{|a_n| + |a|}{\rho_n + \rho}\,|a_n - a|, \tag{38}$$

and $||D_{-1}||$ can be estimated as follows:

$$\|D_{-1}\| = \sup_{n\ge1} |\rho_n - \rho| \le \gamma \sup_{n\ge1} \frac{|a_n| + |a|}{\rho_n + \rho} < \frac{2}{\rho}\,\gamma. \tag{39}$$

In a similar way we obtain for D_k with $k \ge 0$

$$||D_k|| = \sup_{j\ge0} |v_{j,k+j}| \le 2\gamma\rho^k + \|D_{-1}\| \sum_{l=1}^{k} r_{l-1}\rho^{k-l}, \quad k = 1, 2, \dots\,, \tag{40}$$

and

$$\|D_0\| = \sup_{l\ge0} |v_{j,j}| \le 2\gamma. \tag{41}$$

It follows now from (39)–(41) that series (34) converges in operator norm and

$$\begin{aligned} \|V\| &\le \|D_{-1}\| + 2\gamma + 2\gamma\sum_{k=1}^{\infty} \rho^k + \|D_{-1}\| \sum_{k=1}^{\infty}\sum_{l=1}^{k} r_{l-1}\rho^{k-l} \\ &= \|D_{-1}\| \left\{ 1 + \frac{1}{1-\rho} \sum_{l=1}^{\infty} r_{l-1} \right\} + \frac{2\gamma}{1-\rho}\,. \end{aligned}$$

Taking into account the relations (36) and (39), we come to the conclusion

$$\|V\| \le \frac{6\gamma}{\rho(1-\rho)(|a|-\gamma)^2}\,. \tag{42}$$

Step 3. According to the general result from operator theory the spectra of two normal bounded operators W_1 and W_2 are close to each other whenever their difference $W_1 - W_2$ is small in operator norm. Indeed, for resolvent $R(z, W_1) = (W_1 - zI)^{-1}$ the equality

$$\|R(z, W_1)\| = \frac{1}{\mathrm{dist}\,(z, \sigma(W_1))}$$

holds (cf. [19, p. 277, formula (3.31)]). Let $\mathrm{dist}\,(z, \sigma(W_1)) > \|W_2 - W_1\|$. Then

$$W_2 - zI = W_1 - zI + W_2 - W_1 = (W_1 - zI)[I + R(z, W_1)(W_2 - W_1)],$$

and the operator $W_2 - zI$ is invertible, since

$$\|R(z, W_1)(W_2 - W_1)\| \le \frac{\|W_2 - W_1\|}{\mathrm{dist}\,(z, \sigma(W_1))} < 1,$$

that is, $\sigma(W_2) \in \{z : \mathrm{dist}\,(z, \sigma(W_1)) \le \|W_2 - W_1\|\}$ (cf. [18, Chapter 11, Problem 86]).

In our particular situation we can gather some more quantitative information. We know that $\sigma(U_a) \subset \Delta_\alpha$ as long as $|a - \frac{1}{2}| \le \frac{1}{2}$, and by the Spectral Theorem for the operator U_a we have, as above

$$\|(U_a - I)g\|^2 = \int\limits_{\Delta_\alpha} |\zeta - 1|^2\, d(E_{U_a}(\zeta)g, g) \ge 4\sin^2\frac{\alpha}{2}\,\|g\|^2, \quad g \in \ell^2\,.$$

Let

$$T = \Re\, U + I, \qquad T_a = \Re\, U_a + I,$$

so that $T = T_a + \Re\, V$. Then $(Tg, g) \le 2\cos^2\frac{\alpha}{2}\,\|g\|^2$ holds for all $g \in \ell^2$. Hence by (42) the relation

$$\begin{aligned}(Tg, g) &= (T_a g, g) + (\Re\, V g, g) \\ &\le 2\left(\cos^2\frac{\alpha}{2} + \frac{6\gamma}{\rho(1-\rho)(|a|-\gamma)^2}\right)\|g\|^2 = 2\cos^2\frac{\beta}{2}\,\|g\|^2\end{aligned}$$

is true for all $g \in \ell^2$.

Much the same sort of argument is applicable when $|a - \frac{1}{2}| > \frac{1}{2}$ and $\sigma(U_a) \cap \Gamma_\alpha = \{\lambda\}$ (see Remark 3). In this case

$$(Tg, g) \le 2\cos^2\frac{\beta}{2}\,\|g\|^2, \quad g \perp x_a,$$

where the vector x_a is given by (27). The assertion of Theorem 6 follows now from Theorem 1. ■

Corollary 7. *For each $\beta < \alpha$ there exists a small enough value $\gamma = \gamma(\alpha, \beta)$ such that the arc Γ_β (33) contains no more than one point of* $\operatorname{supp}\mu$ *whenever* $\sup_{j\geq 1} |a_j - a| \leq \gamma$.

R e m a r k 8. By (32) the distance between endpoints of the arc Γ_β, which is free from $\operatorname{supp}\mu$ (at least for $|a - \frac{1}{2}| < \frac{1}{2}$), and the endpoints of the arc Γ_α is $\alpha - \beta = O(\gamma)$, $\gamma \to 0$. Notice that this result is sharp with respect to order. In fact, let $a_j = b$, $j \geq 1$ with $0 < b = a - \gamma$. Then the largest arc, which is free from $\operatorname{supp}\mu$ is exactly Γ_β with $\beta = 2\arcsin b$ and $\alpha - \beta \geq 2\gamma$.

Note that under the premises of Theorem 6 the orthogonality measure μ can have no "large" gaps on Δ_α.

The operator series expansion (34) works in various settings (see, e.g., Lemma 13 below). For instanse, the only slight modification of the argument, applied in Step 2 above, leads to the following

Theorem 9. *Let $U(A)$ be the unitary Hessenberg operator, associated with the sequence $A = \{a_n\}$, which satisfies $|a_n| < 1$ and*

$$0 < A_- \overset{\text{def}}{=} \inf_{n\geq 1} |a_n| \leq \sup_{n\geq 1} |a_n| \overset{\text{def}}{=} A_+ < 1.$$

Then for every sequence $B = \{b_n\}$ which is close enough to A in ℓ^∞ – norm:

$$\gamma \overset{\text{def}}{=} \sup_{n\geq 1} |b_n - a_n| \leq \frac{1}{2} \min(A_-, 1 - A_+),$$

the associated Hessenberg operator $U(B)$ is close to $U(A)$ in operator norm:

$$\|U(B) - U(A)\| \leq C(A)\gamma,$$

where the constant $C(A)$ depends only on A.

