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Point realization of Boolean actions of countable inductive limits of locally compact groups

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Let G be a CILLC-group, i.e., the inductive limit of an increasing sequence of its closed locally compact subgroups. Every nonsingular action of G on a measure space $(X, \mathcal{B}, \mu)$ generates a continuous action of G on the underlying Boolean σ-algebra $\mathcal{M}[\mu] = \mathcal{B}/I_\mu$, where I_μ is the ideal of μ-null subsets. It is known that the converse is true for any locally compact G: every abstract Boolean G-space is associated with some Borel nonsingular action of G. In the present work this assertion is generalized to arbitrary CILLC-groups. In addition, we conctruct a free measure preserving action of G on a standard probability space.

Introduction

By a *large* group we mean a non locally compact (infinite dimensional) one. For the last past decades such groups were studied intensively in representation theory, PDE with infinite variables, measure theory in infinite dimensional vector spaces, probability theory, quantum physics, etc. However, despite the achieved progress, the ergodic theory for these groups is still developed inadequately. Indeed, even the basic notions of ergodic theory such as ergodicity, smoothness

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(type I) of an action, or orbit partition are not as clear as in the classical (locally compact) case, since the classically equivalent definitions can lead now to nonequivalent concepts or fail at all. The present paper is devoted to study and classification of measured actions of large groups.

The principal difference of a large group with a locally compact one is the absence of the Haar measure. This makes impossible many of the classical constructions in ergodic theory. For example, given a measurable action of a large group, we can not use the canonical von Neumann cross-product construction. Thus classically the most important and useful connection of measurable dynamical systems and operator algebras is absent for general non locally compact systems. Among other specific problems we mention the lack of quasiinvariant measures on the orbits [M1, R3] and the lack of the full countable sections for free measurable actions of large groups [FR].

In the present paper we consider only those groups which can be represented as topological inductive limits of increasing sequences of locally compact second countable subgroups, which seem to be the simplest large groups. We shall call them *CILLC-groups*. It is easy to see that a nonsingular Borel action of an CILLC-group G on a measure space $(X, \mathcal{B}, \mu)$ generates an action of G on the Boolean σ-algebra $\mathcal{B}/\mathcal{I}(\mu)$, where $I(\mu)$ is the ideal of μ-null subsets. The standard lifting problem of ergodic theory is formulated as follows: whether or not each abstract Boolean G-space arises in this way, i.e., has point realization, and to what extent these realizations can differ? Very satisfying solution of this problem was obtained by P. Halmos and J. von Neumann [HN] for countable G and later by G. Mackey [M2] and A. Ramsay [R1, R3] for locally compact G. However, in the capacity of the "universal" space for the point realization they chose either $L^2_{loc}(G, \lambda)$ [M2, R1] or the closed unit ball of the algebra of bounded linear operators in $L^2(G, \lambda)$ [R3], where λ is a Haar measure on G, so these constructions are both no longer valid for large CILLC-groups. Another approach to point realization of Boolean finite measure preserving actions was proposed by A.M. Vershik [V]. It is based on two well-known statements of functional analysis: the Minlos theorem about characteristic functionals on nuclear vector spaces and the Gelfand–Kost′uchenko–G.I. Kats theorem about Gårding domains for unitary representations of locally compact groups. The main purpose of the present paper is to develop this method and solve in positive the point realization problem for nonsingular actions of arbitrary CILLC-groups (Theorems 2, 2′, and 7). Notice that the first step toward this end was done in [D], where existence of the special *strong* Gårding domains was proved for unitary representations of CILLC-groups. Thus we may disregard the difference between a nonsingular point action and a Boolean one and use those which is more convenient in the specific situation. For example, in the entropy theory the Boolean algebra point of view seems to be more elegant, but in the orbit theory points are preferable. As a corollary we obtain that the two possible natural definitions of ergodicity coincide for nonsingular actions of CILLC-groups (Corollary 8). Moreover, we construct free finite measure preserving actions for such groups (Example 11). We observe also that the well known results on the classification and structure of locally compact group actions with pure point spectrum are generalized naturally to CILLC-group actions.

1. CILLC-groups

Let $G_1 \subset G_2 \subset \dots$ be a sequence of locally compact second countable groups with G_n being closed in G_{n+1}, $n \in \mathbb{N}$. Then $G \stackrel{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} G_n$ endowed with the inductive limit topology ι is a Hausdorff topological group. We call G a *CILLC-group.* Denote by $\mathcal{B}(\iota)$ the Borel σ-algebra generated by ι. Some facts about ι and $\mathcal{B}(\iota)$ we collect in

Proposition 1. (i) *G is σ-compact, and separable;*

(ii) *G is locally compact if and only if there exists $N \in \mathbb{N}$ with G_{n+1}/G_n is discrete for all $n > N$;*

(iii) *$\mathcal{B}(\iota)$ is standard and generated by the Borel σ-algebras on G_n, $n \in \mathbb{N}$;*

(iv) *every Borel homomorphism from G into a Polish group H is continuous;*

(v) *if G is not locally compact, there exists no locally compact topology τ on G compatible with the group structure and such that $\mathcal{B}(\tau) = \mathcal{B}(\iota)$.*

P r o o f. (i) is trivial.

