Matematicheskaya fizika, analiz, geometriya
2000, v. 7, No. 2, p. 172–183

Upper estimates for entire functions of $L^1(R)$ on real line

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Received May 14, 1999
Communicated by I.V. Ostrovskii

Let $\mathcal{S}_\rho$ be the set of all entire functions of order ρ and normal type such that $f(x) \geq 0$ for $x \in \mathbf{R}$ and $f \in L^1(\mathbf{R})$. We prove that: 1) if $f \in \mathcal{S}_\rho$, then $f(x) = o(|x|^{\rho-1})$, $x \to \pm\infty$, 2) for any sequence $\varepsilon_n \downarrow 0$ there exists a function $f \in \mathcal{S}_\rho$ and a real sequence $b_n \to +\infty$ such that $f(b_n) > b_n^{\rho-1-\varepsilon_n}$. We give a generalization of this result for more general growth scale.

1. Introduction and statement of results

Let us denote by $\mathcal{E}_\rho$ the set of all entire functions of order ρ and normal type which are bounded on the real line. A famous theorem of S.N. Bernstein asserts that the following implication holds

$$F \in \mathcal{E}_1 \Longrightarrow F' \in \mathcal{E}_1 \,,$$

and, moreover,

$$\sup\{|F'(x)| : x \in \mathbf{R}\} \leq \sigma \sup\{|F(x)| : x \in \mathbf{R}\} \,,$$

where $\sigma = \limsup_{r\to\infty} r^{-1} \log M(r, F)$. If $\rho > 1$, then the implication

$$F \in \mathcal{E}_\rho \Longrightarrow F' \in \mathcal{E}_\rho \,,$$

Mathematics Subject Classififcations (1991): 30D10, 30D15, 60E05.

This work is partially supported by INTAS. Grant No. 96-0858.

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is not true, but there is (see M.M. Dzhrbashyan [3]) the following asymptotic estimate

$$|F'(x)| = O(|x|^{\rho-1}), \quad x \to \pm\infty\,. \tag{1}$$

It is easy to check the sharpness of the estimate (1). For example, let $n \geq 2$ be natural number and $F(z) = \exp(iz^n)$. Then, $|F'(x)| = |x|^{n-1}$ for real x.

Let us denote by $\mathcal{S}_\rho$ the set of all entire functions f of order ρ and normal type such that $f(x) \geq 0$ for $x \in \mathbf{R}$ and $f \in L^1(\mathbf{R})$. Obviously, if $f \in \mathcal{S}_\rho$ and $F(z) := \int_0^z f(\zeta)\, d\zeta$, then $F \in \mathcal{E}_\rho$ and therefore,

$$|f(x)| = |F'(x)| = O(|x|^{\rho-1}), \quad x \to \pm\infty\,. \tag{2}$$

The aim of this note is to investigate the question of sharpness of estimate (2) in class $\mathcal{S}_\rho$.

R e m a r k 1. In this connection it should be pointed out the following theorem of S.N. Bernstein [2]:

Let F be a real entire function of exponential type not greater than σ such that $F(x)$ is monotone and $|F(x)| \leq 1$ on the real line. Then $|F'(x)| \leq \sigma/\pi$, $-\infty < x < +\infty$. In addition, the equality in this inequality may be attained only at a single point. If this point is equal 0*, then*

$$F(z) = \pm\frac{2}{\pi}\int_0^{\sigma z} \frac{1-\cos t}{t^2}\, dt\,.$$

An analogous theorem for the entire functions of order $1/2$ was proved by N.I. Akhiezer [1].

Let $\mathcal{T}$ be the set of all positive nondecreasing functions t on the halfline $[0,\infty)$ such that the following two conditions are valid:

$$\frac{t(r)}{r} \uparrow \infty\,, \quad r \to +\infty\,, \tag{3}$$

for every $k > 1$ there exists $K(k) < \infty$ such that

$$\limsup_{r\to+\infty} \frac{t(kr)}{t(r)} = K(k)\,. \tag{4}$$

(It is not hard to see that condition (4) is equivalent to the following one: there exists $k_0 > 1$ such that $\limsup_{r\to+\infty} t(k_0 r)/t(r) < \infty$.)

