Matematicheskaya fizika, analiz, geometriya
2000, v. 7, No. 2, p. 184–195

Averaging technique in the periodic decomposition problem

V.M. Kadets and B.M. Shumyatskiy

Department of Mathematics and Mechanics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail: vkadets@kadets.kharkov.ua

Received May 29, 1998
Communicated by I.V. Ostrovskii

Let T_1, T_2 be a pair of commuting isometries in a Banach space X. Generalizing results of M. Laczkovich and Sz. Revesz we prove that in many cases element x of $\mathrm{Ker}[(\mathrm{I}-\mathrm{T}_1)(\mathrm{I}-\mathrm{T}_2)]$ can be decomposed as a sum $x_1 + x_2$ where $x_k \in \mathrm{Ker}(\mathrm{I}-\mathrm{T_k})$, $k = 1, 2$. Moreover, using an averaging technique we prove the existence of linear operators perfoming such a representation. The results are applicable for decomposition of functions into a sum of periodic ones.

1. Introduction

Let a function $f\colon \mathbb{R} \to \mathbb{R}$ be a sum of finitely many functions $\{f_k\}_1^n$ where f_k is a_k-periodic, $\{a_k\}_1^n \subset \mathbb{R}$. Let Δ_a denote the difference operator,

$$(\Delta_a g)(t) = g(t + a) - g(t) \quad (a \in \mathbb{R}, g : \mathbb{R} \to \mathbb{R}).$$

Then $\Delta_{a_k} f_k = 0$ for every k and $\Delta_{a_1} \cdots \Delta_{a_n} f = 0$ because the difference operators commute. By definition, a class $\mathcal{F}$ of functions has the decomposition property if for every $f \in \mathcal{F}$ and for every $\{a_k\}_1^n \subset \mathbb{R}$ the condition $\Delta_{a_1} \cdots \Delta_{a_n} f = 0$ implies that there are functions $\{f_k\}_1^n \subset \mathcal{F}$ such that $f = f_1 + \cdots + f_n$ with f_k be a_k-periodic.

M. Laczkovich and Sz. Revesz tested some function classes for the decomposition property in [1, 2]. In particular, they proved in [1] that the space $BC(\mathbb{R})$

Mathematics Subject Classififcations (1991): 47A50, 46B20.

The work of the first author was supported in part by INTAS. Grant No. 93-1376.

of bounded continuous real functions on $\mathbb{R}$ possesses the decomposition property. Also a generalized problem for some other operators in Banach and topological vector spaces was treated. The following theorem has been established: if $\{T_k\}_1^n$ is a collection of w^*-continuous commuting power bounded operators in a dual Banach space then

$$\mathrm{Ker}[(I-T_1)\cdots(I-T_n)] = \mathrm{Ker}(I-T_1)+\cdots+\mathrm{Ker}(I-T_n).$$

(Recall that operator T is called power bounded if $\sup\{\|T^k\| : k\in\mathbb{N}\}$ is finite).

In our paper we present a different "averaging" technique which gives both new proofs of the known results and some new results. An advantage of this technique is that it gives linear operators performing the decomposition.

The tool which we use for this is the notion of Banach limit (see, for example, [3], Chapter II). We fix a Banach limit "Lim", which (by definition) assigns to every bounded sequence $(x_n)_1^\infty$ of complex numbers a number $\operatorname*{Lim}_n(x_n)_1^\infty$ and possesses the usual properties of linearity, positivity and invariance under shifts, i.e.,

$$\operatorname*{Lim}_n(x_{n+1})_{n=1}^\infty = \operatorname*{Lim}_n(x_n)_{n=1}^\infty.$$

The idea to apply the Banach limit to the periodic decomposition problem appeared first in [4]. In our paper we will use a bit more general variant of the notion – not only for numerical sequences, but for sequences in a Banach space.