4. Linear difference equations associated with orthogonal polynomials on the unit circle and resolvent for Hessenberg matrix

We adopt here the arguments from [8, Section 2] (cf. also [9]) and start with the vector analogue of equation (1) (see also (3))

$$\vec{X}(z, n) = T_n(z)\,\vec{X}(z, n-1) = \mathcal{T}_n(z)\,\vec{X}(z, 0), \qquad n = 1, 2, \dots\,, \tag{43}$$

and its two solutions

$$\vec{\varphi}\,(z,n) \stackrel{\text{def}}{=} \mathcal{T}_n(z) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{bmatrix}, \ \vec{\psi}\,(z,n) \stackrel{\text{def}}{=} \mathcal{T}_n(z) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{bmatrix}. \tag{44}$$

The Wronskian W of any two solutions $\vec{X}\,(z,n)$ and $\vec{Y}\,(z,n)$ of (43) is defined by

$$W\,[\vec{X}\,(z,n), \vec{Y}\,(z,n)] = \det\,[\vec{X}\,(z,n), \vec{Y}\,(z,n)]. \tag{45}$$

From (43) and the relation $\det T_n(z) = z$ we find

$$W\,[\vec{X}\,(z,n), \vec{Y}\,(z,n)] = zW\,[\vec{X}\,(z,n-1), \vec{Y}\,(z,n-1)] = z^n\,W\,[\vec{X}\,(z,0), \vec{Y}\,(z,0)].$$

The latter implies that $\vec{\varphi}\,(z,n)$ and $\vec{\psi}\,(z,n)$ (43) are linearly independent solutions of (43) for $z \neq 0$.

Another helpful solution of (43) is given by

$$\vec{\varphi}_+\,(z,n) = \begin{bmatrix} \varphi_+^1(z,n) \\ \varphi_+^2(z,n) \end{bmatrix} \stackrel{\text{def}}{=} \frac{1}{2}\left[\vec{\psi}_n\,(z) + F(z)\,\vec{\varphi}_n\,(z)\right], \qquad |z| < 1, \tag{46}$$

with

$$F(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu\,. \tag{47}$$

This solution is known to be the unique (up to a constant factor) square summable solution of (43):

$$|\varphi_+^2(z,n)| < |\varphi_+^1(z,n)|, \qquad \sum_{n=0}^{\infty} |\varphi_+^1(z,n)|^2 < \infty \tag{48}$$

(cf. [8, Theorem 2.4]).

Our immediate objective is to calculate a matrix representation for the resolvent

$$R(z,U) = (U - zI)^{-1} = \|R_{k,j}\|_{k,j=0}^{\infty}, \quad R_{k,j} = R_{k,j}(z) = (R(z)e_j, e_k)\,, \ \ |z| < 1,$$

of the unitary Hessenberg operator U (9)–(10) with $|a_n| < 1,\ n \geq 1$, in the standard basis $\{e_n\}$ in ℓ^2. Such operator is unitarily equivalent to the multiplication operator (7) in the appropriate $L^2(\mu)$ space.

The definition of resolvent $R(z)\,(U - zI) = I$ can be displayed in terms of the matrix entries

$$\sum_{k=0}^{\infty} R_{n,k} u_{k,m} - zR_{n,m} = \delta_{n,m}$$

or by (10)

$$-\sum_{k=0}^{m} R_{n,k}\, a_{m+1}\overline{a}_k \prod_{p=k+1}^{m} \rho_p + R_{n,m+1}\rho_{m+1} - zR_{n,m} = \delta_{n,m},$$

where $\rho_p^2 = 1 - |a_p|^2$. Setting

$$L_{n,m} \stackrel{\text{def}}{=} \sum_{k=0}^{m} R_{n,k}\overline{a}_k \prod_{p=k+1}^{m} \rho_p, \qquad L_{n,0} = R_{n,0} \tag{49}$$

yields

$$\rho_{m+1}R_{n,m+1} = zR_{n,m} + a_{m+1}L_{n,m} - \delta_{n,m}. \tag{50}$$

It is clear from (49) that

$$L_{n,m+1} = \rho_{m+1}L_{n,m} + \overline{a}_{m+1}R_{n,m+1}.$$

If we multiply the latter equality through by ρ_{m+1} and take into account (50), we obtain

$$\begin{aligned} \rho_{m+1}L_{n,m+1} &= \overline{a}_{m+1}\rho_{m+1}R_{n,m+1} + \rho_{m+1}^2 L_{n,m} \\ &= \overline{a}_{m+1}zR_{n,m} + |a_{m+1}|^2 L_{n,m} - \overline{a}_{m+1}\delta_{n,m} + \rho_{m+1}^2 L_{n,m} \\ &= \overline{a}_{m+1}zR_{n,m} + L_{n,m} - \overline{a}_{m+1}\delta_{n,m}. \end{aligned}$$

It is convenient now to combine this relation with (50) in a system

$$\begin{cases} \rho_{m+1}R_{n,m+1} = zR_{n,m} + a_{m+1}L_{n,m}, & n \neq m, \\ \rho_{m+1}L_{n,m+1} = z\overline{a}_{m+1}R_{n,m} + L_{n,m}, & n \neq m, \end{cases} \tag{51}$$

which can be written in the matrix form (see (1))

$$\vec{R}_{n,m+1}(z) \stackrel{\text{def}}{=} \begin{bmatrix} R_{n,m+1}(z) \\ L_{n,m+1}(z) \end{bmatrix} = T_{m+1}(z)\, \vec{R}_{n,m}(z), \quad n \neq m, \quad |z| < 1, \tag{52}$$

and

$$\vec{R}_{n,n+1}(z) = T_{n+1}(z)\, \vec{R}_{n,n}(z) - \begin{bmatrix} \rho_{n+1}^{-1} \\ \overline{a}_{n+1}\rho_{n+1}^{-1} \end{bmatrix}, \quad n = m, \quad |z| < 1. \tag{53}$$