We denote by $\mathcal{T}'$ the subset of the set $\mathcal{T}$ which contains those functions $t \in \mathcal{T}$, for which the following condition is satisfied instead of the condition (3):

$$t(r)/r^a \uparrow \quad \text{on} \quad [r_0,\infty) \tag{5}$$

for some $a > 1$ and $r_0 > 0$ (a and r_0 are dependent on the function t).

It is obvious that the functions $t(r) = r^\rho$, $\rho > 1$, $t(r) = r^{\rho(r)}$, where $\rho(r) \to \rho > 1$ $(r \to \infty)$ is a proximate order, belong to the set $\mathcal{T}'$.

We shall use the following generally accepted notation:

$$M(r, f) = \max\{|f(z)| : |z| = r\}, \quad h_f(\varphi) = \limsup_{r\to\infty} r^{-\rho} \log |f(re^{i\varphi})| .$$

Suppose $t \in \mathcal{T}$; then by $\mathcal{S}[t]$ we denote the set of all entire functions f such that: 1) $M(r, f) \le \exp(t(r))$ for sufficiently large r, 2) $f(x) \ge 0$ for all real x, 3) $f \in L^1(\mathbf{R})$. Thus $\mathcal{S}[r^\rho] = \mathcal{S}_\rho$.

Theorem 1. *Let $t \in \mathcal{T}$. If $f \in \mathcal{S}[t]$, then the following estimate*

$$f(x) = o\left(\frac{t(x)}{|x|}\right), \quad x \to \pm\infty , \tag{6}$$

is valid.

We cannot prove the sharpness of the estimate (6) in the following sense: if $t \in \mathcal{T}$ and $\Delta_n \downarrow 0$ $(n \to \infty)$ are given, then there exists a function $f \in \mathcal{S}[t]$ such that

$$f(b_n) \ge \Delta_n \frac{t(b_n)}{b_n}$$

for some sequence $b_n \to +\infty$ $(n \to \infty)$. But we can prove such a result.

Theorem 2. *Let $t \in \mathcal{T}'$. Let there be given an arbitrary sequence $\varepsilon_n \downarrow 0$. Then there exists a function $f \in \mathcal{S}[t]$ such that*

$$f(b_n) \ge \frac{(t(b_n))^{1-\varepsilon_n}}{b_n} \tag{7}$$

for some sequence $b_n \to +\infty$.

In particular, setting $t(r) = r^\rho$, $\rho > 1$, we obtain from Theorems 1 and 2

Corollary 1. *Let $\rho > 1$. If $f \in \mathcal{S}_\rho$, then the following estimate holds*

$$f(x) = o\left(|x|^{\rho-1}\right), \quad x \to \pm\infty .$$

For any sequence $\varepsilon_n \downarrow 0$, there exists a function $f \in \mathcal{S}_\rho$ such that

$$f(b_n) > b_n^{\rho-1-\varepsilon_n}$$

for some sequence of real numbers $b_n \to +\infty$ $(n \to \infty)$.

Let us formulate also the following evident corollary from Theorems 1 and 2.

Corollary 2. *Let $t \in \mathcal{T}$. If $f \in \mathcal{S}[t]$, then the following inequality holds*

$$\limsup_{x\to\pm\infty} \frac{\log f(x)}{\log\left(t(x)/|x|\right)} \le 1\,.$$

Let $t \in \mathcal{T}'$. Then there exists $f \in \mathcal{S}[t]$ such that

$$\limsup_{x\to\pm\infty} \frac{\log f(x)}{\log\left(t(x)/|x|\right)} = 1\,.$$

R e m a r k 2. We shall see from the proof of Theorem 1 that the condition $f(x) \ge 0$, $-\infty < x < +\infty$, may be omitted in this theorem.

The proof of Theorems 1 and 2 will be given in the next section.