Let E be a Banach space. For a bounded sequence $(x_n^*)_1^\infty$ in the dual space E^* we define a Banach limit $\mathrm{M}_n(x_n^*)_1^\infty\in E^*$ as follows:

$$(\mathrm{M}_n(x_n^*)_1^\infty, x) = \operatorname*{Lim}_n((x_n^*, x))_1^\infty.$$

Evidently M_n is a well-defined bounded linear and shift-invariant map from $\ell_\infty(E^*)$ to E^*. In the sequel we shall use the concept for $E=X^*$, $E^*=X^{**}$.

Definition 1.1. *We say that a family of commuting linear continuous operators $\{R_k\}_1^n$ in a topological vector space X has the decomposition property (DePr) if*

$$\mathrm{Ker}(R_1R_2\cdots R_n)\subset \mathrm{Ker}R_1+\mathrm{Ker}R_2+\cdots+\mathrm{Ker}R_n$$

(the inverse inclusion is evident).

Definition 1.2. *The family $\{R_k\}_1^n$ has the linear decomposition property (LDePr) if there are linear operators $G_k\colon X\to \mathrm{Ker}R_k$, $k=1,2,...,n$ such that*

$$G_1x+G_2x+\cdots+G_nx = x \tag{1}$$

for every $x\in\mathrm{Ker}(R_1R_2\cdots R_n)$.

Definition 1.3. *We say that a Banach space X possesses the DePr (LDePr) if for every family $\{T_k\}_1^n$ of commuting power bounded operators in X, the family $\{R_k\}_1^n$, $R_k = I - T_k$, has the DePr (LDePr).*

We prove that every reflexive space has the linear decomposition property. On the other hand, separable spaces containing subspaces, isomorphic to c_0, does not possess even the original decomposition property. The following questions look interesting but we don't know the answers:

P r o b l e m. What are the characterizations of DePr and LDePr? In particular, does the quasireflexive James space J have one of the properties? Is it true that DePr and LDePr of a space are the same property? Do the properties for pairs of operators imply the properties for finite families of operators?

2. Positive results

Lemma 1 *(the key lemma). Let $\{R_k\}_1^n$ be a family of commuting linear operators in a linear topological space X. Denote $Y = \mathrm{Ker} R_1 R_2 \cdots R_n$. Let there exist linear operators $P_k\colon X \to X$ commuting with all of R_j such that $P_k(Y) = \mathrm{Ker} R_k$, $P_k x = x$ for every $x \in \mathrm{Ker} R_k$ (remark that P_k may be non-commuting one with another). Then the family $\{R_k\}_1^n$ has the LDePr.*

P r o o f. We will prove that the operators $G_k = P_k(I - P_{k-1})(I - P_{k-2}) \cdots (I - P_1)$ satisfy the conditions. Evidently G_k are linear and $G_k(Y) \subset P_k(Y)$, so $G_k(Y) \subset \mathrm{Ker} R_k$ and we need only to check (1). Let $x \in Y$, i.e.,

$$R_1 R_2 \cdots R_n x = 0.$$

Then $(R_2 \cdots R_n)x \in \mathrm{Ker} R_1$ and

$$(I - P_1)(R_2 \cdots R_n)x = 0.$$

By commutativity we have

$$(R_2 \cdots R_n)(I - P_1)x = 0,$$

and by the same reasons as before we get

$$(R_3 \cdots R_n)(I - P_2)(I - P_1)x = 0.$$

Proceeding we receive

$$(I - P_n)(I - P_{n-1}) \cdots (I - P_1)x = 0,$$

which after opening the parentheses gives the decomposition (1).