(ii) The "if" part is obvious and we prove the "only if" one. Let G be locally compact and let λ be a Haar measure on it. Then there exists $N \in \mathbb{N}$ with $\lambda(G_N) > 0$. It follows that G_N is open in G as well as in G_n for all $n > N$.

(iii) Follows directly from [M1].

(iv) Let $\phi : G \to H$ be a Borel homomorphism. Then $\phi \restriction G_n$ is also Borel and hence continuous since G_n is a Polish group for all $n \in \mathbb{N}$. By the definition of ι we have that ϕ is continuous.

(v) It follows readily from (iv) that ι is stronger than τ. Since τ is locally compact, there is $n \in \mathbb{N}$ with $\lambda(G_n) > 0$, where λ is a Haar measure on $(G, \mathcal{B}(\tau))$. Then G_n is τ-open and hence ι-open. But this contradicts to (ii). ■

2. Statement of the main theorem. Auxiliary lemmas

Let $(X, \mathcal{B})$ be a standard Borel space, G a CILLC-group, and $T : G \times X \ni (g, x) \mapsto T_g x \in X$ a Borel action. As usual, $G \times X$ is endowed with the product Borel structure. Then $(X, \mathcal{B})$ is called a *standard Borel G-space.* Now let μ be a nonatomic probability measure on $(X, \mathcal{B})$. Denote by $\mathcal{M}[\mu]$ the Boolean μ-measurable σ-algebra, i.e., the quotient $\mathcal{B}/I(\mu)$, where $I(\mu)$ is the ideal of μ-null Borel subsets. The quotient homomorphism $\mathcal{B} \to \mathcal{M}[\mu]$ will be denoted by π. It is known that $\mathcal{M}[\mu]$ endowed with the natural metric is a Polish space. Moreover, $\mathcal{M}[\mu]$ is isomorphic as topological σ-algebra to the Boolean σ-algebra of projectors of the von Neumann algebra $L^\infty(X,\mu)$ acting on $L^2(X,\mu)$ via multiplication [M2, R1]. We call $\mathcal{M}[\mu]$ a *Boolean G-space* if G acts on the left on $\mathcal{M}[\mu]$ and the map $G \ni g \mapsto gB \in \mathcal{M}[\mu]$ is Borel for each $B \in \mathcal{M}[\mu]$.

It is easy to see that if $(X, \mathcal{B})$ is a standard Borel G-space and μ is a quasiinvariant measure on it, i.e., $\mu \circ T_g \sim \mu$ for all $g \in G$, then

$$\widetilde{T}_g \pi(B) = \pi(T_g B),\ B \in \mathcal{B},\ g \in G,$$

defines an action $\widetilde{T}$ of G on $\mathcal{M}[\mu]$ so that $\mathcal{M}[\mu]$ is a Boolean G-space (for the case of locally compact G we refer to [M2]). Our purpose is to prove that each Boolean G-space arises this way, i.e., to generalize Theorem 1 of [M2] and Theorem 3.3 of [R1] to arbitrary CILLC-groups.

Theorem 2. *Let G be a CILLC-group and $\mathcal{M}[\mu]$ a Boolean G-space. Then there is a Borel μ-nonsingular action T of G on X (i.e., μ is quasiinvariant) with $\widetilde{T}_g B = gB$ for all $B \in \mathcal{M}[\mu]$ and $g \in G$.*

It is convenient for us to rephrase this theorem in equivalent terms. We denote by $\mathcal{U}$ the unitary group of the (real) Hilbert space $L^2(X,\mu)$ and set

$$\mathcal{U}_+(X,\mu) = \{V \in \mathcal{U} \mid VPV^* \in \mathcal{M}[\mu] \quad \text{for all } P \in \mathcal{M}[\mu] \text{ and } V1 \in L^2_+(X,\mu)\},$$

$$\mathcal{U}_{+,0}(X,\mu) = \{V \in \mathcal{U}_+(X,\mu) \mid V1 = 1\},$$

where $L^2_+(X,\mu)$ is the (closed) cone of nonnegative L^2-functions. It is known that $\mathcal{U}_+(X,\mu)$ and $\mathcal{U}_{+,0}(X,\mu)$ are the closed subgroups of $\mathcal{U}$ equipped with the weak (or, equivalently, $*$-strong) operator topology. Notice that they are homeomorphic to the groups of μ-nonsingular and μ-preserving transformations (μ-mod 0) of X respectively equipped with the weak automorphism topology [HO]. Each nonsingular action T of G on X generates a Borel representation

$$U_T : G \ni g \mapsto U_T(g) \in \mathcal{U}_+(X,\mu)$$

as follows:

$$(U_T(g)f)(x) = f(T_g^{-1}x)\sqrt{\frac{d\mu \circ T_g^{-1}}{d\mu}(x)}, \qquad x \in X,\ g \in G,\ f \in L^2(X,\mu). \tag{2.1}$$