As the vectors $\vec{\varphi}\,(z,0)$ and $\vec{\varphi}_+\,(z,0)$ are linearly independent we have from (52)

$$\vec{R}_{n,s}(z) = A_n(z)\, \vec{\varphi}\,(z,s) + B_n(z)\, \vec{\varphi}_+\,(z,s), \quad s = 0,1,\ldots,n, \quad |z| < 1, \tag{54}$$

for certain scalar functions A_n and B_n. Next, the vectors $\vec{\varphi}\,(z,n+1)$ and $\vec{\varphi}_+\,(z,n+1)$ are also linearly independent unless $z=0$:

$$W\,[\vec{\varphi}\,(z,n+1),\vec{\varphi}_+\,(z,n+1)] = -z^{n+1}, \tag{55}$$

and hence for $|z|<1,\ z\neq 0$

$$\vec{R}_{n,s}\,(z) = C_n(z)\ \vec{\varphi}\,(z,s) + D_n(z)\ \vec{\varphi}_+\,(z,s), \quad s=n+1,n+2,\dots, \tag{56}$$

for certain scalar functions C_n and D_n. From (53), (54) and (43) we conclude

$$\begin{aligned}\vec{R}_{n,n+1}\,(z) &= T_{n+1}(z)\Big(A_n(z)\ \vec{\varphi}\,(z,n) + B_n(z)\ \vec{\varphi}_+\,(z,n)\Big) - \begin{bmatrix}\rho_{n+1}^{-1}\\ \overline{a}_{n+1}\rho_{n+1}^{-1}\end{bmatrix}\\ &= A_n(z)\ \vec{\varphi}\,(z,n+1) + B_n(z)\ \vec{\varphi}_+\,(z,n+1) - \begin{bmatrix}\rho_{n+1}^{-1}\\ \overline{a}_{n+1}\rho_{n+1}^{-1}\end{bmatrix}.\end{aligned}$$

If we compare the latter equality with (56) for $s=n+1$, we come to the relation

$$(A_n(z)-C_n(z))\ \vec{\varphi}\,(z,n+1) + (B_n(z)-D_n(z))\ \vec{\varphi}_+\,(z,n+1) = \begin{bmatrix}\rho_{n+1}^{-1}\\ \overline{a}_{n+1}\rho_{n+1}^{-1}\end{bmatrix}, \tag{57}$$

which holds for $|z|<1,\ z\neq 0$.

To take advantage of (54) and (56) let us determine the functions $A_n,\ B_n,\ C_n$ and D_n. It follows from (56) that

$$R_{n,s}(z) = C_n(z)\,\varphi_s(z) + D_n(z)\,\varphi_+^1(z,s), \quad s\geq n+1.$$

Since $\{\varphi_+^1(z,k)\}\in\ell^2$ for $|z|<1$ (see (48)) and

$$\sum_{k=0}^{\infty}|R_{n,k}(z)|^2 = \sum_{k=0}^{\infty}|\,(R(z)e_k,e_n)\,|^2 = \|R^*(z)e_n\|^2 < \infty, \qquad |z|<1,$$

then $\{\varphi_k(z)\}\in\ell^2$ unless $C_n(z)=0$. But this is not the case under the basic assumption (6) (cf., e.g., [12, Theorem 21.1]). Hence $C_n(z)=0,\ |z|<1$.

Next, by (49) $L_{n,0}=R_{n,0}$, and (54) with $s=0$ provides

$$A_n(z) + B_n(z)\,\frac{1+F(z)}{2} = R_{n,0}(z) = L_{n,0}(z) = A_n(z) + B_n(z)\,\frac{-1+F(z)}{2},$$

so that $B_n(z)=0$.

We can now solve (57) for A_n and D_n to obtain

$$\begin{aligned} A_n(z) &= \frac{\varphi_+^2(z,n+1) - \overline{a}_{n+1}\varphi_+^1(z,n+1)}{\rho_{n+1}\, W\,[\vec{\varphi}\,(z,n+1), \vec{\varphi}_+\,(z,n+1)]}\,, \\ D_n(z) &= \frac{\varphi_{n+1}^*(z) - \overline{a}_{n+1}\varphi_{n+1}(z)}{\rho_{n+1}\, W\,[\vec{\varphi}\,(z,n+1), \vec{\varphi}_+\,(z,n+1)]}\,, \qquad |z|<1,\ z\neq 0. \end{aligned} \tag{58}$$

It is easy to see from (43) that for arbitrary solution $\vec{X}\,(z,n) = (X_1(z,n), X_2(z,n))'$ of (43) the relation

$$X_2(z,n+1) - \overline{a}_{n+1}\, X_1(z,n+1) = \rho_{n+1}\, X_2(z,n)$$

holds. Hence the right-hand side in (58) can be presented in the form

$$A_n(z) = \frac{\varphi_+^2(z,n)}{W\,[\vec{\varphi}\,(z,n+1), \vec{\varphi}_+\,(z,n+1)]}\,,$$

$$D_n(z) = \frac{\varphi_n^*(z)}{W\,[\vec{\varphi}\,(z,n+1), \vec{\varphi}_+\,(z,n+1)]}\,.$$

In view of (54)–(56) we have now

$$\vec{R}_{n,s}\,(z) = \begin{cases} -z^{-n-1}\,\varphi_+^2(z,n)\ \vec{\varphi}\,(z,s), & s=0,1,\dots,n, \\ -z^{-n-1}\,\varphi_n^*(z)\ \vec{\varphi}_+\,(z,s), & s=n+1,n+2,\dots \end{cases}$$

or for the first element

$$R_{n,m}(z) = \begin{cases} -z^{-n-1}\,\varphi_+^2(z,n)\,\varphi_m(z), & m\le n, \\ \\ -z^{-n-1}\,\varphi_n^*(z)\,\varphi_+^1(z,m), & m>n. \end{cases}$$

It is worth summing up the results obtained above in the following statement.

Theorem 10. *Let $R(z,U) = (U-zI)^{-1}$ be the resolvent of the unitary Hessenberg operator U (9)–(10) with $|a_n|<1$, which is unitarily equivalent to the multiplication operator in the appropriate $L^2(\mu)$ space. Then its matrix entries in the standard basis take the form*

$$R_{n,m}(z) = \begin{cases} -\frac{1}{2}z^{-n-1}\,\varphi_m(z)\left(F(z)\varphi_n^*(z) - \psi_n^*(z)\right), & m\le n, \\ \\ -\frac{1}{2}z^{-n-1}\,\varphi_n^*(z)\left(F(z)\varphi_m(z) + \psi_m(z)\right), & m>n, \end{cases} \tag{59}$$

for $|z|<1,\ z\neq 0$. Here φ_n and ψ_n are orthonormal polynomials of the first and second kind, respectively, and F is C–function (47).

R e m a r k 11. According to the known property of orthonormal polynomials (cf. [11, p. 11, formula (1.19)]) the expressions in the right-hand side (59) have removable singularities at $z = 0$.