Theorem 1. *Let X be a Banach space complemented in its second dual and let $P_X : X^{**} \to X$ be the projector. Let also $\{T_k\}_1^n$ be a family of commuting power bounded operators such that T_k^{**} commute with P_X. Then the family $\{R_k\}_1^n$, $R_k = I - T_k$, has the LDePr.*

P r o o f. To apply Lemma 1 we construct operators P_k:

$$P_k x = P_X \operatorname{M}_m\{(T_k^{**})^m \pi x\}, x \in X,$$

where $\pi : X \to X^{**}$ is the canonical embedding, M_m is the Banach limit in X^{**}. Every P_k is clearly linear and bounded, acts in X and leaves elements of the X_k unchanged. Prove that P_k maps X into X_k, i.e., $T_k P_k = P_k$:

$$T_k P_k x = T_k P_X \operatorname{M}_m\{(T_k^{**})^m \pi x\} = P_X T_k^{**} \operatorname{M}_m\{(T_k^{**})^m \pi x\}$$

and

$$P_k x = P_X \operatorname{M}_m\{(T_k^{**})^m \pi x\}.$$

We are to show that expressions under P_X are equal:

$$(T_k^{**} \operatorname{M}_m\{(T_k^{**})^m \pi x\}, x^*) = (\operatorname{M}_m\{(T_k^{**})^m \pi x\}, T_k^* x^*).$$

By the definition of M this is equal to

$$\operatorname{M}_m\{((T_k^{**})^m \pi x, T_k^* x^*)\} = \operatorname{M}_m\{((T_k^{**})^{m+1} \pi x, x^*)\},$$

by the invariance of M this is equal to

$$\operatorname{M}_m\{((T_k^{**})^m \pi x, x^*)\} = (\operatorname{M}_m\{(T_k^{**})^m \pi x\}, x^*).$$

Therefore P_k is a projector from X on X_k.

Now prove that P_k commutes with T_j, $1 \le k,\ j \le n$:

$$P_k T_j x = P_X \operatorname{M}_m\{(T_k^{**})^m T_j^{**} x\} = P_X \operatorname{M}_m\{T_j^{**} (T_k^{**})^m x\}$$

and

$$T_j P_k x = T_j P_X \operatorname{M}_m\{(T_k^{**})^m x\} = P_X T_j^{**} \operatorname{M}_m\{(T_k^{**})^m x\}.$$

The same steps as above show the equality between expressions under P_X.

R e m a r k 1. The theorem above covers, in particular, the case of a dual space X (X equals some Y^*) with w^*-continuous operators T_k, because operator $P_X = \pi^*$, where π: $X \to X^{**}$ is the canonical embedding, gives the projector of X^{**} onto X we need.

Corollary 1. *Every reflexive Banach space X has LDePr.*

Theorem 1 is applicable in other cases (for example, L_1 is complemented in its bidual), but the same averaging idea as in the proof of Theorem 1 is applicable for some operators even when the conditions of Theorem 1 are not satisfied. Of course, for such applications we need some other restrictions on T_k.

Theorem 2. *Let $\{T_k\}_1^n$ be a commuting family of operators in a Banach space X, $R_k = I - T_k$ and every element of the* $\operatorname{Ker} \prod\limits_{k=1}^{n} R_k$ *has relatively weakly compact orbits under each of T_k. Then the family $\{R_k\}_1^n$ has the DePr. If orbits of all elements of X are relatively weakly compact then the family $\{R_k\}_1^n$ has the LDePr.*

P r o o f. Define operators P_k in X^{**} by the formula

$$P_k x^{**} = \mathrm{M}_m\{(T_k^{**})^m x^{**}\}, \quad x^{**} \in X^{**}.$$

By the same reasons as in the proof of Theorem 1 we may apply lemma for X^{**}, R_k^{**} and P_k and obtain that for every element $x^{**} \in \operatorname{Ker} \prod\limits_{k=1}^{n} R_k^{**}$ the operators $G_k = P_k(I - P_{k-1})(I - P_{k-2}) \cdots (I - P_1)$ perform the decomposition

$$x^{**} = G_1 x^{**} + G_2 x^{**} + \cdots + G_1 x^{**}. \tag{2}$$

Consider X as a subspace of X^{**}. Apply P_k for an element $x \in X$. Evidently $P_k x$ belongs to the w^*-closure in X^{**} of the set $\operatorname{conv}\{T_k^m x,\ m \in \mathbb{N}\}$. This orbit being relatively weakly compact, $P_k x$ in fact belongs to X. Thus we have that for $x^{**} = x \in \operatorname{Ker} \prod\limits_{k=1}^{n} R_k$ all the summands in (2) belong to X which gives the DePr. If the relative weak compactness of orbits is known for every $x \in X$ we get that G_k's map X into X, which proves the LDePr.