By Proposition 1(iv), U_T is continuous. Given a continuous representation $U : G \to \mathcal{U}_+(\mathcal{X},\mu)$, we have the structure of Boolean G-space on $\mathcal{M}[\mu]$:

$$gP = U(g)PU(g^{-1}), \qquad P \in \mathcal{M}[\mu],\ g \in G. \tag{2.2}$$

Conversely, it is easy to deduce from [R2] that each Boolean G-space determines a continuous unitary representation $U : G \to \mathcal{U}_+(X,\mu)$ so that (2.2) is satisfied. Thus Theorem 2 can be reformulated as follows

Theorem 2′. *Let G be an CILLC-group and $U : G \ni g \mapsto U(g) \in \mathcal{U}_+(X,\mu)$ a continuous representation of G in $L^2(X,\mu)$. Then there is a Borel nonsingular action T of G on X with $U(g) = U_T(g)$ for all $g \in G$. Moreover, if $U(g) \in \mathcal{U}_{+,0}(X,\mu)$ for some $g \in G$, then the associated transformation T_g preserves μ.*

We preface the proof of Theorem 2′ with some auxiliary statements. For the terminology and background of the locally convex spaces we refer to [S]. In the present paper we consider only vector spaces over the real field $\mathbb{R}$. The following assertion was proved in somewhat more strong formulation as Main Theorem of [D].

Lemma 3. *Let $U : G \ni g \mapsto U(g)$ be a strongly continuous unitary representation of G in a separable Hilbert space $\mathcal{H}$. Then there exist a separable nuclear reflexive space $\mathcal{F}$ and continuous one-to-one linear map $J : \mathcal{F} \to \mathcal{H}$ such that the following properties are satisfied:*

(i) *the dual vector space $\mathcal{F}'$ endowed with the strong topology $\beta(\mathcal{F}',\mathcal{F})$ is a separable Fréchet space;*

(ii) $\operatorname{Im} J$ *is dense in* $\mathcal{H}$*;*

(iii) $U(g)\operatorname{Im} J = \operatorname{Im} J$ *for all* $g \in G$*;*

(iv) $L_g \stackrel{\text{def}}{=} J^{-1}U(g)J \in \mathcal{L}(\mathcal{F},\mathcal{F})$ *for all $g \in G$, where $\mathcal{L}(\mathcal{F},\mathcal{F})$ is the algebra of continuous linear operators in $\mathcal{F}$;*

(v) $G \ni g \mapsto L_g \in \mathcal{L}(\mathcal{F},\mathcal{F})$ *is a continuous map if $\mathcal{L}(\mathcal{F},\mathcal{F})$ is endowed with the weak operator topology.*

We shall prove here that the action of G on $\mathcal{F}'$ generated by the conjugate representation to $U(G) \restriction \operatorname{Im} J$ is Borel.

Let $\mathcal{E}$ be a vector space. Given a locally convex topology ι on $\mathcal{E}$, we denote by $\mathcal{B}(\iota)$ the Borel σ-algebra generated by ι.

Lemma 4. *Let $\langle \mathcal{E},\mathcal{F}\rangle$ be a duality system and let $\mathcal{E}$ endowed with the Mackey topology $\tau(\mathcal{E},\mathcal{F})$ be second countable. Then $\mathcal{B}(\iota) = \mathcal{B}(\tau(\mathcal{E},\mathcal{F}))$ for each topology ι compatible with $\langle \mathcal{E},\mathcal{F}\rangle$.*

P r o o f. Since by the Mackey theorem $\sigma(\mathcal{E},\mathcal{F}) \le \iota \le \tau(\mathcal{E},\mathcal{F})$, it suffices to prove that $\mathcal{B}(\tau(\mathcal{E},\mathcal{F})) = \mathcal{B}(\sigma(\mathcal{E},\mathcal{F}))$. The inclusion $\supset$ is evident. On the other hand, consider a subset $O \in \tau(\mathcal{E},\mathcal{F})$. Since $\mathcal{E}$ endowed with $\tau(\mathcal{E},\mathcal{F})$ is metrizable and separable, $O = \bigcup_{n\in\mathbb{N}}(e_n + O_n)$, where $e_n \in \mathcal{E}$ and O_n is a convex

$\tau(\mathcal{E},\mathcal{F})$-closed neighborhood of zero, $n \in \mathbb{N}$. By a corollary from the Hahn-Banach theorem every convex $\tau(\mathcal{E},\mathcal{F})$-closed subset is $\sigma(\mathcal{E},\mathcal{F})$-closed. Hence $O_n \in \mathcal{B}(\sigma(\mathcal{E},\mathcal{F}))$ for each $n \in \mathbb{N}$. Therefore $O \in \mathcal{B}(\sigma(\mathcal{E},\mathcal{F}))$. ■

Note that for a separable Banach space $\mathcal{E}$ the fact $\mathcal{B}(\sigma(\mathcal{E},\mathcal{E}')) = \mathcal{B}(\tau(\mathcal{E},\mathcal{E}'))$ was proved earlier by E. Mourier.