2. Proof of results

P r o o f o f T h e o r e m 1. Let $t \in \mathcal{T}$ and $f \in \mathcal{S}[t]$. We prove that (6) is true. Let us consider the function

$$F(z) := \int\limits_0^z f(\zeta)\,d\zeta - \int\limits_0^{+\infty} f(x)\,dx\,. \tag{8}$$

Evidently, F is an entire function, the estimate

$$M(r,F) \le rM(r,f) + \text{const} < \exp\left((2t(r)\right) \tag{9}$$

holds for sufficiently large r, and $F(x) \to 0$ as $x \to +\infty$. We shall estimate $|F(z)|$ for z close to the real positive ray. Let $D_R := \{z : \operatorname{Im} z > 0, |z-R| < R/2\}$ be the half-disk and let

$$\omega_R(z) = \frac{2}{\pi} \arg \frac{z-3R/2}{z-R/2} - 1, \quad z \in D_R\,,$$

be the harmonic measure of the segment $[R/2, 3R/2]$ with respect to D_R. It is not difficult to show that

$$\omega_R(R+iy) = 1 - \frac{8y}{\pi R}\left(1 + \beta\left(\frac{y}{R}\right)\right)\,, \tag{10}$$

where $\beta(\tau) \to 0$ as $\tau \to 0$. Let $\varepsilon > 0$ be a small number. Then

$$|F(x)| < \varepsilon, \quad x \in [R/2, 3R/2]\,,$$

for sufficiently large R. With the aid of (9) and (4) we obtain for $z \in \partial D_R \setminus [R/2, 3R/2]$ that

$$|F(z)| \le M(3R/2, F) \le \exp\left(2t(3R/2)\right) \le \exp\left(B_1 t(R)\right) ,$$

where B_1 is a positive constant. Therefore, using the theorem on two constants we have

$$|F(R+iy)| \le \varepsilon^{1 - \frac{8y}{\pi R}(1+\beta(\frac{y}{R}))} \exp\left(B_1 t(R) \frac{8y}{\pi R}\left(1+\beta(\frac{y}{R})\right)\right) \qquad (11)$$

for $0 \le y \le R$. The same estimate holds for $-R \le y \le 0$. We must only replace y by $|y|$ in the right-hand side of (11). This yields that

$$|F(R+iy)| \le \sqrt{\varepsilon} \exp\left(B_2 \frac{t(R)}{R}|y|\right) ,$$

if $|y| \le 1$ and R is sufficiently large. Therefore, if we introduce the domain G as

$$G := \left\{ z = x + iy : x > R_0, |y| < \frac{x}{B_2 t(x)} \right\} ,$$

then the inequality

$$|F(z)| \le \sqrt{\varepsilon}\, e, \quad z \in G\,, \qquad (12)$$

holds. Let us denote by C_x the disk $\left\{\zeta : |\zeta - x| \le \frac{\gamma}{B_2} \frac{x}{t(x)}\right\}$, where γ is a small positive constant and B_2 is the same constant as that in the definition of G. It is easy to see that if γ is sufficiently small, then, by (3) and (4), $C_x \subset G$ for x large enough. Therefore, by the Cauchy formula and by (12) we obtain

$$|f(x)| = |F'(x)| = \left| \frac{1}{2\pi i} \int\limits_{\partial C_x} \frac{F(\zeta)}{(\zeta - x)^2}\, d\zeta \right| \le B_3 \sqrt{\varepsilon}\, \frac{t(x)}{x} .$$

This proves (6) for $x \to +\infty$. The proof for $x \to -\infty$ is analogous. ■

In the course of the proof of Theorem 2 we shall use some methods of paper [4]. The following three lemmas are needed for the sequel.

Lemma 1. *For every number β, $1 < \beta < 2$, there exists an entire even function $\theta_\beta(z)$ of order β and normal type such that:*

1) *the indicator of θ_β equals*

$$h_{\theta_\beta}(\varphi) = A_\beta \cos\beta\,(\varphi - \pi/2)\,, \quad 0 \le \varphi \le \pi, \qquad (13)$$

where A_β is positive constant,

2) $\theta_\beta(x) \geq 0$ *for* $x \in \mathbf{R}$,

3) $\theta_\beta(0) = 1$ *and* $\int\limits_{-\infty}^{\infty} \theta_\beta(x)\,dx = 1$.