The next results show how one can replace the topological conditions by some algebraic ones to obtain the LDePr.

Lemma 2. *Let X be a topological vector space and let $T\colon X \to X$ be a linear operator with bounded orbits. Then* $\operatorname{Ker}[(I - T)^n] = \operatorname{Ker}(I - T)$, $n \in \mathbb{N}$.

P r o o f. It is sufficient to prove the lemma for the $n = 2$. Let $(I - T)^2 x = 0$ but $y = (I - T)x \neq 0$. Then $y = Ty$, $Tx = x - y$, $T^m x = x - my$. But this is a contradiction because the left side is bounded while the right is not.

Definition 2.1. *Operators T, $S\colon X \to X$ are called commensurable if there exist n, $m \in \mathbb{N}$ such that $T^n = S^m$.*

Theorem 3. *Let operators T_k, $1 \le k \le n$ in a linear topological space X be commuting, pairwise commensurable and have bounded orbits. Then the family $\{R_k\}_1^n$, $R_k = I - T_k$, has the LDePr.*

P r o o f. It is not difficult to see that there exist T: $X \to X$ and $\{l_k\}_1^n \subset \mathbb{N}$ such that $T = T_k^{l_k}$ for all k. Explicitly by the conditions there exists $\{m_k^j\}_{k,j=1}^n \subset \mathbb{N}$ such that $T_k^{m_k^j} = T_j^{m_j^k}$, $1 \le k, j \le n$, choose $l_k = \prod\limits_{j=1}^{k-1} m_{j+1}^j \cdot \prod\limits_{j=k}^{n-1} m_j^{j+1}$. Denote the least common multiple of $\{l_k\}_1^n$ as L.

Let $x \in \operatorname{Ker} \prod\limits_{k=1}^n R_k$, then the above means that $x \in \operatorname{Ker}(I-T)^n = \operatorname{Ker}(I-T)$ (T obviously has bounded orbits). Consider operators $P_k = \frac{1}{L/l_k} \sum\limits_{j=1}^{L/l_k} T_k^j = \frac{1}{L} \sum\limits_{j=1}^{L} T_k^j$. Their restrictions on $\operatorname{Ker}(I-T)$ are clearly projectors on $P_k(\operatorname{Ker}(I-T))$ which equals $\operatorname{Ker}(I-T_k)$, they commute with $\{T_k\}_1^n$ and by Lemma 1 this is sufficient for the LDePr.

Although the space $BC(\mathbb{R})$ of all bounded continuous real-valued functions with sup-norm is not complemented in its second dual and the orbits of shift operators are not weakly compact in general, the technique introduced above is applicable to prove a result from [1] in a simpler way.

Theorem 4 *(M. Laczkovich, Sz. Revesz). Let $\{T_k\}$, $k = 1, 2$, be the shift operators in $BC(\mathbb{R})$: $(T_k f)(x) = f(x + a_k)$. Then the pair of difference operators $R_k = I - T_k$ has the DePr.*

P r o o f. Let T_1 be an a_1-shift, T_2 be an a_2-shift. If a_1 and a_2 are commensurable numbers the previous theorem covers the case.