Now we use the notations of Lemma 3. Having in mind the duality system $\langle\mathcal{F},\mathcal{F}'\rangle$, we denote by $V(g)$ the adjoint operator to L_g. Let us endow $\mathcal{F}'$ with $\beta(\mathcal{F}',\mathcal{F}) = \tau(\mathcal{F}',\mathcal{F})$. Then $V(g) \in \mathcal{L}(\mathcal{F}',\mathcal{F}')$ and $V(gh) = V(h)V(g)$ for all $g, h \in G$.

Lemma 5. *The map* $V : G \times \mathcal{F}' \ni (g,f) \mapsto V(g)f \in \mathcal{F}'$ *is continuous.*

P r o o f. It follows from Lemma 3 that the map $G \ni g \mapsto V(g)f \in \mathcal{F}'$ is weakly continuous for each $f \in \mathcal{F}'$. Since $\mathcal{F}'$ is a separable Fréchet space, it follows from Lemma 4 that V is Borel in the variable g. Moreover, V is continuous in the variable f because of $V(g) \in \mathcal{L}(\mathcal{F}',\mathcal{F}')$. The Lebesgue–Kuratovski theorem [K, § 31.V, Theorem 2] implies that $V \restriction (G_n \times \mathcal{F}')$ is jointly Borel for all $n \in \mathbb{N}$. It follows from [Mo, Proposition 1.4] that $V \restriction (G_n \times \mathcal{F}')$ is continuous for all n, as desired. ■

3. Proof of the main theorem

This section is devoted to the proof of Theorem 2′. Let

$$U : G \ni g \mapsto U(g) \in \mathcal{U}_+(X,\mu)$$

be a continuous homomorphism. Then by Lemma 3 there exists a separable nuclear reflexive space $\mathcal{F}$ and a continuous one-to-one map $J : \mathcal{F} \to \mathcal{H} \stackrel{\text{def}}{=} L^2(X,\mu)$ such that the properties (i)–(v) of Lemma 3 are satisfied. Denote by S_g the adjoint operator to L_g^{-1}. It follows from Lemma 5 that

$$S : G \times \mathcal{F}' \ni (g,f) \mapsto S_g f \in \mathcal{F}'$$

is a continuous action of G on $\mathcal{F}'$. So, $(\mathcal{F}', \mathcal{B}(\beta(\mathcal{F}',\mathcal{F})))$ is a standard Borel G-space. The map $\chi : \mathcal{F} \to \mathbb{C}$ given by

$$\chi(f) = \int\limits_X \exp(i(Jf)(x))\,d\mu(x)$$

is a continuous positively defined functional with $\chi(0) = 1$. Since $\mathcal{F}$ is nuclear, by the Minlos theorem [VTC, Chapter VI, Theorem 4.3] there exists a probability measure ν on $\mathcal{F}'$ such that

$$\chi(f) = \int\limits_{\mathcal{F}'} \exp(i\langle f,y\rangle)\,d\nu(y)$$

for all $f \in \mathcal{F}$. It is easy to derive from [VTC, Chapter IV, Theorem 2.5] that there exists a Borel onto map $\varphi : X \to \mathcal{F}'$ such that $\nu = \mu \circ \varphi^{-1}$ and $Jf(x) = \langle f, \varphi(x) \rangle$ at μ-a.e. x for all $f \in \mathcal{F}$. Then we have

$$\int_X |Jf(x)|^2 \, d\mu(x) = \int_{\mathcal{F}'} |\langle f, y \rangle|^2 \, d\nu(y), \quad f \in \mathcal{F}.$$

It follows that each continuous linear functional on $\mathcal{F}'$ is square ν-integrable. Since $\operatorname{Im} J$ is dense in $L^2(X, \mu)$, φ is one-to-one on a Borel μ-conull subset and $\mathcal{F}$ is dense in $L^2(\mathcal{F}', \nu)$. Changing φ on a μ-null subset, we may regard φ as a Borel isomorphism of X onto $\mathcal{F}'$. So, φ generates an isomorphism of $\mathcal{M}[\mu]$ onto $\mathcal{M}[\nu]$ (as Boolean σ-algebras) and a unitary operator $\Phi : L^2(\mathcal{F}', \nu) \to L^2(X, \mu)$ with $\Phi^{-1}\mathcal{U}_+(X, \mu)\Phi = \mathcal{U}_+(\mathcal{F}', \nu)$, $\Phi 1_{\mathcal{F}'} = 1_X$ and $\Phi f = Jf$ for all $f \in \mathcal{F}$. It is easy to verify that $\Phi^{-1}U(g)\Phi f = L_g f$ for all $f \in \mathcal{F}$ and $g \in G$. Let $\{ f_n \mid n \in \mathbb{N} \}$ be a dense subset in $\mathcal{F}$. Then there exists a ν-nonsingular transformation D_g of $\mathcal{F}'$ and a Borel ν-conull subset C_g with

$$\begin{aligned} \langle f_n, S_g y \rangle &= \langle L_g^{-1} f_n, y \rangle = (\Phi^{-1} U(g^{-1}) \Phi f_n)(y) \\ &= f_n(D_g y)\rho(g, y) = f_n(\rho(g, y) \cdot D_g y) \end{aligned}$$

for all $y \in C_g$ and $n \in \mathbb{N}$, $g \in G$, where $\rho(g, y) = \sqrt{\frac{d\nu \circ D_g}{d\nu}(y)}$ [HN, M2]. Since $\{f_n \mid n \in \mathbb{N}\}$ as a family of functions on $\mathcal{F}'$ separates points it follows that

$$S_g y = \rho(g, y) \cdot D_g y \qquad \text{for a.e. } y \in \mathcal{F}'. \tag{3.1}$$