P r o o f. We set

$$\theta_{1,\beta}(w) = \prod_{n=1}^{\infty} \left(1 - \frac{w}{n^{2/\beta}}\right).$$

Since $\beta < 2$, it follows (see, e.g., [6], 8.6.4) that

$$\log \left|\theta_{1,\beta}\left(Re^{i\psi}\right)\right| \sim \frac{\pi}{\sin \pi\beta/2} \cos \frac{\beta}{2}(\psi - \pi) R^{\beta/2}, \quad R \to +\infty, \tag{14}$$

for all $\psi \neq 0$. Let us set

$$\theta_{2,\beta}(z) := \left(\theta_{1,\beta}(z^2)\right)^2.$$

Taking into account (14), we obtain

$$h_{\theta_{2,\beta}}(\varphi) = \frac{2\pi}{\sin \pi\beta/2} \cos \beta(\varphi - \pi/2), \quad 0 \leq \varphi \leq \pi.$$

Clearly, $\theta_{2,\beta}(x) \geq 0$ for $x \in \mathbf{R}$ and $\theta_{2,\beta}(0) = 1$. Since $h_{\theta_{2,\beta}}(0) = h_{\theta_{2,\beta}}(\pi) < 0$ we have $\int_{-\infty}^{\infty} \theta_{2,\beta}(x)\,dx < \infty$. We put

$$\theta_\beta(z) := \theta_{2\beta}(c_\beta z), \quad c_\beta = \int\limits_{-\infty}^{+\infty} \theta_{2\beta}(x)\,dx.$$

It is obvious that the conditions 2) and 3) of Lemma 1 hold and indicator of θ_β is equal to (13) with $A_\beta = 2\pi c_\beta^{\beta/2} / \sin(\pi\beta/2)$. ■

We need an estimate from above of $\max\{|\theta_\beta(z - b)| : |z| \leq R\}$ for any $b > 0$ and $R > 0$.

Lemma 2. *Let θ_β be the function constructed in Lemma 1. Then, for every β, $1 < \beta < 2$, there exists a number $\delta_\beta > 0$ such that for all $b \geq 0$ the following inequalities hold*

$$\max_{|z| \leq R} |\theta_\beta(z - b)| \leq \begin{cases} C_\beta \exp\left(-d_\beta b^\beta\right), & \text{if} \quad 0 \leq R \leq \delta_\beta b, \\ C_\beta \exp\left(D_\beta R^\beta\right), & \text{if} \quad R > \delta_\beta b, \end{cases} \tag{15}$$

where $C_\beta, d_\beta, D_\beta$ are positive constants which are dependent only on β but independent of b and R.

P r o o f. We have (see (13)) $h_{\theta_\beta}(0) = h_{\theta_\beta}(\pi) < 0$ for every $\beta \in (1, 2)$. Therefore, we can choose a number η_β so that $0 < \eta_\beta < |h_{\theta_\beta}(0)|$. By definition, we put for $\varphi \in [0, \pi]$

$$H_\beta(\varphi) := h_{\theta_\beta}(\varphi) + \eta_\beta = A_\beta \cos\beta\,(\varphi - \pi/2) + \eta_\beta\,,$$
$$H_\beta(\pi + \varphi) = H_\beta(\varphi)\,.$$

