Matematicheskaya fizika, analiz, geometriya
2000, v. 7, No. 4, p. 380–386

# Solutions of algebraic equations with analytic almost periodic coefficients

V.V. Brytik, S.Yu. Favorov

*Department of Mathematics and Mechanics, V.N. Karazin Kharkov National University*
*4 Svobody Sq., Kharkov, 61077, Ukraine*
E-mail:favorov@ilt.kharkov.ua

Received February 3, 2000
Communicated by I.V. Ostrovskii

·

We prove that continuous or meromorphic with a small number of poles solutions of algebraic equations with the analytic almost periodic coefficients are almost periodic, too.

It is well known that solutions $w(z)$ of the equation

$$a_m(z)w^m + a_{m-1}(z)w^{m-1} + \ldots + a_1(z)w + a_0(z) = 0 \tag{1}$$

often inherit properties of the coefficients $a_j(z)$, $j = 0, \ldots, m$. As an example, suppose that these coefficients are almost periodic functions on the axis, $a_m(z) = 1$, and the discriminant $D(z)$ of the polynomial in (1) satisfies the condition

$$|D(z)| \geq \gamma > 0; \tag{2}$$

then each solution of (1) is an almost periodic function, too [1, 2]. Nevertheless, one cannot replace condition (2) by the weaker condition

$$D(z) \neq 0 \tag{3}$$

even for the equation

$$w^2 - a_0(z) = 0 \tag{4}$$

·This research was supported by INTAS-99-00089.

[3]. However for analytic almost periodic coefficients $a_j(z)$, $j=0,\dots,m$, on a strip $S$, the conditions $a_m(z)=1$ and (3) imply that every continuous solution of (1) is an analytic almost periodic function on this strip [4].

Note also that one can formulate classical Bohr's theorem on division of analytic almost periodic functions (see for example [5]) in the following way: an analytic solution of (1) for $m=1$ and analytic almost periodic functions $a_1(z)$, $a_0(z)$ on a strip is an almost periodic function on this strip.

It is natural to consider analytic solutions of (1) with analytic almost periodic coefficients without any restriction on the discriminant $D(z)$. We know only one result of this kind: namely, an analytic solution of (4) with an analytic almost periodic function $a_0(z)$ on a strip is almost periodic as well. However, by our opinion, the proof of this result in [6] is not perfect.

Recall that a function $f(z)$ is said to be *almost periodic on the real axis* $\mathbb{R}$ if $f(z)$ belongs to the closure of the set of finite exponential sums

$$\sum a_n e^{i\lambda_n z}, \quad a_n \in \mathbb{C}, \quad \lambda_n \in \mathbb{R}, \tag{5}$$

with respect to the topology of uniform convergence on $\mathbb{R}$. Further, let $S$ be a strip $\{z\in\mathbb{C} : a<\operatorname{Im} z<b\}$ ($a$ can be $-\infty$ and $b$ can be $+\infty$). We write $S'\subset\subset S$ if $S'=\{z\in C : a'<\operatorname{Im} z<b'\}$, $a<a'<b'<b$. A function $f(z)$ is said to be *analytic almost periodic on a strip* $S$ if $f(z)$ belongs to the closure of the set of sums (5) with respect to the topology of uniform convergence on every substrip $S'\subset\subset S$. The equivalent definitions are the following: the family $\{f(z+h)\}_{h\in\mathbb{R}}$ is a relative compact set with respect to the topology of uniform convergence on $\mathbb{R}$ (for almost periodic functions on the axis) or with respect to the topology of uniform convergence on every substrip $S'\subset\subset S$ (for analytic almost periodic functions).

By $AP(S)$ we denote the space of all analytic almost periodic functions on $S$ equipped with the topology of uniform convergence on every substrip $S'\subset\subset S$; the zero set of a function $f\in AP(S)$ is denoted by $Z(f)$.