We employ Theorem 10 in the case $U = U_a$ (16) and $\mu = \mu_a$. Now $\varphi_n = \widehat{\varphi}_n$, $\psi_n = \widehat{\psi}_n$ are the Geronimus orthonormal polynomials of the first and second kind, respectively. An explicit expression for such polynomials and their *–reversed can be found in [15, formulas (17)–(20), p. 399]: $\widehat{\varphi}_0 = \widehat{\psi}_0 = \widehat{\varphi}_0^* = \widehat{\psi}_0^* = 1$,

$$\begin{aligned} \widehat{\varphi}_n(z) &= \frac{z+a}{\rho}\,\Lambda_n(z) - z\Lambda_{n-1}(z), \\ \widehat{\psi}_n(z) &= \frac{z-a}{\rho}\,\Lambda_n(z) - z\Lambda_{n-1}(z), \end{aligned} \tag{60}$$

for $n = 1, 2, \dots$, and

$$\begin{aligned} \widehat{\varphi}_n^*(z) &= \frac{1+\overline{a}z}{\rho}\,\Lambda_n(z) - z\Lambda_{n-1}(z), \\ \widehat{\psi}_n^*(z) &= \frac{1-\overline{a}z}{\rho}\,\Lambda_n(z) - z\Lambda_{n-1}(z). \end{aligned} \tag{61}$$

Here

$$\Lambda_n(z) \stackrel{\text{def}}{=} \frac{r_+^n(z) - r_-^n(z)}{r_+(z) - r_-(z)}, \quad n \geq 1, \qquad \Lambda_0 \stackrel{\text{def}}{=} 0, \tag{62}$$

and the functions $r_\pm$ are defined in (20). We are able to evaluate matrix entries in (59) by using (22) and the limit relation

$$F(z) = F_a(z) = \lim_{n\to\infty} \frac{\widehat{\psi}_n^*(z)}{\widehat{\varphi}_n^*(z)} = \frac{1 - \overline{a}z - \rho r_-(z)}{1 + \overline{a}z - \rho r_-(z)} \tag{63}$$

for C–function (47) (cf. [11, p. 11, formulas (1.16)] and (28)). We have for $n,\ m \geq 1$

$$\begin{aligned} & F_a(z)\,\widehat{\varphi}_n^*(z) - \widehat{\psi}_n^*(z) \\ =\ & \frac{1}{\rho}\Lambda_n(z)\left(F_a(z)(1+\overline{a}z) - (1-\overline{a}z)\right) - z\Lambda_{n-1}(z)\left(F_a(z) - 1\right) \\ =\ & \frac{-2\overline{a}zr_-(z)\Lambda_n(z) + 2\overline{a}z^2\Lambda_{n-1}(z)}{1+\overline{a}z-\rho r_-(z)} = -\frac{2\overline{a}z}{1+\overline{a}z-\rho r_-(z)}\,r_-^n(z); \end{aligned} \tag{64}$$

$$\begin{aligned} & F_a(z)\,\widehat{\varphi}_m(z) + \widehat{\psi}_m(z) \\ =\ & \frac{1}{\rho}\Lambda_m(z)\left(F_a(z)(z+a) + (z-a)\right) - z\Lambda_{m-1}(z)\left(F_a(z)+1\right) \\ =\ & \frac{2z}{1+\overline{a}z-\rho r_-(z)}\left\{\Lambda_m(z)(\rho - r_-(z)) - \Lambda_{m-1}(z)(1-\rho r_-(z))\right\}. \end{aligned}$$

Since by (22)

$$\Lambda_m(\rho - r_-) - \Lambda_{m-1}(1-\rho r_-) = \frac{(r_-^2 - 2\rho r_- + 1)r_-^{m-1}}{r_+ - r_-} = (\rho - r_-)r_-^{m-1}$$

we obtain finally

$$F_a(z)\,\widehat{\varphi}_m(z) + \widehat{\psi}_m(z) = \frac{2z(\rho - r_-(z))}{1+\overline{a}z - \rho r_-(z)}\, r_-^{m-1}(z). \tag{65}$$

Note that (64) and (65) are valid for $n = m = 0$ as well.

According to (59) we now have

$$\widehat{R}_{n,m}(z) = \begin{cases} f^{-1}(z)\,\overline{a}z^{-n}r_-^n(z)\widehat{\varphi}_m(z), & m \le n, \\ f^{-1}(z)\,(r_-(z)-\rho)\,z^{-n}\widehat{\varphi}_n^*(z)r_-^{m-1}(z), & m > n, \end{cases} \tag{66}$$

for $|z| < 1,\ z \ne 0$, where $f(z) \overset{\text{def}}{=} 1 + \overline{a}z - \rho r_-(z)$.

Of particular interest is the location of zeros of the function f. By (22)

$$\begin{aligned} f(z) &= 1 + \overline{a}z - (z + 1 - \rho r_+(z)) = (\overline{a}-1)z + \rho r_+(z) \\ &= (\overline{a}-1)\,r_+(z)r_-(z) + \rho r_+(z) = \rho_+(z)((\overline{a}-1)r_-(z) + \rho), \end{aligned}$$

and as $r_+(z) \ne 0$ in the closed unit disk, then $f(w) = 0$ is equivalent to

$$r_-(w) = \rho(1-\overline{a})^{-1}.$$

Note that

$$\frac{\rho^2}{|1-\overline{a}|^2} - 1 = \frac{1-|a|^2}{|1-\overline{a}|^2} - 1 = \frac{2}{|1-\overline{a}|^2}\left(\frac{1}{4} - |a - \frac{1}{2}|^2\right).$$

For $\rho > |1-\overline{a}|$, that is $\left|a - \frac{1}{2}\right| < \frac{1}{2}$, the function f is never zero in $\overline{\mathbb{D}}$. Let now $\rho = |1-\overline{a}|$ or $\left|a - \frac{1}{2}\right| = \frac{1}{2}$, that is equivalent to $|a|^2 = \Re a$.* Only two points meet this condition and $|a| = \sin\frac{\alpha}{2}$, namely $a = a_\pm = \sin^2\frac{\alpha}{2} \pm i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$. For these points the relation

$$\begin{aligned} \frac{\rho}{1-\overline{a}} &= \frac{\rho(1-a)}{1+|a|^2 - 2\Re a} = \frac{\rho(1-a)}{1-|a|^2} = \frac{1-a}{\rho} \\ &= \frac{1}{\rho}\left(1 - \sin^2\frac{\alpha}{2} \mp i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right) = \cos\frac{\alpha}{2} \mp i\sin\frac{\alpha}{2} = e^{\mp i\frac{\alpha}{2}} \end{aligned}$$

*We refer to this case as the *special Geronimus polynomials*.

holds. Hence

$$\frac{\rho}{1-\overline{a}} = \begin{cases} \exp\left(-i\frac{\alpha}{2}\right), & \text{for } \left|a-\frac{1}{2}\right| = \frac{1}{2},\ \Im a > 0, \\ \exp\left(i\frac{\alpha}{2}\right), & \text{for } \left|a-\frac{1}{2}\right| = \frac{1}{2},\ \Im a < 0. \end{cases}$$

Thus for the special Geronimus polynomials

$$f(e^{-i\alpha}) = 0,\ \Im a > 0; \qquad f(e^{i\alpha}) = 0,\ \Im a < 0. \tag{67}$$

Finally, when $\rho < |1-\overline{a}|$ or $\left|a-\frac{1}{2}\right| > \frac{1}{2}$, the only zero of the function f agrees with the eigenvalue λ (25) (cf. (63)), which lies on the arc Γ_α.