Consider now incommensurable a_1 and a_2. $BC(\mathbb{R})$ is a subspace of $L_\infty(\mathbb{R})$ which is a dual space to $L_1(\mathbb{R})$. The shift operators are defined on $L_\infty(\mathbb{R})$ as well as on $BC(\mathbb{R})$ and are w^*-continuous. So by the Remark 1 (or by the theorem from [2], cited in the introduction) for every function $f \in BC(\mathbb{R})$, $f \in \operatorname{Ker} R_1 R_2$, there are functions f_1, $f_2 \in L_\infty(\mathbb{R})$ such that

$$f = f_1 + f_2 \quad \text{almost everywhere}$$

and f_k is a_k-periodic, $k = 1, 2$.

We have to prove that f_1, f_2 may be chosen to be continuous. Consider everywhere defined functions

$$f_k^+(x) = \operatorname{vrai} \limsup_{t \to x} f_k(t),$$

$$f_k^-(x) = \operatorname{vrai} \liminf_{t \to x} f_k(t), \quad k = 1, 2.$$

Because f is continuous then for all $x \in \mathbb{R}$

$$f_1^+(x) + f_2^-(x) = f(x),$$
$$f_1^-(x) + f_2^+(x) = f(x). \qquad (3)$$

Subtracting we get

$$f_1^+ - f_1^- = f_2^+ - f_2^-.$$

The left hand is a_1-periodic, the right one is a_2-periodic so both sides have both periods. Let prove that $f_1^+ - f_1^-$ is a constant.

For $c \in \mathbb{R}$ consider sets

$$A_c = \{t \in \mathbb{R} : f_1^+(t) - f_1^-(t) \geq c\}.$$

f_1^+ is upper semicontinuous, f_1^- is lower semicontinuous so the difference is upper semicontinuous and A_c is a closed set. But A_c is invariant under two incommensurable shifts a_1 and a_2 so A_c can be or the empty set or the whole $\mathbb{R}$. This proves that $f_1^+ - f_1^-$ is a constant as well as $f_2^+ - f_2^-$.

This gives equations

$$f_1^+ = f_1^- + \text{const},$$
$$f_2^+ = f_2^- + \text{const},$$

where left hands are upper semicontinuous and right hands are lower semicontinuous. Therefore all the functions are continuous and (3) gives the required decomposition.

3. Limitations for the decomposition property

It is very easy to show that power boundness condition cannot be omitted even in finite dimensional spaces.

E x a m p l e 1. Let $X = \text{lin}\{e^t, e^{-t}, te^t\}$, T_1, T_2: $X \to X$, $T_1 f = f'$, $T_2 f = f''$.

X is finite dimensional, T_1 and T_2 commute but for $f(t) = te^t$

$$f \in \text{Ker}(I - T_1)(I - T_2),$$

$$f \notin \text{Ker}(I - T_1) + \text{Ker}(I - T_2).$$

The next example shows the existence of a shift-invariant Banach space of functions on $\mathbb{R}$ such that all the shifts are isometries but there is a pair of shift operators without the DePr.

Introduce first the following system of subsets in $\mathbb{N}$:

$$A_{\varepsilon,\{r_j\}_1^n} = \{k \in \mathbb{N}\colon \max_{j \leq n} |e^{ikr_j} - 1| < \varepsilon\},$$

where $n \in \mathbb{N}$, $\varepsilon > 0$, and $r_j \in \mathbb{R}$. It is not hard to see that all $A_{\varepsilon,\{r_j\}_1^n}$ are infinite. They form a base of a filter $\mathcal{F}$. Consider now the following space of functions:

$$X = \{x = \sum_{k\in\mathbb{N}} 2^{-k}(x_k^1 e^{ikt} + x_k^2 e^{i\pi kt}) : \sup_{k\in\mathbb{N}, j=1,2} |x_k^j| < \infty, \lim_{\mathcal{F}} x_k^1 = \lim_{\mathcal{F}} x_k^2\},$$

where x_k^1 and x_k^2 are sequences of reals ("coordinates" of x). Define

$$\|x\| = \sup_{k\in\mathbb{N}, j=1,2} |x_k^j| < \infty.$$

By the standard argument X is a Banach space (isomorphic to some $C(K)$ space). X is shift-invariant because $\mathcal{F}$ has been constructed to satisfy

$$\lim_{k\in\mathcal{F}} e^{irk} = 1, \quad r \in \mathbb{R}.$$

All the shift operators are commuting invertible isometries of X but

$$\mathrm{Ker}(I - T_2)(I - T_{2\pi}) = X,$$

$$\mathrm{Ker}(I - T_2) + \mathrm{Ker}(I - T_{2\pi}) = \{x \in X : \lim_{\mathcal{F}} x_k^1 = \lim_{\mathcal{F}} x_k^2 = 0\}.$$