**Theorem 1.** *Let $w(z)$ be a continuous solution of (1) in $S$ and $a_k(z)\in AP(S)$, $0\le k\le m$. Then $w(z)\in AP(S)$.*

Proof of this theorem makes use of the following simple lemmas on roots of polynomials

$$Q(w)=w^m+b_{m-1}w^{m-1}+\ldots+b_1 w+b_0. \tag{6}$$

**Lemma 1.** *For any $N<\infty$, $\varepsilon>0$, there exists a constant $\nu>0$ depending on $N$ and $\varepsilon$ only such that the roots $w_j$, $j=1,\dots,m$, and $\tilde w_j$, $j=1,\dots,m$, of any polynomials $Q$, $\tilde Q$ of the form (6) with $\max_j|b_j|\le N$, $\max_j|\tilde b_j|\le N$,*

$\max_j |b_j - \tilde{b_j}| \le \nu$, *satisfy, under a suitable numeration, the conditions* $|w_j - \tilde{w}_j| < \varepsilon,\ j = 1 \ldots m$.

P r o o f. Assume the contrary. Then for some $N < \infty,\ \varepsilon_0 > 0$ there exist two sequences of polynomials

$$Q_n(w) = w^m + b_{m-1}^{(n)} w^{m-1} + \ldots + b_0^{(n)}, \quad \tilde{Q}_n(w) = w^m + \tilde{b}_{m-1}^{(n)} w^{m-1} + \ldots + \tilde{b}_0^{(n)}$$

such that $\max_j |b_j| \le N,\ \max_j |\tilde{b}_j| \le N,\ \max_j |b_j - \tilde{b_j}| \to 0$ as $n \to \infty$, and

$$\max_j |w_j^{(n)} - \tilde{w}_j^{(n)}| > \varepsilon_0 \tag{7}$$

under every numeration of roots $w_j^{(n)},\ \tilde{w}_j^{(n)},\ j = 1, \ldots, m$, of the polynomials $Q_n$, $\tilde{Q}_n$, respectively. Without loss of generality it can be assumed that

$$b_j^{(n)} \to \overline{b}_j, \quad \tilde{b}_j^{(n)} \to \overline{b}_j, \quad j = 0, \ldots, m-1, \quad \text{as } n \to \infty.$$

Hence the sequences of the polynomials $Q_n,\ \tilde{Q}_n$ converge to the same polynomial

$$\overline{Q}(w) = w^m + \overline{b}_{m-1} w^{m-1} + \ldots + \overline{b}_0$$

with respect to the uniform convergence on every compact subset of $\mathbb{C}$.

Let $C_j,\ j = 1, \ldots, p,\ p \le m$, be disjoint disks of radius $r < \varepsilon_0/2$ with the centers at the roots of the polynomial $\overline{Q}(w)$. By Hurwitz's theorem, for $n$ large enough all roots of the polynomials $Q_n(w)$, $\tilde{Q}_n(w)$, lie in these disks and a number of roots of the polynomial $Q_n(w)$ in a disk $C_j$ coincides with a number of roots of the polynomial $\tilde{Q}_n(w)$ in the same disk for each $j = 1, \ldots, p$. Therefore there exists a numeration of roots of the polynomials $Q_n$, $\tilde{Q}_n$ such that (7) is false. This contradiction proves the lemma.

**Lemma 2.** *The distance between any two roots of a polynomial $Q$ of the form (6) with the discriminant $d(Q) \ne 0$ is greater than a constant $\tau > 0$ depending on $|d(Q)|$ and $\max_j |b_j|$ only.*

P r o o f. Assume the contrary. Then there exists a sequence of polynomials

$$Q_n(w) = w^m + b_{m-1}^{(n)} w^{m-1} + \ldots + b_0^{(n)}$$

such that $\max_j |b_j| \le N < \infty,\ |d(Q_n)| \ge \delta > 0$, and the distance between some two roots of a polynomial $Q_n$ tends to zero as $n \to \infty$. Without loss of generality it can be assumed that $b_j^{(n)} \to \overline{b}_j,\ j = 0, \ldots, m-1$, as $n \to \infty$; hence the discriminants $d(Q_n)$ converge to the discriminant $d(\overline{Q})$ of the polynomial

$$\overline{Q}(w) = w^m + \overline{b}_{m-1} w^{m-1} + \ldots + \overline{b}_0$$

and $d(\overline{Q}) \neq 0$. Using Lemma 1 for $\overline{Q}$ and $Q_n$ with $n$ large enough, we obtain that the distance between some two roots of the polynomial $\overline{Q}$ is arbitrary small, i.e., this polynomial has a multiple root. This contradicts the assertion $d(\overline{Q}) \neq 0$. The lemma is proved.