From Lemma 1 it follows (see, e.g., [5], p. 71) that for all $r \geq 0$ and $\varphi \in [0, 2\pi]$ the inequality

$$|\theta_\beta(re^{i\varphi})| \leq C_\beta \exp(H_\beta(\varphi) r^\beta)\,, \tag{16}$$

holds, where C_β is a positive constant. Let us show that for all $z = re^{i\varphi} = x + iy$ the inequality

$$H_\beta(\varphi) r^\beta \leq -l_\beta |x|^\beta + L_\beta |y|^\beta\,, \tag{17}$$

is satisfied, where l_β and L_β are some positive constants. Let us denote by ψ_β the zero of the function $H_\beta(\varphi)$ in interval $(0, \pi/2)$. It is sufficient to prove (17) in two cases: 1) $|\varphi| \leq \psi_\beta/2$, 2) $|\varphi - \pi/2| \leq \pi/2 - \psi_\beta/2$. In the first case, we have

$$H_\beta(\varphi) r^\beta \leq H_\beta(\psi_\beta/2) r^\beta \leq -l_\beta x^\beta\,, \tag{18}$$

where $l_\beta = |H_\beta(\psi_\beta/2)|$. In the second case, the inequality $r \leq y/\sin(\psi_\beta/2)$ holds, so we have

$$H_\beta(\varphi) r^\beta \leq H_\beta(\pi/2) r^\beta \leq L_{1,\beta} y^\beta\,,$$

where $L_{1,\beta} = H_\beta(\pi/2)\,(\sin(\psi_\beta/2))^{-\beta}$. Since $|x| \leq y/\tan(\psi_\beta/2)$ in the case under consideration, we can write

$$H_\beta(\varphi) r^\beta \leq L_{1,\beta} y^\beta + l_\beta |x|^\beta - l_\beta |x|^\beta \leq L_\beta y^\beta - l_\beta |x|^\beta\,, \tag{19}$$

where $L_\beta = L_{1,\beta} + l_\beta\,(\tan(\psi_\beta/2))^{-\beta}$ and l_β is the constant from (18). According to (18) the estimate (19) is true also in the case $|\varphi| \leq \psi_\beta/2$. Thus (17) is proved.

We proceed to the proof of (15). Let l_β and L_β be constants from (17). Let us take a small number δ_β, $0 < \delta_\beta < 1$, such that

$$L_\beta(\tan(\arcsin\delta_\beta))^\beta \leq l_\beta/2, \qquad (1 - \delta_\beta)^\beta \geq 1/2\,. \tag{20}$$

Let $b > 0$ and $0 \leq R \leq \delta_\beta b$. For any $z = x + iy$ such that $|z| \leq R$, we have

$$\begin{aligned} |\theta_\beta(z - b)| &\leq C_\beta \exp\left(L_\beta |y|^\beta - l_\beta (b - x)^\beta\right) \\ &\leq C_\beta \exp\left(\left(L_\beta(\tan(\arcsin\delta_\beta))^\beta - l_\beta\right)(b - x)^\beta\right) \\ &\leq C_\beta \exp\left(-\frac{l_\beta}{2}(b - \delta_\beta b)^\beta\right) \leq C_\beta \exp\left(-\frac{l_\beta}{4} b^\beta\right)\,. \end{aligned}$$

Here we have used (17) in the first inequality, estimate $|y| \le (b-x)\tan(\arcsin \delta_\beta)$ — in the second inequality, the first condition from (20) and estimate $|x| \le R \le \delta_\beta b$ — in the third inequality, the second condition from (20) — in the fourth inequality. This proves the first inequality in (15) with $d_\beta = l_\beta/4$.

Now, let $R > \delta_\beta b$. For any $z = x+iy$, $|z| \le R$, it follows from (16) and (17) that

$$|\theta_\beta(z-b)| \le C_\beta \exp\left\{L_\beta |y|^\beta - l_\beta |x-b|^\beta\right\} \le C_\beta \exp\left(L_\beta R^\beta\right),$$

This proves the second inequality in (15) with $D_\beta = L_\beta$. ∎

Let $\{\beta_n\}_{n=1}^\infty$ be a sequence of numbers from the interval $(1,2)$, $\theta_{\beta_n}(z)$ be the function constructed in Lemma 1, $\{a_n\}$, $\{b_n\}$, $\{q_n\}$ be sequences of positive numbers. Let us consider the function

$$f(z) := \sum_{n=1}^\infty a_n \theta_{\beta_n}\left(q_n(z-b_n)\right). \tag{21}$$

According to (15) the series in the right-hand side of (21) converges uniformly on every disk of the complex plane if $\{b_n\}$, $\{q_n\}$ tend to infinity sufficiently rapidly. The following easy lemma gives a condition of integrability on the real line for the function of the form (21).