The example above shows in particular the significance of complementation in X^{**} condition in Theorem 1. The next two examples show that the spaces c_0 and ℓ_1 do not have the DePr.

E x a m p l e 2. c_0 does not have the DePr.

Consider the following compact subset K of $[-1, 1]$:

$$K = \{0, 1, -1, 1/2, -1/2, 1/4, -1/4, ...\}.$$

The space $C(K)$ of continuous functions on K is evidently isomorphic to c_0.

Consider two functions on K:

$$f_1(t) = \begin{cases} 1 \ , \ t \leq 0, \\ e^{it}, \ t > 0, \end{cases}$$

$f_2(t) = f_1(-t)$. Let T_1, T_2 be the corresponding multiplication operators:

$$T_k g = f_k \cdot g, \quad k = 1, 2.$$

Both of T_k are invertible isometries of $C(K)$ but

$$\mathrm{Ker}(I - T_1) + \mathrm{Ker}(I - T_2) = \{f \in C(K) : \ f(0) = 0\},$$

$$\mathrm{Ker}(I - T_1)(I - T_2) = C(K).$$

So the pair (T_1, T_2) does not have the DePr.

E x a m p l e 2. ℓ_1 has no the DePr.

Denote the standard basis of ℓ_1 by $\{e_k\}_0^\infty$ and introduce the following operators: T_1, T_2: $\ell_1 \to \ell_1$,

$$T_1 e_0 = e_0 + e_2, \quad T_1 e_k = \begin{cases} e_{k+2} - e_2\,, & k \text{ even}, k \geq 2, \\ e_k & , k \text{ odd}, \end{cases}$$

$$T_2 e_0 = e_0 + e_1, \quad T_2 e_k = \begin{cases} e_{k+2} - e_1\,, & k \text{ odd}, k \geq 1, \\ e_k & , k \text{ even}. \end{cases}$$

They are commuting power bounded operators (it is easy to check because in ℓ_1 the formula

$$\|T\| = \sup_k \|Te_k\|$$

for the operator norm is true). Further,

$$\mathrm{Ker}(I - T_1) = \mathrm{span}\{e_1, e_3, e_5, ...\},$$

$$\mathrm{Ker}(I - T_2) = \mathrm{span}\{e_2, e_4, e_6, ...\}.$$

So $\mathrm{Ker}(I - T_1) + \mathrm{Ker}(I - T_2)$ does not contain e_0 but

$$(I - T_1)(I - T_2)e_0 = 0.$$

R e m a r k 2. On the space ℓ_1.

It is well known that if K is a countable compact then the dual to $C(K)$ is isometric to ℓ_1. So ℓ_1 is isomorphic to duals of a variety of spaces and there are many w^*-topologies on ℓ_1. For w^*-continuous operators the DePr is true so the example above shows that there exists a pair of commuting power bounded operators such that for every w^*-topology on ℓ_1 at least one of them is not w^*-continuous. Moreover, if P is a projector from ℓ_1^{**} onto ℓ_1 then at least one of T_k^{**} above does not commute with P. May be both of them have to be discontinuous in every w^*-topology but we don't know whether it is true. But the symmetry shows easily that the operator T: $\ell_1 \to \ell_1$, $Te_k = e_{k+1} - e_1$ has the property: for every projector P from ℓ_1^{**} onto ℓ_1 $T^{**}P \neq PT^{**}$.