P r o o f o f T h e o r e m 1. We may assume that $a_m(z) \not\equiv 0$. First let us suppose that the discriminant $D(z) \not\equiv 0$. The solution $w(z)$ is bounded on a neighborhood of any point $z' \in S$; moreover $w(z)$ is analytic at any point $z' \in S$ such that $a_m(z') \neq 0$, $D(z') \neq 0$. Since zeros of $a_m(z)$ and $D(z)$ are isolated, $w(z)$ is analytic on $S$.

Let us show that for an arbitrary sequence $\{h_n\} \subset \mathbb{R}$ there exists a subsequence $\{h_{n'}\}$ such that the functions $w(z + h_{n'})$ form a Cauchy sequence in the space $AP(S)$. It is sufficient to check that these functions converge uniformly on each substrip $S_0 \subset\subset S_1 \subset\subset S$. We may assume that the functions $a_j(z+h_n)$ converge to functions $\overline{a_j}(z)$ in the space $AP(S)$ for $n \to \infty$ and each $j = 1, \ldots, m$; then the functions $D(z + h_n)$ converge in this space to the discriminant $\overline{D}(z)$ of the right hand side of the equation

$$\overline{a}_m(z)w^m + \ldots + \overline{a}_1(z)w + \overline{a}_0(z)w = 0. \tag{8}$$

Let $U_r$ be the $r$-neighborhood of the set $[Z(\overline{a}_m) \bigcup Z(\overline{D})] \bigcap S_1$. We claim that for sufficiently small $r$ there exist closed rectangles $\Pi_l$ such that $S_0 \subset \bigcup_l \Pi_l \subset S_1$ and $\partial\Pi_l$ disjoint with $U_r$ for all $l \in \mathbb{N}$.

Since the functions $\overline{D}(z)$ and $\overline{a}_m(z)$ belong to $AP(S)$, the numbers of their zeros inside a rectangle $\{z \in S_1 : |\mathrm{Re} z - t| < 1\}$ are bounded by a number $K$ independent of $t \in \mathbb{R}$ (see [5]). Hence for $r < 1/4K$ and for all $t \in \mathbb{R}$ there exists $c_t \in \mathbb{R}$ such that $|c_t - t| < 1$ and the straight line $\mathrm{Re} z = t$ does not intersect the set $U_r$. Then there exists a sequence of rectangles $\{z \in S_1 : c_l \leq \mathrm{Re} z \leq c'_l\}$ overlapping the strip $S_1$ whose lateral sides are disjoint with $U_r$. Furthermore, suppose $r < (8K)^{-1} \inf\{|z - z'| : z \in S_0,\ z' \notin S_1\}$, then there exist segments $\{z : \mathrm{Im} z = d_l, c_l \leq \mathrm{Re} z \leq c'_l\} \subset S_1 \setminus S_0$ disjoint with $U_r$ as well. Thus the rectangles $\Pi_l = \{z : c_l \leq \mathrm{Re} z \leq c'_l,\ d_l \leq \mathrm{Im} z \leq d'_l\}$ with suitable $d_l$, $d'_l$ are just required.