Let $\sim$ denote the equivalence relation on $\mathcal{F}' - \{0\}$ defined by $y_1 \sim y_2$ if $y_2 = \lambda y_1$ for some $\lambda > 0$, and $\pi : \mathcal{F}' \to Z$ the $\sim$-quotient mapping. Notice that Z endowed with the quotient topology is homeomorphic to the unit sphere in $\mathcal{F}'$. It is easy to see that π intertwines S with some Borel (even continuous) action $\widetilde{S} = \{\widetilde{S}_g\}$ of G on Z. Since $\mathcal{F}$ is dense in $L^2(\mathcal{F}', \nu)$, there exists a sequence $\{f_n\}_n \subset \mathcal{F}$ such that $\lim_{n\to\infty}\langle f_n, y \rangle = 1$ for ν-a.e. $y \in \mathcal{F}'$. At the same time it is clear that the pointwise limit of f_n is a Borel linear functional defined on a Borel ν-conull linear subspace $\mathcal{E} \subset \mathcal{F}'$. Therefore if $y_1, y_2 \in \mathcal{E}$, $y_1 \sim y_2$, $y_1 \neq y_2$, and $\lim_{n\to\infty}\langle f_n, y_1 \rangle = 1$, we have $\lim_{n\to\infty}\langle f_n, y_2 \rangle \neq 1$. Thus there exists a Borel ν-conull subset of $\mathcal{F}'$ which meets each $\sim$-equivalent class at most once. Hence changing π on a ν-null subset, we can transform it into a Borel isomorphism of $\mathcal{F}'$ onto Z. Moreover, it follows from (3.1) that $\widetilde{S}_g$ coincides $\nu \circ \pi^{-1}$-almost everywhere with a nonsingular transformation — the "image" of D_g under π for all $g \in G$. Now the desired action $T = \{T_g\}_{g \in G}$ may be defined as follows: $T_g = \varphi^{-1} \circ \pi^{-1} \circ \widetilde{S}_g \circ \pi \circ \varphi$.

Remark 6. Notice that we have proved more than claimed in the statement of Theorem 2: every Boolean G-action may be realized as a continuous (not only Borel) pointwise nonsingular action of G on a Polish space.

4. Essential uniqueness of point realization

Let μ and ν be probability measures on standard Borel spaces $(X, \mathcal{B})$ and $(Y, \mathcal{G})$ respectively. Given a Borel map $\varphi : X \to Y$ with $\nu \sim \mu \circ \varphi^{-1}$, we denote by $\varphi^* : \mathcal{M}[\nu] \to \mathcal{M}[\mu]$ the associated homomorphism of the underlying Boolean σ-algebras [M2, R1]. The following assertion is a generalization of [M2, Theorem 2] to CILLC-groups.

Theorem 7. *Let μ and ν be quasiinvariant probability measures on standard Borel G-spaces $(X, \mathcal{B})$ and $(Y, \mathcal{G})$ respectively. If there is a G-equivariant isomorphism ϑ of the underlying Boolean σ-algebra $\mathcal{M}[\nu]$ onto $\mathcal{M}[\mu]$, then there exist G-invariant Borel conull subsets $X_0 \subset X$ and $Y_0 \subset Y$ and a Borel G-equivariant isomorphism $\varphi : X_0 \to Y_0$ such that $\varphi^* = \vartheta$.*

P r o o f. It follows from [M2, Theorem 2] that for each $n \in \mathbb{N}$ there are G_n-invariant Borel conull subsets $X_n \subset X$ and $Y_n \subset Y$ and a Borel G_n-equivariant map $\varphi_n : X_n \to Y_n$ with $\varphi_n^* = \varphi$. We let

$$C_n = \{\, x \in X_n \cap X_{n+1} \mid \varphi_n(x) = \varphi_{n+1}(x) \,\}.$$

It is obvious that C_n and $\varphi_n(C_n)$ are Borel conull and G_n-invariant. Now we set $A_k = \bigcap_{n \geq k} C_n$, $X_0 = \bigcup_{k \in \mathbb{N}} A_k$, and $\varphi(x) = \varphi_k(x)$ for all $x \in A_k$, $k \in \mathbb{N}$. A simple verification shows that X_0 and φ are as desired. ■

Corollary 8. *Let $(X, \mathcal{B})$ be a standard Borel G-space and μ a finite invariant measure on it. If E is a Borel subset of X with $\mu(E \triangle gE) = 0$ for all $g \in G$, there is a Borel G-invariant subset E_0 such that $\mu(E \triangle E_0) = 0$.*

The proof of this statement coincides almost literally with that of Theorem 3 from [M2]. So, the two natural definitions of ergodicity are equivalent for nonsingular actions of CILLC-groups. Notice however that the statement of Corollary 8 is not valid for actions of general groups.