One more problem appears if one is interested in constructing of an "optimal" decomposition in some sense. The theorem below shows that even for shifts in $\mathbb{Z}$ it is impossible in general to obtain the decomposition (1) with $\|G_k\| = 1$.

For $n \in \mathbb{N}$ denote the space of all n-periodic sequences as $\mathrm{Per}(n)$, the n-shift operator as T_n. Notice that $\mathrm{Per}(n)$ is invariant under T_m for every n, m.

Also denote the standard averaging projector from $\mathrm{Per}(mn)$ on $\mathrm{Per}(n)$ as P_n^{mn}, $P_n^{mn} = \frac{1}{m}\sum_{k=1}^{m} T_{nk} = \frac{1}{mn}\sum_{k=1}^{mn} T_{nk}$.

Let n_1, $n_2 \in \mathbb{N}$. Denote the greatest common divisor of n_1 and n_2 as n and the least common multiple as N. Let $n_1/n \geq 3$, $n_2/n \geq 3$, $X = \mathrm{lin}\{\mathrm{Per}(n_1), \mathrm{Per}(n_2)\} \subset \mathrm{Per}(N)$. Equip X with the sup-norm.

Theorem 5. *There are no linear operators Q_1 and Q_2, $Q_1\colon X \to \mathrm{Per}(n_1)$, $Q_2\colon X \to \mathrm{Per}(n_2)$ satisfying $Q_1 + Q_2 = I$, $\|Q_1\| \leq 1$, $\|Q_2\| \leq 1$.*

P r o o f. We argue by contradiction: let such Q_1 and Q_2 exist. Averaging Q_1 and Q_2 by shifts, obtain $\tilde{Q}_1 = \frac{1}{N}\sum_{k=1}^{N} T_k Q_1 T_{-k}$ and $\tilde{Q}_2 = \frac{1}{N}\sum_{k=1}^{N} T_k Q_2 T_{-k}$. Now $\tilde{Q}_1$ and $\tilde{Q}_2$ are linear, $\tilde{Q}_1 + \tilde{Q}_2 = I$, $\|\tilde{Q}_1\| \leq 1$, $\|\tilde{Q}_2\| \leq 1$, $\tilde{Q}_1 : X \to \mathrm{Per}(n_1)$, $\tilde{Q}_2 : X \to \mathrm{Per}(n_2)$. Also $\tilde{Q}_1$ and $\tilde{Q}_2$ are commuting with shifts and projectors.

Denote $P_{n_1}^N$ as P_1, $P_{n_2}^N$ as P_2, P_n^N as P_0.

Applying P_1 and P_2 to the equality $\tilde{Q}_1 + \tilde{Q}_2 = I$, we have

$$P_1\tilde{Q}_1 + P_1\tilde{Q}_2 = P_1,$$

$$P_2\tilde{Q}_1 + P_2\tilde{Q}_2 = P_2,$$

or

$$\tilde{Q}_1 - P_1 = -P_0\tilde{Q}_2,$$
$$\tilde{Q}_2 - P_2 = -P_0\tilde{Q}_1,$$

i.e., $\tilde{Q}_1$ and P_1, $\tilde{Q}_2$ and P_2 coincide on the $\mathrm{Ker}P_0$. Reveal how they can differ on the $\mathrm{Im}P_0$.

First consider the case $n = 1$, then n_1, $n_2 \geq 3$. Then $\mathrm{Im}P_0 = \mathrm{lin}\{\mathbf{1}\}$ and $\tilde{Q}_1\mathbf{1} = \tilde{Q}_1 P_0\mathbf{1} = P_0\tilde{Q}_1\mathbf{1} \in Per(n)$, i.e., $\tilde{Q}_1\mathbf{1} = c\mathbf{1}$, $\tilde{Q}_2\mathbf{1} = (1-c)\mathbf{1}$. This implies $0 \leq c \leq 1$. Consequently

$$\tilde{Q}_1 = \tilde{Q}_1 P_0 + \tilde{Q}_1(I - P_0) = cP_0 + P_1(I - P_0) = P_1 + (c-1)P_0$$

and

$$\tilde{Q}_2 = \tilde{Q}_2 P_0 + \tilde{Q}_2(I - P_0) = (1-c)P_0 + P_2(I - P_0) = P_2 - cP_0.$$