It follows from properties of analytic almost periodic functions (see [5]) that $|\overline{D}(z)| \geq \eta$, $|\overline{a_m}(z)| \geq \eta$ for $z \in S_1 \setminus U_r$, where $\eta$ is a strictly positive constant. Hence for $n \geq N$ and $z \in S_1 \setminus U_r$, we have $|D(z+h_n)| \geq \eta$, $|a_m(z+h_n)| \geq \eta$. Besides, the functions $a_j(z), 0 \leq j \leq m-1$, are uniformly bounded on $S_1$. Applying Lemma 2, we get that the distance between any two roots of the polynomial

$$Q_n(w) = w^m + \frac{a_{m-1}(z + h_n)}{a_m(z + h_n)} w^{m-1} + \ldots + \frac{a_0(z + h_n)}{a_m(z + h_n)}$$

is greater than $\tau > 0$. Note that the constant $\tau$ is the same for all $z \in S_1 \setminus U_r$, $n \geq N$. Further, the functions

$$\frac{a_j(z+h_n)}{a_m(z+h_n)}$$

form a Cauchy sequence with respect to the uniform convergence on the set $S_1 \setminus U_r$ for every $j = 0, \dots, m-1$. This yields that the polynomials $Q_n(w)$, $Q_k(w)$ satisfy the conditions of Lemma 1 with $\varepsilon = \tau/3$ and $n, k \geq N_1(\varepsilon)$ for all $z \in S_1 \setminus U_r$. Hence for every fixed $z \in S_1 \setminus U_r$ there exists a solution $\tilde{w}(z)$ of the equation $Q_n(w) = 0$ such that

$$|w(z+h_k) - \tilde{w}(z)| \leq \varepsilon.$$

Now we have two possibilities for each $z \in S_1 \setminus U_r$ : either $\tilde{w}(z) = w(z+h_n)$ and

$$|w(z+h_k) - w(z+h_n)| \leq \frac{\tau}{3}, \tag{9}$$

or $|\tilde{w}(z) - w(z+h_n)| \geq \tau$ and

$$|w(z+h_n) - w(z+h_k)| \geq |w(z+h_n) - \tilde{w}(z)| - |\tilde{w}(z) - w(z+h_k)| \geq \frac{2}{3}\tau.$$

Fix an arbitrary point $z_0 \in \partial\Pi_1$. The coefficients of the polynomials $Q_n(w)$ are bounded at this point, therefore the sequence $w(z_0 + h_n)$ is also bounded. Without loss of generality it can be assumed that this sequence converges, hence inequality (9) is true for $z = z_0$. Since the set $\bigcup_l \partial\Pi_l$ is connected, we see that (9) holds on this set. Using the Maximum Principle, we obtain that (9) is true for all $z \in \bigcup_l \Pi_l \supset S_0$. Hence we have $\tilde{w}(z) = w(z+h_n)$ for all $z \in S_0$. Thus the functions $w(z+h_n)$ form a Cauchy sequence with respect to the uniform convergence on $S_0$ and $w(z)$ is an almost periodic function on $S$.

If the discriminant of the polynomial $P(w) = a_m(z)w^m + \ldots + a_1(z)w + a_0(z)w$ is zero, then the equations $P(w) = 0$ and $P'(w) = ma_m(z)w^m + \ldots + a_1(z) = 0$ have a common solution for each fixed $z \in S$. Using the Euclid algorithm, we get

$$P(w) = Q(w)R(w), \qquad\qquad P'(w) = T(w)R(w),$$

where the coefficients of $Q(w)$, $T(w)$, $R(w)$ lie in the quotient field of $AP(S)$. Besides, if $w(z)$ is a solution of (1) for fixed $z \in S$, then $w(z)$ is an ordinary solution of the equation $Q(w) = 0$ whenever all the coefficients of $Q(w)$ are finite at this point $z$. Multiplying $Q(w)$ by a suitable function from $AP(S)$, we obtain a polynomial $\tilde{Q}(w)$ with the coefficients from $AP(S)$ such that $w(z)$ is an ordinary solution of the equation $\tilde{Q}(w) = 0$ for all $z \in S$ outside of some discrete set. Hence the discriminant of $\tilde{Q}(w)$ does not vanish and we can use the previous result. The theorem is proved.