Lemma 3. *Let f be a function of the form* (21)*. Then $f \in L^1(\mathbf{R})$ if and only if*

$$\sum_{n=1}^\infty \frac{a_n}{q_n} < \infty. \tag{22}$$

P r o o f. Since all terms of the series in right-hand side of (21) are nonnegative and according to the equality $\int_{-\infty}^{\infty} \theta_\beta(x)\,dx = 1$, we have

$$\int_{-\infty}^{+\infty} f(x)\,dx = \sum_{n=1}^\infty a_n \int_{-\infty}^{+\infty} \theta_{\beta_n}(q_n(x-b_n))\,dx = \sum_{n=1}^\infty \frac{a_n}{q_n}.$$

This gives the desired assertion. ∎

P r o o f o f T h e o r e m 2. Let $t \in \mathcal{T}'$ and $a > 1$ be a number such that $t(r)/r^a \uparrow$ on $[r_0, \infty)$, $r_0 > 0$. We fix a sequence $\{\beta_n\}_{n=1}^\infty$ so that two following conditions are valid for all n: 1) $1 < \beta_n < \min(2,a)$, 2) $1/\beta_n > 1 - \varepsilon_n/2$, where $\{\varepsilon_n\}$ is the same sequence of positive numbers as in (7). Let θ_{β_n} be the function constructed in Lemma 1. The desired function f we shall construct in the form (21), where the sequences of positive numbers $\{a_n\}$, $\{b_n\}$, $\{q_n\}$ are to be determined. The sequences $\{b_n\}$ and $\{q_n\}$ will tend to infinity very rapidly. The sequence $\{a_n\}$ must possess the property

$$a_n \le b_n^L, \quad n \ge n_0\,, \tag{23}$$

for some $L > 0$. Indeed, because $\theta_\beta(x)$ and a_n are nonnegative and according to the condition $\theta_\beta(0) = 1$, we have $a_n \le f(b_n)$. Since $t \in \mathcal{T}' \subset \mathcal{T}$ it follows from (4) that $t(r) \le r^p$, $r \ge r_0$, for some $p > 0$. Thus we obtain (23) from Theorem 1.

For brevity we introduce the following notation (see the right-hand side of (15)):

$$m_\beta(R,b) := \begin{cases} -d_\beta b^\beta & \text{for} \quad 0 \le R \le \delta_\beta b, \\ D_\beta R^\beta & \text{for} \quad R > \delta_\beta b\,. \end{cases} \tag{24}$$

Lemma 2 asserts that

$$\max_{|z|\le R} |\theta_\beta(z-b)| \le C_\beta \exp\left(m_\beta(R,b)\right)\,. \tag{25}$$

Using (25), we get

$$M(R,f) \le \sum_{n=1}^{\infty} a_n \max_{|z|\le R} |\theta_{\beta_n}(q_n(z-b_n))| \le \sum_{n=1}^{\infty} a_n C_{\beta_n} \exp(m_{\beta_n}(q_n R, q_n b_n))$$

for every $R \ge 0$. We shall show that the sequences $\{a_n\}$, $\{b_n\}$, $\{q_n\}$ can be chosen so that the inequality

$$a_n C_{\beta_n} \exp(m_{\beta_n}(q_n R, q_n b_n)) \le 2^{-n} \exp(t(R)) \tag{26}$$

is valid for any $n = 1, 2, \dots$ and $R \ge 0$. From (26) we shall get the convergence of the series in the right-hand side of (21) and the estimate $M(R,f) \le \exp(t(R))$.