Denote $\Gamma_\alpha(a) \stackrel{\text{def}}{=} \Gamma_\alpha \backslash \{\lambda\}$ for $\left|a-\frac{1}{2}\right| > \frac{1}{2}$ and $\Gamma_\alpha(a) \stackrel{\text{def}}{=} \Gamma_\alpha$ otherwise. It is clear that the functions $\widehat{R}_{n,m}$ as well as the functions in the right-hand side of (66) are analytic on $\Gamma_\alpha(a)$ and therefore the latter formula remains valid for $z \in \Gamma_\alpha(a)$. We can estimate from above the matrix entries $\widehat{R}_{n,m}$ of the resolvent $R(z, U_a)$ on this set with the help of (60)–(62) and (22). Indeed,

$$\begin{aligned} r_-^m(z)\widehat{\varphi}_m(z) &= \frac{z+a}{\rho}\sum_{k=0}^{m-1}(r_-(z)r_+(z))^k r_-(z)^{2m-2k-1} \\ &\quad -z\sum_{k=0}^{m-2}(r_-(z)r_+(z))^k r_-(z)^{2m-2k-2} \\ &= \sum_{k=0}^{m-1} z^k r_-(z)^{2m-2k-1} - z\sum_{k=0}^{m-2} z^k r_-(z)^{2m-2k-2}, \end{aligned}$$

so that $\left|r_-^m\widehat{\varphi}_m\right| \le K_1(m+1)$ for $z \in \Gamma_\alpha$, $m \ge 0$ with some constant $K_1 = K_1(a)$. Taking into account (66), we end up with the following bound for $\widehat{R}_{n,m}$:

$$\left|\widehat{R}_{n,m}(z)\right| \le 2K_1(\min(n,m)+1)\,\frac{|r_-(z)|^{|n-m|}}{|1+\overline{a}z-\rho r_-(z)|}, \qquad z \in \Gamma_\alpha(a), \quad m,n \ge 0. \tag{68}$$

Our consideration in Section 5 is based on the asymptotic behavior of the resolvent $R(z, U_a)$ as z approaches the endpoints $e^{\pm i\alpha}$ along Γ_α, so it seems advisable to remove from Γ_α a small vicinity of the eigenvalue λ, which does not contain the endpoints (and exerts no influence on the asymptotic behavior of the resolvent near the endpoints). Denote this set by $E_\alpha(a)$. As f is bounded away from zero on $E_\alpha(a)$ for $\left|a-\frac{1}{2}\right| \ne \frac{1}{2}$, we come to the relation, which holds for $z \in E_\alpha(a)$ with some constant $K_2 = K_2(a)$:

$$\left|\widehat{R}_{n,m}(z)\right| \le K_2(\min(n,m)+1)\,|r_-(z)|^{|n-m|}, \quad m,n \ge 0, \quad \left|a-\frac{1}{2}\right| \ne \frac{1}{2}. \tag{69}$$

Note that (69) is valid on the whole arc Γ_α as long as $\left|a-\frac{1}{2}\right| < \frac{1}{2}$.

5. Trace class perturbations and finiteness of discrete spectrum

Under the assumption $\lim_{n\to\infty} a_n = a,\ 0<|a|<1$, the structure of $\operatorname{supp}\mu$ on the arc Γ_α was completely discribed by Ya.L. Geronimus [10, Theorem 1', p. 205]: the set $\operatorname{supp}\mu\cap\Gamma_\beta$ is finite for every $0<\beta<\alpha$ (see also [15, Theorem 3] for the operator theoretic proof). The conclusion holds true under the Lopez condition (30) as well.

The following question appears to be quite natural: what kind of assumption imposed on the reflection coefficients a_n provides finitness of the set $\operatorname{supp}\mu\cap\Gamma_\alpha$? Theorem 4 discloses such possibility for the finite perturbations of the constant sequence $\{a\}$. In the present Section we examine certain infinite perturbations, which preserve this property (the similar problem is studied in [23, Theorem 4.2, p. 350] in more general setting, but under the stronger assumption).

Theorem 12. *The set* $\operatorname{supp}\mu\cap\Gamma_\alpha$ *is finite as long as* *

$$A_1 \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} n\,|a_n - a| < \infty. \tag{70}$$

To prove Theorem 12 we proceed in several steps.

Trace class operators. Recall the basic definitions and properties of the trace class operators (cf., e.g., [24, Chapter VI.6]). A bounded linear operator acting on a Hilbert space H is said to belong to the *trace class* $\mathcal{S}_1$ if for every orthonormal basis $\{h_n\}$

$$\|T\|_1 \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (|T|\,h_n, h_n) < \infty, \qquad |T| = (T^*T)^{\frac{1}{2}} \tag{71}$$

(the value in the right-hand side does not actually depend on the choice of basis). The trace class $\mathcal{S}_1$ forms a self adjoint norm–ideal in the algebra of all bounded linear operators, that is, $\mathcal{S}_1$ is a linear space; $T\in\mathcal{S}_1$ if and only if $T^*\in\mathcal{S}_1$; $T\in\mathcal{S}_1$ implies $ATB\in\mathcal{S}_1$ with

$$\|ATB\|_1 \le \|A\|\,\|T\|_1\,\|B\| \tag{72}$$

for every bounded operators A and B. Furthermore, the trace class endowed with norm $\|\cdot\|_1$ (71) is a Banach space.

Of particular interest and paramount importance is the following functional on $\mathcal{S}_1$, which is known as the *trace*:

$$\operatorname{tr} T \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (Th_n, h_n)$$

*For the continuous analogue of the condition (70) in the spectral theory of the Schrödinger operator see, for instance, [1, Chapter II, formula (2.1.2), p. 37].