Example 9[V]. Let $X = \{0, 1\}^{\mathbb{N}}$, S_∞ be the Polish group of bijections of $\mathbb{N}$ acting on X via permutations of coordinates, and μ_1 and μ_2 two distinct Bernoulli measures on X. Since $\mu_1 \perp \mu_2$, there is a Borel subset $E \subset X$ with $\mu_1(E) = 1$ and $\mu_2(E) = 0$. We set $\mu = 0.5\mu_1 + 0.5\mu_2$. It is clear that μ is S_∞-invariant and $\mu(gE \triangle E) = 0$ for all $g \in S_\infty$. Since S_∞ acts transitively on a μ-conull Borel subset, there is no any S_∞-invariant subset E_0 with $\mu(E_0) = \mu(E) = 0.5$.

We consider also an application to actions with pure point spectrum. A finite measure preserving action of an Abelian CILLC-group G on a standard measure space $(X, \mathfrak{B}, \mu)$ is said to have *pure point spectrum* if the linear span of all eigenfunctions of the associated unitary representation U_T is dense in $L^2(X, \mu)$. Denote by $\widehat{G}$ the dual group of G (i.e., the group of continuous characters of G) and by $\operatorname{Sp} T$ the set of all eigenvalues of U_T. It is clear that $\operatorname{Sp} T$ is a countable subgroup of $\widehat{G}$. Notice that if $G = \operatorname{inj\,lim}_{n\to\infty} G_n$, G_n is locally compact, then $\widehat{G} = \operatorname{proj\,lim}_{n\to\infty} \widehat{G}_n$, where the canonical projection $\widehat{G}_{n+1} \to \widehat{G}_n$ is associated to the embedding $G_n \to G_{n+1}$, $n \in \mathbb{N}$.

Theorem 10. (i) *Two ergodic G-actions T and S with pure point spectrum are conjugate if and only if* $\operatorname{Sp} T = \operatorname{Sp} S$;

(ii) *For every countable subgroup Γ of $\widehat{G}$ there exists an ergodic action T of G with pure point spectrum such that* $\operatorname{Sp} T = \Gamma$.

P r o o f follows easily from Theorem 7, the Pontryagin's duality theorem for Abelian CILLC-groups — $\widehat{\widehat{G}}$ is canonically isomorphic to G [Sa], and the fact that the multiplicity of each eigenvalue of an ergodic G-action is one. ■

5. Free actions of CILLC-groups

It is well known that an arbitrary locally compact group admits free finite measure preserving actions. The main purpose of this section is to demonstrate

Example 11. For an arbitrary CILLC-group $G = \operatorname{inj\,lim}_{n\to\infty} G_n$ there exists a free measure preserving Borel G-action on a standard probability space.

Notice that the below argument is a modification of that from [Fed, § 6] where the case of locally compact groups and their *measured* actions was studied. Our *Borel* approach requires the use of new tools. We proceed in several steps.

1^0. We let $G_0 = \{e\}$, $X_n = G_{n-1} \backslash G_n$, and $e_n = G_{n-1} \in X_n$ for all $n \in \mathbb{N}$. Then $X = \prod_{n=1}^{\infty} X_n$ endowed with the product topology is a Polish space. To define a G-action T on X we first choose Borel cross-sections $s_n : X_n \to G_n$ of the quotient maps $G_n \to X_n$ with $s_n(e_n) = e$, $n \in \mathbb{N}$, and then set

$$T_g x = (y_1, \ldots, y_n, x_{n+1}, x_{n+2}, \ldots), \quad x = (x_n) \in X, \quad g \in G_n \subset G,$$

where $s_1(x_1) \ldots s_n(x_n) g^{-1} = s_1(y_1) \ldots s_n(y_n)$. It is routine to verify that $T = \{T(g)\}_{g \in G}$ is well defined, Borel, and free. Notice that the G-orbit of a point $x = (x_n)$ is exactly the set $\{y = (y_n) \in X \mid \exists N \in \mathbb{N} \text{ with } y_n = x_n \text{ for all } n > N\}$. Let μ_n be a G_n-quasiinvariant measure on X_n, $n \in \mathbb{N}$. Then $\mu = \bigotimes_{n=1}^{\infty} \mu_n$ is a G-quasiinvariant measure on X, since the map

$$X \ni x = (x_k) \mapsto s_1(x_1) \ldots s_n(x_n) \in G_n$$

intertwines the action $\{T_g\}_{g\in G_n}$ with the shiftwise action of G_n on itself and takes μ to a Haar-equivalent measure. It follows that for each $n \in \mathbb{N}$ there exists a closed $U(G_n)$-invariant subspace $\mathcal{H}_n \subset L^2(X,\mu)$, so that $U_T(G_n) \upharpoonright \mathcal{H}_n$ is unitarily equivalent to the left regular representation of G_n. So, U_T is faithful.