We shall denote later by $C_j(n)$ the constants which are uniquely determined by the choice of β_n and independent of b_n and q_n. We shall denote by B_j the constants which are independent of n. From now on we shall write also $\delta(n)$, $d(n)$, $D(n)$ instead of δ_{β_n}, d_{β_n}, D_{β_n}, respectively (see (15)).

Taking into account (23), we see that (26) is a consequence of the following inequality:

$$B_1 \log b_n + C_1(n) + m_{\beta_n}(q_n R, q_n b_n) \le t(R), \quad R > 0\,, \tag{27}$$

where $C_1(n) = \log C_{\beta_n} + n \log 2$. We consider two cases: 1) $0 \le R \le \delta(n) b_n$, 2) $R > \delta(n) b_n$. In the first case, we see by (24) that (27) is equivalent to the inequality:

$$B_1 \log b_n + C_1(n) - d(n) b_n^{\beta_n} q_n^{\beta_n} \le t(R)\,. \tag{28}$$

Since $t(R) \geq 0$ and $q_n \geq 1$, (28) follows from the inequality

$$B_1 \log b_n + C_1(n) \leq d(n) b_n^{\beta_n} ,$$

which is true if b_n is sufficiently large.

In the second case, (27) is equivalent to the inequality

$$B_1 \log b_n + C_1(n) + D(n) R^{\beta_n} q_n^{\beta_n} \leq t(R), \quad R \geq \delta(n) b_n . \tag{29}$$

Using the evident inequality

$$x^\gamma + y^\gamma + z^\gamma < 3(x+y+z)^\gamma, \quad x, y, z, \gamma > 0 ,$$

we shall estimate from above the left-hand side of (29). Since $3 < 3^{\beta_n}$ and $(\log b_n)^{1/\beta_n} < \log b_n$, it follows that the left-hand side of (29) does not exceed

$$\begin{aligned} &3\{(B_1 \log b_n)^{1/\beta_n} + (C_1(n))^{1/\beta_n} + (D(n))^{1/\beta_n} R q_n\}^{\beta_n} \\ &\leq \{C_2(n) \log b_n + C_3(n) + D_1(n) R q_n\}^{\beta_n} , \end{aligned}$$

where $C_2(n) = 3B_1^{1/\beta_n}$, $C_3(n) = 3(C_1(n))^{1/\beta_n}$, $D_1(n) = 3(D(n))^{1/\beta_n}$. This yields that (29) follows from the inequality

$$C_2(n) \log b_n + C_3(n) + D_1(n) R q_n \leq (t(R))^{1/\beta_n}, \quad R > \delta(n) b_n . \tag{30}$$

Obviously, (30) is a consequence of the following two inequalities:

$$C_2(n) \log b_n + C_3(n) < \frac{1}{2} t(R)^{1/\beta_n} \quad \text{for} \quad R > \delta(n) b_n \tag{31}$$

and

$$C_3(n) R q_n \leq \frac{1}{2} (t(R))^{1/\beta_n} \quad \text{for} \quad R > \delta(n) b_n . \tag{32}$$

Let us estimate from below the right-hand side of (31). Since $t \in \mathcal{T}'$, then (see (5)) $t(R) > R^a$ for $R \geq r_0$. Since the sequence $\delta(n)$ is already fixed, the inequality $\delta(n) b_n > r_0$ will be valid, if we take b_n sufficiently large. Therefore, for any $R \geq \delta(n) b_n$ the right-hand side of (31) is greater than $1/2(\delta(n))^{a/\beta_n} b_n^{a/\beta_n}$. Thus (31) is a consequence of the inequality

$$C_2(n) \log b_n + C_3(n) < C_4(n) b_n^{a/\beta_n} ,$$

where $C_4(n) = 1/2(\delta(n))^{1/\beta_n}$. This inequality is true if b_n is large enough.