(the latter does not depend on the choice of basis). As a matter of fact, $\operatorname{tr} T$ is a bounded linear functional on the space $\mathcal{S}_1$ which satisfies

$$|\operatorname{tr} T| \le \|T\|_1, \qquad \operatorname{tr} T^* = \overline{\operatorname{tr} T}, \qquad \operatorname{tr} TA = \operatorname{tr} AT \tag{73}$$

for every bounded linear operator A (cf. [24, Theorem VI.25, p. 212]).

The following sufficient condition for an operator T to belong to $\mathcal{S}_1$ proves useful thereafter.

Lemma 13. *Let T be a bounded linear operator on H. If for some orthonormal basis $\{h_n\}$*

$$\sum_{n,m=0}^{\infty} |(Th_n, h_m)| < \infty \tag{74}$$

holds, then $T \in \mathcal{S}_1$ and

$$\|T\|_1 \le \sum_{n,m=0}^{\infty} |(Th_n, h_m)|\,. \tag{75}$$

P r o o f. The argument below is based on the series expansion similar to that in (34). Let $T = \{t_{k,j}\}_{k,j=0}^{\infty}$ be the matrix representation of T in the basis $\{h_n\}$. The (formal) operator series

$$T = \sum_{k=1}^{\infty} (S^*)^k B_{-k} + \sum_{m=0}^{\infty} B_m S^m \tag{76}$$

arises naturally. Here B_n are diagonal operators

$$\begin{aligned} \operatorname{diag} B_{-k} &= (t_{k,0}, t_{k+1,1}, \ldots)\,, \qquad k = 1, 2, \ldots\,, \\ \operatorname{diag} B_m &= (t_{0,m}, t_{1,m+1}, \ldots)\,, \qquad m = 0, 1, \ldots\,. \end{aligned}$$

Since

$$\|B_m\|_1 = \sum_{n=0}^{\infty} |t_{n,m+n}|, \quad m = 0, 1, \ldots, \qquad \|B_{-k}\|_1 = \sum_{n=0}^{\infty} |t_{k+n,n}|, \quad k = 1, 2, \ldots,$$

series (76) is easily seen to converge in the trace class norm under assumption (74), and (75) holds, as claimed. ■

Let us now go back to the Hessenberg operators U (9)–(10) and U_a (16). Perturbation V (29) is known to be compact whenever $\lim_{n\to\infty} a_n = a$. We show that $V \in \mathcal{S}_1$ under certain restriction on the rate of convergence.

Lemma 14. *Let*

$$A_0 \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} |a_n - a| < \infty. \tag{77}$$

Then V belongs to the trace class.

P r o o f. Much as in the proof of Theorem 6 we consider series expansion (34) and show that (77) ensures convergence of this series in the trace class norm.

Assume first that $r \stackrel{\text{def}}{=} \sup \rho_n < 1$, $\rho_n^2 = 1 - |a_n|^2$ (in other words, $a_n \neq 0$). As in (37), (38) we have for $k = -1, 0, 1, \ldots$

$$|v_{j,k+j}| \le (|a_{k+j+1} - a| + |a_j - a|)\, r^k + r^{k-1} \sum_{l=1}^{k} |\rho_{l+j} - \rho| \le C_1 r^k \sum_{l=0}^{k+1} |a_{l+j} - a|, \tag{78}$$

and hence

$$\|D_k\|_1 = \sum_{j=0}^{\infty} |v_{j,k+j}| \le C_2 A_0 k r^k,$$

where C_m, $m = 1, 2, \ldots$ stand for positive constants which depend only on the sequence $\{a_n\}$. Therefore (34) converges in the trace class norm, as needed.

To remove the assumption $r < 1$ pick $N \in \mathbb{N}$ such that $a_n \neq 0$ for $n \geq N$, and form a new sequence $\{b_n\}$ defined by

$$b_n = a_n, \quad n \geq N, \qquad b_1 = b_2 = \ldots = b_{N-1} = \frac{1}{2}.$$

Let $U(B)$ be the corresponding Hessenberg operator. Then $\sup(1 - |b_n|^2) < 1$ and hence $U(B) - U_a \in \mathcal{S}_1$. Next, the operator $U - U(B)$ is of finite rank. Thus $V = U - U(B) + U(B) - U_a \in \mathcal{S}_1$ that completes the proof. ■

Note that much the same method enables one to prove the general statement: V belongs to the normed ideal $\mathcal{S}_p$ whenever $a_n - a \in \ell^p$, $1 \le p < \infty$ (the space c_0 is to be taken instead of ℓ^∞ for $p = \infty$).

As a straightforward consequence of the trace class theorems of scattering theory (cf. [4]) we have the following

Theorem 15. *Under the premises of Lemma* 14 *the absolutely continuous spectrum of the operator U fills the arc Δ_α.*

Finiteness of discrete spectrum. In order to evaluate the number of the points in $\operatorname{supp} \mu \cap \Gamma_\alpha$ (or, in other words, the number of eigenvalues of the operator U on the arc Γ_α) let us go over to the much better explored situation of the trace class perturbations of a bounded self-adjoint operator.

Consider the Zhukovsky transform

$$J \overset{\text{def}}{=} \frac{U+U^*}{2} = \frac{U+U^{-1}}{2}, \qquad J_a \overset{\text{def}}{=} \frac{U_a+U_a^*}{2} = \frac{U_a+U_a^{-1}}{2}. \tag{79}$$

The essential spectrum of J is exactly the interval $[-1, \cos\alpha]$. According to the Spectral Mapping Theorem, a number $\cos\beta$, $\beta < \alpha$ is the eigenvalue of J if and only if one of the numbers $\exp\{\pm i\beta\}$ is the eigenvalue of U. Hence the set $\operatorname{supp}\mu \cap \Gamma_\alpha$ is finite if and only if the number of the eigenvalues of J above the essential spectrum is finite.