2^0. Now we remind one of the simplest constructions of a faithful unitary representation of the (real) unitary group $\mathcal{U}(L^2(X,\mu))$ with the image in $\mathcal{U}_{+,0}$. Let κ be a Haussian measure on $\mathbb{R}$ with $d\,\kappa(t) = \pi^{-1/2}\exp(-t^2)\,dt$ and $(Y,\lambda) = (\mathbb{R},\kappa)^{\mathbb{N}}$. Choose an orthonormal basis $\{f_n\}_{n=1}^{\infty}$ in $L^2(X,\mu)$. It is clear that the subset

$$D \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty}\bigcap_{N=1}^{\infty}\bigcup_{M=1}^{\infty}\bigcap_{M\le p\le q}\left\{(\gamma,y)\in\mathcal{U}\times Y\ :\ \left|\sum_{k=p}^{q}\langle f_n,\gamma f_k\rangle y_k\right| < 1/N\right\}$$

is Borel. We define a Borel function

$$A : \mathcal{U}\times Y \ni (\gamma,y) \mapsto A(\gamma,y) = (A(\gamma,y)_n) \in Y$$

as follows:

$$A(\gamma,y)_n = \begin{cases} \sum_{k=1}^{\infty}\langle f_n,\gamma f_k\rangle y_k, & \text{if } (\gamma,y)\in D\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to deduce from [SF, § 9.1] that

(i) for every $\gamma\in\mathcal{U}$ we have $A(\gamma^{-1},A(\gamma,y)) = y$ for λ-a.e. y;

(ii) for every $\gamma\in\mathcal{U}$ we have $\lambda\circ\gamma^{-1} = \lambda$, where

$$\lambda\circ\gamma^{-1}(B) = \lambda(\{y \mid A(\gamma,y)\in B\})$$

for each Borel subset $B\subset Y$.

Furthermore, using the similar argument, one can show that

(iii) for every $\gamma_1,\ \gamma_2\in\mathcal{U}$ we have $A(\gamma_1\gamma_2,y) = A(\gamma_1,A(\gamma_2,y))$ for λ-a.e. y.

It follows from [R1, Lemma 3.1] that A well defines a unitary representation (see (2.1)) $U_A : \mathcal{U}\to\mathcal{U}_{+,0}(Y,\lambda)$. Moreover, U_A is clearly continuous and faithful.

3^0. Let H be a locally compact second countable group and $(Z,\mathcal{B})$ a standard Borel H-space. For every point $z\in Z$ denote by $H(z)$ the stability group at z. It follows from [Va, Theorem 3.2] that $H(z)$ is closed for all $z\in Z$. Denote by 2^H the space of closed subsets of H. In the topology introduced by J.M.G. Fell [Fel] it is compact and metric. Furthermore, the map $Z\ni z\mapsto H(z)\in 2^H$ is Borel [AM, Proposition 2.3; R1, Lemma 9.4].

Lemma 12. (i) *For each subset $B\in 2^H$ the function*

$$f_B : 2^H \ni A \mapsto A\cap B \in 2^H$$

is Borel.

(ii) *Let ν be an invariant probability measure on $(Z, \mathcal{B})$ so that the associated unitary representation of H is faithful. Then for each element $h \neq e \in H$ there is a compact neighborhood W of h such that the subset*

$$D_W = \{z \in Z \mid H(z) \cap W = \emptyset\}$$

is of positive ν-measure.

P r o o f. (i) The Fell topology on 2^H has as subbase all sets of the form $O^+ = \{A \in 2_H \mid A \cap O \neq \emptyset\}$, where O is an open subset of H, plus all sets of the form $C^- = \{A \in 2^H \mid A \cap C = \emptyset\}$, where C is a compact subset of H [Fel]. Let us choose a sequence of open subsets $B_n \subset H$ with $\bigcap_{n=1}^{\infty} B_n = B$. It is easy to see that $f_B^{-1}(C^-) = (C \cap B)^-$ and $f_B^{-1}(O^+) = \bigcap_{n=1}^{\infty}(O \cap B_n)^+$. The assertion of (i) follows.

(ii) Suppose the contrary. Let W_n, $n \in \mathbb{N}$, be a fundamental sequence of compact neighborhoods of h and $D_{W_n} = \{z \in Z \mid H(z) \cap W_n = \emptyset\}$. Then the subset $M \stackrel{\text{def}}{=} \bigcap_n (Z - D_{W_n})$ is Borel by (i) and ν-conull. It follows that $h \in H(z)$ for a. a. $z \in Z$ and hence h belongs to the kernel of the associated representation, contrary to the hypotheses of the lemma. ∎

It is worthwhile to observe that f_B may be discontinuous.