Construct functions $g_j \in \mathrm{Per}(n_j)$, $j = 1, 2$. $g_j(m)$ will be equal 1 for $1 \leq m \leq n_j - 1$ and -1 for $m = n_j$, $j = 1, 2$. It is obvious that $\|g_j\| = 1$, $P_j g_j = g_j$ and $P_0 g_j = \frac{n_j - 2}{n_j} \cdot \mathbf{1}$. Hence

$$\tilde{Q}_1 g_1 = g_1 + (1-c)\frac{n_1 - 2}{n_1} \cdot \mathbf{1}$$

and

$$\tilde{Q}_2 g_2 = g_2 - c\frac{n_2 - 2}{n_2} \cdot \mathbf{1}.$$

Since g_j changes sign while $\mathbf{1}$ does not, these equalities mean

$$\|\tilde{Q}_1\| \geq 1 + (1 - c)\frac{n_1 - 2}{n_1}$$

and

$$\|\tilde{Q}_2\| \geq 1 + c\frac{n_2 - 2}{n_2}.$$

Therefore $c = 1$ and $c = 0$ but this condition is contradictory.

The case $n > 1$ can be reduced to the treated one by dividing $\mathbb{Z}$ into parts $m + n\mathbb{Z}$, $1 \leq m \leq n - 1$.

References

[1] *M. Laczkovich and Sz. Revesz*, Periodic decompositions of continuous functions. — Acta. Math. Hung. (1989), v. 54 (3–4), p. 329–341.

[2] *M. Laczkovich and Sz. Revesz*, Decompositions into the sum of periodic functions belonging to a given Banach space. — Acta. Math. Hung. (1990), v. 55 (3–4), p. 353–363.

[3] *S. Banach*, Theory of Linear Operations. North-Holland Mathematical Library (1987), v. 38.

[4] *Z. Gajda*, Note on decomposition of bounded functions into the sum of periodic terms. — Acta Math. Hung. (1992), v. 59 (1-2), p. 103–106.

Техника усреднения в задаче периодического разложения

В.М. Кадец, Б.М. Шумяцкий

Пусть T_1, T_2 — пара коммутирующих линейных изометрий в банаховом пространстве X. Обобщая результат М. Лацковича и С. Ревеса, доказываем, что во многих случаях элемент x из $\mathrm{Ker}[(\mathrm{I} - \mathrm{T}_1)(\mathrm{I} - \mathrm{T}_2)]$ может быть разложен в сумму $x_1 + x_2$, где $x_k \in \mathrm{Ker}(\mathrm{I} - \mathrm{T}_k)$, $k = 1, 2$. Более того, используя технику усреднения, доказываем существование линейных операторов, осуществляющих такое разложение. Эти результаты применимы к задаче разложения функций в сумму периодических функций.

Техніка усереднення у задачі періодичного розкладання

В.М. Кадець, Б.М. Шумяцький

Нехай T_1, T_2 — пара комутуючих лінійних ізометрій у банаховому просторі X. Узагальнюючи результат М. Лацковича і С. Ревеса, доводимо, що у багатьох випадках элемент x з $\mathrm{Ker}[(\mathrm{I}-\mathrm{T}_1)(\mathrm{I}-\mathrm{T}_2)]$ може бути розкладений в суму x_1+x_2, де $x_k \in \mathrm{Ker}(\mathrm{I}-\mathrm{T_k})$, $k=1,2$. Більш того, використовуючи техніку усереднення, доводимо існування лінійних операторів, що здійснюють таке розкладання. Ці результати застосовуються до задачі розкладання функцій в суму періодичних функцій.