Denote $N_+(\beta)$ the number of eigenvalues of J, which are greater than $\cos\beta$. Our further consideration is based on the following result due to J. Geronimo [7, Theorem IV.3, p. 262]:

$$N_+(\beta) \le \operatorname{tr}\,[(J-J_a)\,R(\cos\beta, J_a)]^2\,. \tag{80}$$

The key idea is to argue that

$$\operatorname{tr}\,[(J-J_a)\,R(\cos\beta, J_a)]^2 = O(1), \qquad \beta \to \alpha + 0. \tag{81}$$

Under condition (77) the perturbation

$$J - J_a = \frac{U-U_a}{2} + \frac{U^*-U_a^*}{2} = \frac{V+V^*}{2}$$

belongs to the trace class, so that the value in the right-hand side of (80) is finite. Next, it is actually not hard to relate the resolvents of the operators J_a and U_a. Indeed,

$$R(\cos\beta, J_a) = \left(\frac{U_a+U_a^{-1}}{2} - \cos\beta\, I\right)^{-1} = 2U_a\left(U_a^2 - 2\cos\beta U_a + I\right)^{-1}.$$

Since $t^2 - 2\cos\beta t + 1 = \left(t-e^{i\beta}\right)\left(t-e^{-i\beta}\right)$, we have by partial fraction expansion

$$R(\cos\beta, J_a) = (1 - i\cot\beta) R\left(e^{i\beta}, U_a\right) + (1 + i\cot\beta) R\left(e^{-i\beta}, U_a\right). \tag{82}$$

Finally

$$\{(J-J_a)R(\cos\beta, J_a)\}^2$$
$$= -\frac{1}{4\sin^2\beta}\{(V+V^*)\left(e^{i\beta}R\left(e^{i\beta}, U_a\right) - e^{-i\beta}R\left(e^{-i\beta}, U_a\right)\right)\}^2. \tag{83}$$

We will establish (81) first under two additional restrictions on the sequence a_n:

$$\left|a - \frac{1}{2}\right| \ne \frac{1}{2} \qquad \text{and} \qquad r = \sup_n \rho_n < 1. \tag{84}$$

The first one makes it possible to apply (69) and the second one simplifies the calculations (cf. the proof of Lemma 14). The restrictions will be removed at the end of the section.

Put

$$W(\zeta) \stackrel{\text{def}}{=} VR(\zeta,U_a) = \{w_{n,m}(\zeta)\}_{n,m=0}^{\infty},\ W_*(\zeta) \stackrel{\text{def}}{=} R(\zeta,U_a)V^* = \{w^*_{n,m}(\zeta)\}_{n,m=0}^{\infty},$$

so that

$$w_{n,m}(\zeta) = \sum_{j=n-1}^{\infty} v_{n,j}\widehat{R}_{j,m}(\zeta), \qquad w^*_{n,m}(\zeta) = \sum_{j=m-1}^{\infty} \widehat{R}_{n,j}(\zeta)\overline{v}_{m,j}.$$

As $|r_-(\zeta)| < 1$ on $E_\alpha(a)$, then by (69)

$$\left|\widehat{R}_{j,m}(\zeta)\right| \le K_2\ (\min(j,m)+1)$$

uniformly on $E_\alpha(a)$. For $m < n$ we have

$$|w_{n,m}(\zeta)| \le K_2(m+1) \sum_{j=n-1}^{\infty} |v_{n,j}|\,.$$

Similarily, for $m \ge n$

$$|w_{n,m}(\zeta)| \le K_2 \sum_{j=n-1}^{m-1} j\,|v_{n,j}| + K_2(m+1)\sum_{j=m}^{\infty}|v_{n,j}| \le K_2(m+1) \sum_{j=n-1}^{\infty} |v_{n,j}|\,.$$

It follows from (78) that

$$\sum_{j=n-1}^{\infty} |v_{n,j}| \le C_1 \sum_{j=n-1}^{\infty} r^{j-n} \sum_{l=0}^{j-n+1} |a_{l+n}-a| \le C_3 \sum_{l=0}^{\infty} r^l\,|a_{l+n}-a|\,.$$

Hence

$$|w_{n,m}(\zeta)| \le C_4(m+1)\,\omega(n), \qquad \omega(k) \stackrel{\text{def}}{=} \sum_{l=0}^{\infty} r^l\,|a_{l+k}-a| \tag{85}$$

uniformly on $E_\alpha(a)$.

In exactly the same way we can make sure that

$$\left|w^*_{n,m}(\zeta)\right| \le C_4(n+1)\,\omega(m). \tag{86}$$

We are able now to estimate uniformly for $\zeta_{1,2} \in E_\alpha(a)$ the trace of "typical" operators

$$VR(\zeta_1,U_a)VR(\zeta_2,U_a), \qquad V^*R(\zeta_1,U_a)V^*R(\zeta_2,U_a), \tag{87}$$

which occur in the right-hand side of (83). Indeed, $VR(\zeta_1, U_a)VR(\zeta_2, U_a) = W(\zeta_1)W(\zeta_2)$ and by (85)

$$\sum_{k=0}^{\infty} |w_{n,k}(\zeta_1)w_{k,n}(\zeta_2)| \le C_5(n+1)\,\omega(n)\sum_{k=0}^{\infty}(k+1)\,\omega(k),$$

so that uniformly for $\zeta_1,\ \zeta_2 \in E_\alpha(a)$

$$|\mathrm{tr}\ VR(\zeta_1, U_a)VR(\zeta_2, U_a)| \le \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|w_{n,k}(\zeta_1)w_{k,n}(\zeta_2)| \le C_6\left(\sum_{n=0}^{\infty} n\omega(n)\right)^2.$$

Note that the latter series converges whenever (70) holds. In fact, the inequality

$$\sum_{n=0}^{q} nr^{q-n} \le q\sum_{n=0}^{q} r^{q-n} < (1-r)^{-1}q$$

yields

$$\sum_{q=0}^{\infty} q\,|a_q - a| > (1-r)\sum_{q=0}^{\infty} q\,|a_q - a|\sum_{n=0}^{q} nr^{q-n} = \sum_{n=0}^{\infty} n\omega(n),$$

as claimed. Thus, under assumption (70) $|\mathrm{tr}\ VR(\zeta_1, U_a)VR(\zeta_2, U_a)| < \infty$ uniformly on $E_\alpha(a)$. The same goes for the second operator in (87), since due to (73)

$$\mathrm{tr}\,V^*R(\zeta_1, U_a)V^*R(\zeta_2, U_a) = \mathrm{tr}\,R(\zeta_1, U_a)V^*R(\zeta_2, U_a)V^* = \mathrm{tr}\,W_*(\zeta_1)W_*(\zeta_2).$$