Now, we consider (32). By (5) we have for $\beta_n < a$ that $r^{-1}(t(r))^{1/\beta_n} \uparrow$ on $[r_0, \infty)$. If b_n is large enough, then $\delta(n) b_n > r_0$. Therefore, (32) is equivalent to the following inequality:

$$q_n \leq \frac{1}{2C_3(n)} \frac{(t(\delta(n) b_n))^{1/\beta_n}}{\delta(n) b_n} . \tag{33}$$

We set with agreement with (33)

$$q_n := C_4(n)\frac{(t(\delta(n)b_n))^{1/\beta_n}}{b_n}, \tag{34}$$

where $C_4(n) = (2C_3(n)\delta(n))^{-1}$. Then (32) is valid. Therefore, if the sequence $\{a_n\}$ is such that (23) is true for some L, then the function f will be entire and the estimate $M(r, f) \le \exp(t(r))$ will be valid.

Let us choose $\{a_n\}$. We set

$$a_n := \frac{(t(b_n))^{1/\beta_n - \varepsilon_n/2}}{b_n}. \tag{35}$$

Since $t \in \mathcal{T}'$, we see that condition (23) is satisfied. Let us check that the condition (22) is also satisfied. It follows from (34) and (35) that

$$\sum_{n=1}^{\infty} \frac{a_n}{q_n} = \sum_{n=1}^{\infty} \frac{1}{C_4(n)} \left(\frac{t(b_n)}{t(\delta(n)b_n)}\right)^{1/\beta_n} \frac{1}{(t(b_n))^{\varepsilon_n/2}}. \tag{36}$$

Using (4), we get $t(b_n)/t(\delta(n)b_n) \le K(1/\delta(n)) + 1$, if b_n is sufficiently large. Therefore, we see from (36) that

$$\sum_{n=1}^{\infty} \frac{a_n}{q_n} \le \sum_{n=1}^{\infty} C_5(n)\frac{1}{(t(b_n))^{\varepsilon_n/2}} < \infty,$$

if the sequence $\{b_n\}$ tends to infinity sufficiently rapidly (we can take here $C_5(n) = C_4^{-1}(n)\left(K(\delta^{-1}(n)) + 1\right)^{1/\beta_n}$). From (35) and condition $\beta_n^{-1} > 1 - \varepsilon_n/2$ we obtain

$$f(b_n) \ge a_n \ge \frac{(t(b_n))^{1-\varepsilon_n}}{b_n},$$

which completes the proof of Theorem 2. ■

Acknowledgement. I am deeply grateful to Prof. I.V. Ostrovskii for suggesting this problem and for pointing out simplifications in the proofs of Lemmas 1 and 2.

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Оценки сверху на вещественной оси целых функций из $L^1(R)$

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Пусть $\mathcal{S}_\rho$ — множество целых функций порядка ρ и нормального типа таких, что $f(x) \geq 0$ для $x \in \mathbf{R}$ и $f \in L^1(\mathbf{R})$. В статье доказано: 1) если $f \in \mathcal{S}_\rho$, то $f(x) = o(|x|^{\rho-1})$, $x \to \pm\infty$, 2) для любой последовательности $\varepsilon_n \downarrow 0$ существуют функция $f \in \mathcal{S}_\rho$ и вещественная последовательность $b_n \to +\infty$ такие, что $f(b_n) > b_n^{\rho-1-\varepsilon_n}$. Приведено обобщение этого результата для более общих шкал роста.

Оцінки зверху на дійсній осі цілих функцій з $L^1(R)$

О.І. Ільїнський

Нехай $\mathcal{S}_\rho$ — множина цілих функцій порядку ρ та нормального типу таких, що $f(x) \geq 0$, $x \in \mathbf{R}$, та $f \in L^1(\mathbf{R})$. В статті доведено: 1) якщо $f \in \mathcal{S}_\rho$, тоді $f(x) = o(|x|^{\rho-1})$, $x \to \pm\infty$, 2) для усякої послідовності $\varepsilon_n \downarrow 0$ існують функція $f \in \mathcal{S}_\rho$ та дійсна послідовність $b_n \to +\infty$ такі, що $f(b_n) > b_n^{\rho-1-\varepsilon_n}$. Наведено узагальнення цього результату у випадку більш загальних шкал зростання.