4^0. It follows from 1^0, 2^0, and Theorem $2'$, and Remark 6 that there is a continuous action T of G on a Polish space Z and a finite invariant Borel measure ν on Z such that the associated unitary representation is faithful. So, we may apply Lemma 12(ii): there are two sequences of compact subsets $W_k \subset G$ and Borel subsets $A_k \subset Z$ with $\bigcup_{k=1}^{\infty} W_k = G - \{e\}$, $\mu(A_k) > 0$, and $T_g z \neq z$ for all $g \in W_k$, $z \in A_k$, $k \in \mathbb{N}$. Consider the diagonal action D of G on the Polish space $\Omega = Z^{\mathbb{N}}$:

$$(D_g \omega)_l = T_g \omega_l, \qquad l \in \mathbb{N}, \quad \omega = (\omega_l) \in \Omega, \quad g \in G.$$

Clearly, D is continuous and preserves the measure $\kappa = \nu^{\mathbb{N}}$. Moreover, the Borel subset $A = \bigcup_{k=1}^{\infty}(Z - A_k)^{\mathbb{N}}$ is κ-null. If $\omega = (\omega_l) \notin A$ and $g \in G - \{e\}$, there is $k \in \mathbb{N}$ with $g \in W_k$ and $\omega \notin (\Omega - A_k)^{\mathbb{N}}$. Hence $T_g \omega_n \neq \omega_n$ for some $n \in \mathbb{N}$ and therefore $D_g \omega \neq \omega$. Thus the Borel invariant subset

$$\Omega_0 = \{\omega \in \Omega \mid \text{the stabilizer } G_n(\omega) \text{ is trivial for all } n \in \mathbb{N}\}$$

contains a conull subset and we are done.

As a corollary we obtain

Example 13. For each CILLC-group G and a countable group Γ there is a free, Borel, finite measure preserving action of $G \times \Gamma$ so that the Γ-action is ergodic. In particular, G can be embedded into the normalizer of the type II_1

full group $[\Gamma]$ so that the G-action is strictly outer (for the definitions we refer to [HO, GS]).

Notice that for locally compact groups this was done in [GS, Example 2.13] and [Fed, Corollary 6.8]. We use a slight modification of those arguments. Let (X, μ) be a free standard Borel G-space with invariant probability measure μ. Without loss in generality we may assume that each G-orbit is μ-null — otherwise consider the diagonal action of G on the Cartesian product $X \times X$. Form the space $(Y, \nu) = (X, \mu)^{\Gamma}$. Then the diagonal action of G on Y is clearly commutes with the Bernoulli action of Γ determined by

$$(\gamma y)_\delta = y_{\delta\gamma}, \qquad \text{for } y = (y_\delta)_{\delta \in \Gamma} \in Y, \quad \gamma \in \Gamma.$$

Thus Y is a Borel $(G \times \Gamma)$-space and the Γ-action is ergodic. Set

$$B_{\gamma,\epsilon} = \{y = (y_\delta) \mid y_\gamma \in G y_\epsilon\}.$$

Then $B_{\gamma,\epsilon}$ is Borel, G-invariant, and ν-null. It is clear that $\delta B_{\gamma,\epsilon} = B_{\gamma\delta^{-1},\epsilon\delta^{-1}}$. Therefore the subset $Y_0 = Y - \bigcup_{\gamma,\epsilon \in \Gamma} B_{\gamma,\epsilon}$ is $(G \times \Gamma)$-invariant and ν-conull. Now if $gy = \gamma x$ for some $g \in G$, $\gamma \in \Gamma$, and $y \in Y_0$, we have that g and γ are the identities in G and Γ respectively.

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Точечная реализация булевских действий счетных индуктивных пределов локально компактных групп

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Пусть G — CILLC-группа, т.е. индуктивный предел возрастающей последовательности замкнутых локально компактных подгрупп. Любое несингулярное действие G на пространстве с мерой $(X, \mathcal{B}, \mu)$ индуцирует непрерывное действие G на булевской σ-алгебре $\mathcal{M}[\mu] = \mathcal{B}/I_\mu$, где I_μ — идеал μ-нулевых подмножеств. Известно, что обратное утверждение верно для любой локально компактной группы G: всякое абстрактное булевское G-пространство ассоциировано с некоторым борелевским несингулярным действием G. В настоящей работе это утверждение обобщается для произвольных CILLC-групп. Кроме того, построено свободное сохраняющее меру действие группы G на стандартном вероятностном пространстве.

Точкова реалізація булівських дій зчисленних індуктивних границь локально компактних груп

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Нехай G — CILLC-група, тобто індуктивна границя зростаючої послідовності замкнених локально компактних підгруп. Кожна несингулярна дія G на просторі з мірою $(X, \mathcal{B}, \mu)$ индукує неперервну дію G на булівській σ-алгебрі $\mathcal{M}[\mu] = \mathcal{B}/I_\mu$, де I_μ — ідеал μ-нульових підмножин. Відомо, що зворотнє твердження є вірним для кожної локально компактної групи G: довільний абстрактний булівський G-простір є асоційованим з якоюсь борелівською несингулярною дією G. В цій роботі це твердження узагальнюється для довільних CILLC-груп. Крім того, на стандартному ймовірносному просторі побудовано свободну дію групи G, що зберігає міру.