There are two more "typical" operators of the form

$$V_1(\zeta_1, \zeta_2) \stackrel{\mathrm{def}}{=} VR(\zeta_1, U_a)V^*R(\zeta_2, U_a) = VW_*(\zeta_1)R(\zeta_2) = \{v^1_{n,m}(\zeta_1, \zeta_2)\}_{n,m=0}^{\infty},$$
$$V_2(\zeta_1, \zeta_2) \stackrel{\mathrm{def}}{=} V^*R(\zeta_1, U_a)VR(\zeta_2, U_a),$$

for which the computation is somewhat more intricate. Put

$$W_1(\zeta) \stackrel{\mathrm{def}}{=} VW_*(\zeta, U_a) = \{w^1_{n,m}(\zeta)\}_{n,m=0}^{\infty},$$

so that

$$w^1_{n,m}(\zeta) = \sum_{j=n-1}^{\infty} v_{n,j}w^*_{j,m}(\zeta).$$

By (86) and (78)

$$\begin{aligned}
\left|w^1_{n,m}(\zeta)\right| &\le C_7\omega(m)\sum_{j=n-1}^{\infty} r^{j-n}(j+1)\sum_{l=0}^{j-n+1}|a_{l+n}-a| \\
&= C_7\frac{\omega(m)}{r}\sum_{q=0}^{\infty} r^q(q+n)\sum_{l=0}^{q}|a_{l+n}-a| \\
&= C_7\frac{\omega(m)}{r}\sum_{l=0}^{\infty}|a_{l+n}-a|\sum_{q=l}^{\infty} r^q(q+n) \\
&\le C_8\,\omega(m)\sum_{l=0}^{\infty}(l+n)r^l\,|a_{l+n}-a| = C_8\,\omega(m)\widehat{\omega}(n),
\end{aligned}$$

where

$$\widehat{\omega}(n) \stackrel{\text{def}}{=} \sum_{j=n}^{\infty} jr^{j-n}\,|a_j-a|\,.$$

Hence under assumption (70)

$$\begin{aligned}
\left|v^1_{n,n}(\zeta_1,\zeta_2)\right| &\le \sum_{k=0}^{\infty}\left|w^1_{n,k}(\zeta_1)\widehat{R}_{k,n}(\zeta_2)\right| \\
&\le C_9\sum_{k=0}^{n-1}(k+1)\left|w^1_{n,k}(\zeta_1)\right| + C_9(n+1)\sum_{k=n}^{\infty}\left|w^1_{n,k}(\zeta_1)\right| \\
&\le C_{10}\,\widehat{\omega}(n)\sum_{k=0}^{n-1}(k+1)\,\omega(k) + C_{10}(n+1)\,\widehat{\omega}(n)\sum_{k=n}^{\infty}\omega(k) \\
&\le C_{10}\,\widehat{\omega}(n)\sum_{k=0}^{\infty}(k+1)\,\omega(k) \le C_{11}\,\widehat{\omega}(n)
\end{aligned}$$

and

$$|\operatorname{tr} V_1(\zeta_1,\zeta_2)| \le \sum_{n=0}^{\infty}\left|v^1_{n,n}(\zeta_1,\zeta_2)\right| \le C_{12}\sum_{n=0}^{\infty}\widehat{\omega}(n).$$

It remains only to notice that

$$\sum_{n=0}^{\infty}\widehat{\omega}(n) = \sum_{n=0}^{\infty}\sum_{j=n}^{\infty} jr^{j-n}\,|a_j-a| = \sum_{j=0}^{\infty} j\,|a_j-a|\sum_{n=0}^{j} r^{j-n} < \frac{A_1}{1-r} < \infty.$$

Since by (73) $\operatorname{tr} V_2(\zeta_1,\zeta_2) = \operatorname{tr} V_1(\zeta_2,\zeta_1)$, we also have $|\operatorname{tr} V_2(\zeta_1,\zeta_2)| < \infty$ uniformly on $E_\alpha(a)$.

Thus, the limit relation (81) is verified under conditions (84).

To remove the second assumption in (84) we proceed as in the proof of Lemma 14: pick $N \in \mathbb{N}$ such that $a_n \neq 0$ for $n \geq N$, and form a new sequence $\{b_n\}$ defined by

$$b_n = a_n, \quad n \geq N, \qquad b_1 = b_2 = \ldots = b_{N-1} = \frac{1}{2}.$$

The support of the corresponding measure $\mu(B)$ is finite on Γ_α. As the initial sequence $\{a_n\}$ is a finite perturbation of the sequence $\{b_n\}$, the set $\operatorname{supp}\mu \cap \Gamma_\alpha$ is also finite according to [22, Theorem 4.1].

It remains to handle the special sequences $\{a_n\}$ with $|a - 1/2| = 1/2$. The key idea here has nothing to do with the spectral theory of unitary operators. We shall appeal to the second kind sequence $\{-a_n\}$, which corresponds to the second kind polynomials ψ_n and the second kind measure τ. It is clear that this sequence is no longer special and it satisfies the condition analogous to (70). Hence $\operatorname{supp}\tau \cap \Gamma_\alpha$ is finite.

Denote by $G(z)$ the second kind C–function (see (47)). We see that $G(z)$ is analytic on $\Gamma_\alpha \backslash \operatorname{supp}\tau$ and has a finite number of simple poles at $\operatorname{supp}\tau \cap \Gamma_\alpha$. Furthermore, $\Re G(t) = 0, \quad t \in \Gamma_\alpha \backslash \operatorname{supp}\tau$.

It was proved first in [14, p. 130–131] that $G(z) = F^{-1}(z)$, so we can examine the first kind C–function F on the arc Γ_α. It is well known that the zeros of G interlace its poles on Γ_α,* so that $F = G^{-1}$ has a finite number of poles on this arc. Next, $\Re F(t) = 0$ for all $t \in \Gamma_\alpha$ but a finite number of points. Therefore $\operatorname{supp}\mu \cap \Gamma_\alpha$ is a finite set, as claimed.

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*This property is commonly related to Nevanlinna functions on the upper half plane which are analytic and real on some interval of the real line (cf., e.g., [3, Appendix II, Theorem II.3.1, p. 439]).

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Теоретико-операторный подход к исследованию ортогональных полиномов на дуге единичной окружности

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Изучаются вероятностные меры на единичной окружности и операторы умножения на независимую переменную в соответствующих пространствах L^2. Если такая мера не удовлетворяет условию Сеге, то ортонормированные полиномы образуют ортонормированный базис в этом гильбертовом пространстве. В этом случае оператор умножения представим матрицей Хессенберга. Основной результат работы касается некоторых бесконечномерных возмущений "постоянной" матрицы Хессенберга, которые имеют конечное число собственных значений вне существенного спектра.

Теоретико-операторний підхід до вивчання ортогональних поліномів на дузі одиничного кола

Л.Б. Голінський

Вивчаються імовірносні міри на одиничному колі та оператори множення на незалежну змінну у відповідних просторах L^2. Якщо така міра не задовольняє умові Сеге, то ортонормовані поліноми утворюють ортонормований базис у цьому гільбертовому просторі. В даному випадку оператор множення зображується матрицею Хессенберга. Основний результат роботи стосується деяких нескінченновимірних збурень "постійної" матриці Хессенберга, які мають скінченну кількість власних значень поза істотним спектром.