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The numerical range of linear relations in spaces with an indefinite metric

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It is shown that the numerical range of linear relations in spaces with an indefinite metric is always convex.

Let H be a linear space over the field $\mathbf{C}$ of complex numbers. By definition a linear relation in H is a linear subspace of the Cartesian product $H \times H$. An element in $H \times H$ is written as a pair $\{x, y\}, x, y \in H$. For example, the graph of a linear operator T in H, $\{\{x, Tx\} | x \in \operatorname{dom} T\}$, is a linear relation in H. A linear relation θ is the graph of a linear operator if and only if $\{0, x\} \in \theta$ implies $x = 0$.

The basic notions connected with geometry of spaces with an indefinite metric are considered in [1, 2].

Let θ be a linear relation in H and let $[\cdot, \cdot]$ be an inner product on H.

Definition. *The set*

$$W(\theta) = \{[x', x] | \ \{x, x'\} \in \theta, [x, x] = 1\}$$

is called the numerical range of the linear relation θ.

Theorem 1. *The numerical range of a linear relation is always convex in* $\mathbf{C}$.

P r o o f. We follow the proof of Theorem 1 from [3]. Let $\lambda_1 = [x_1', x_1]$, $\lambda_2 = [x_2', x_2]$ are different elements of $W(\theta)$; then we have $[x_1, x_1] = [x_2, x_2] = 1$, $\{x_1, x_1'\} \in \theta$, $\{x_2, x_2'\} \in \theta$. Suppose that x_1 and x_2 are linearly dependent,

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i.e., $x_2 = \alpha x_1$ for some $\alpha \in \mathbf{C}$ with $|\alpha| = 1$. It is clear that $\{\alpha x_1, x_2'\} \in \theta$, $\{\alpha x_1, \alpha x_1'\} \in \theta$. Setting $h = x_2' - \alpha x_1'$, we obtain

$$\{0, h\} \in \theta, \qquad [h, x_1] \neq 0.$$

Thus for every $\xi \in \mathbf{C}$, it follows $\{x_1, x_1' + \xi h\} \in \theta$, consequently, $W(\theta) \supset \{[x_1' + \xi h, x_1] | \ \xi \in \mathbf{C}\} = \mathbf{C}$, i.e., $W(\theta) = \mathbf{C}$.

Now, we suppose that x_1 and x_2 are linearly independent and let

$$L = \{\alpha_1 x_1 + \alpha_2 x_2 | \ \alpha_1, \alpha_2 \in \mathbf{C}\}.$$

Define a linear operator $T : L \to H$ by the formula

$$Tx_1 = x_1', \qquad Tx_2 = x_2'.$$

Then

$$\lambda_1 = [Tx_1, x_1], \quad \lambda_2 = [Tx_2, x_2], \quad [x_1, x_1] = [x_2, x_2] = 1.$$

Let $\lambda = (1-t)\lambda_1 + t\lambda_2 \ \ (0 < t < 1)$. By Theorem 1 from [4], there exists an element $\alpha_1 x_1 + \alpha_2 x_2 \in L$ such that

$$\lambda = [T(\alpha_1 x_1 + \alpha_2 x_2), \alpha_1 x_1 + \alpha_2 x_2], \ \ [\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 x_1 + \alpha_2 x_2] = 1.$$

Hence

$$\lambda = [\alpha_1 x_1' + \alpha_2 x_2', \alpha_1 x_1 + \alpha_2 x_2],$$
$$\{\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 x_1' + \alpha_2 x_2'\} \in \theta,$$

consequently, $\lambda \in W(\theta)$, therefore $W(\theta)$ is convex.

The theorem is proved.

R e m a r k. This theorem was proved for Hilbert spaces in [3]. Let $D(\theta) = \{x \in H | \ \{x, x'\} \in \theta\}$ be the domain of the linear relation θ.

Theorem 2. *If* $\dim D(\theta) < \infty$, $D(\theta)$ *is indefinite and non-degenerate and* $W(\theta) \neq \mathbf{C}$, *then the numerical range of* θ *coincides with the numerical range of some linear operator in* $D(\theta)$.

P r o o f. It is well-known that every finite-dimensional non-degenerate subspace of an inner product space is ortho-complemented (cf. [1]). Thus the space H can be represented as the orthogonal direct sum of $D(\theta)$ and $D(\theta)^{\perp}$:

$$H = D(\theta)[\dot{+}]D(\theta)^{\perp}.$$

Define a linear operator T in $D(\theta)$ by the formula

$$Tx = Px' \qquad (\{x, x'\} \in \theta),$$

where P is the ortho-projection of H on $D(\theta)$.

Now we prove that this definition is correct. If $h \notin D(\theta)^{\perp}$, then there exists $x_1 \in D(\theta)$ such that $[x_1, x_1] = 1, [h, x_1] \neq 0$. In fact, we choose $x_2 \in D(\theta)$ with $[x_2, x_2] = 1$. Then every $x \in D(\theta)$ can be represented as the sum of two positive elements of $D(\theta)$, i.e., we have

$$x = (x - tx_2) + tx_2,$$

where $[tx_2, tx_2] > 0, [x - tx_2, x - tx_2] > 0$ for sufficiently large real number t. Hence, there exists $x_3 \in D(\theta)$ such that $[x_3, x_3] > 0, [h, x_3] \neq 0$.
Setting

$$x_1 = \frac{x_3}{\sqrt{[x_3, x_3]}} \in D(\theta),$$

we obtain $[x_1, x_1] = 1, [h, x_1] \neq 0$.

Furthermore, if $\{0, h\} \in \theta$ and $h \notin D(\theta)^{\perp}$ then as shown in the proof of the preceding theorem, we conclude that $W(\theta) = \mathbf{C}$, contrary to the assumption. Thus $\{0, h\} \in \theta$ implies $h \in D(\theta)^{\perp}$.

Finally, we prove that the definition of T is correct.
Let $\{x, x_1'\} \in \theta, \{x, x_2'\} \in \theta$, then $\{0, x_1' - x_2'\} \in \theta$, hence $x_1' - x_2' \in D(\theta)^{\perp}$. Since the subspace $D(\theta)$ is non-degenerate and

$$[Px_1' - Px_2', z] = [x_1' - x_2', z] = 0 \qquad (z \in D(\theta)),$$

we conclude that $Px_1' = Px_2'$. The theorem is proved.

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Числовая область линейных отношений в пространствах с индефинитной метрикой

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Показано, что числовая область линейных отношений в пространствах с индефинитной метрикой всегда выпукла.

Числова область лінійних відношень у просторах з індефінітною метрикою

Ц. Баясгалан

Доведено, що числова область лінійних відношень у просторах з індефінітною метрикою завжди опукла.