**Theorem 2.** *Suppose $w(z)$ is a meromorphic solution of (1) with $a_j(z) \in AP(S)$, $j = 0, \dots, m$, and*

$$\operatorname{card}\{z \in S' : |\operatorname{Re} z| < t,\ w(z) = \infty\} = o(t) \quad \textit{as} \quad t \to \infty$$

*for each $S' \subset\subset S_1$. Then $w(z) \in AP(S)$.*

P r o o f . Let $S_0 \subset\subset S_1 \subset\subset S$. It can be easily seen that for all $t \in \mathbb{R}$ there exists a rectangle $\{z \in S_1 : |\mathrm{Re} z - h| < t\}$ without poles of $w(z)$. Hence there exists a sequence of rectangles $\{z \in S_1 : |\mathrm{Re} z - h_n| < t_n\}$, $t_n \to \infty$, without poles of $w(z)$. We may assume $a_m(z) \not\equiv 0$, $D(z) \not\equiv 0$ and the sequences of the functions $a_j(z + h_n)$, $j = 0, \dots, m$, $D(z + h_n)$ converge in the space $AP(S)$ to functions $\overline{a}_j(z)$, $\overline{D}(z)$ respectively. Note that all poles of $w(z)$ lie in the set $Z(a_m)$. Applying the arguments of Theorem 1, we obtain that the sequence $w(z + h_n)$ converges uniformly on the set $\bigcup_l \partial\Pi_l$.

Let $\overline{w}(z)$ be the limit of the sequence. Since every rectangle $\Pi_l$ lies inside the set $\{z \in S_1 : |z| < t_n\}$ for $n \geq n(l)$, we see that the functions $w(z + h_n)$ converge on $\Pi_l$ to an analytic function, therefore $\overline{w}(z)$ is analytic on $S_0$. Now Theorem 1 implies that $\overline{w}(z)$ is an almost periodic solution of (8).

Furthermore,

$$\sup_{S'} |\overline{a_j}(z - h_n) - a_j(z)| = \sup_{S'} |a_j(z + h_n) - \overline{a_j}(z)| \to 0 \quad \text{as} \quad n \to \infty$$

for each $S' \subset\subset S$ and $j = 0, \dots, m$. Applying the above arguments, we see that the sequence of the functions $\overline{w}(z - h_n)$ converges in the space $AP(S_0)$ to an analytic solution $\overline{\overline{w}}(z)$ of (1).

Since $S_0$ is an arbitrary substrip of $S$, we only need to prove that $\overline{\overline{w}}(z) = w(z)$. Assume the contrary. Let $S' \subset\subset S_0$ be an arbitrary substrip, $\tilde{U}_r$ be the $r-$neighborhood of the set $[Z(a_m) \cup Z(D)] \cap S'$. Applying the above arguments and Lemma 2, we see that

$$|\overline{\overline{w}}(z) - w(z)| \geq \tau > 0 \quad \text{for all } z \in S' \setminus \tilde{U}_r \tag{10}$$

with certain $\tau > 0$. On the other hand, we have

$$|w(z + h_n) - \overline{w}(z)| \leq \tau/3 \quad \text{for } n \geq n(\tau),\ z \in \bigcup_l \partial\Pi_l.$$

Therefore, using the uniform convergence of $\overline{w}(z - h_n)$ to $\overline{\overline{w}}(z)$ on $S'$, we obtain

$$|\overline{\overline{w}}(z) - w(z)| \leq |\overline{\overline{w}}(z) - \overline{w}(z - h_n)| + |\overline{w}(z - h_n) - w(z)| \leq \frac{2}{3}\tau \tag{11}$$

for $z \in \bigcup_l \partial\Pi_l - h_n$ and sufficiently large $n$. Arguing as in the proof of Theorem 1, we can see that if $r$ is small enough, then every vertical segment $\{z \in S' : \mathrm{Re} z = t\}$ has common points with $S' \setminus \tilde{U}_r$. Thus inequalities (10) and (11) are simultaneously fulfilled on the nonempty set. This contradiction completes the proof.

The authors are grateful to the late Prof. L.I. Ronkin who had called their attention to the problem considered in this paper.

## References

[1] *R.H. Cameron*, Implicit functions of almost periodic functions. — Bull. Am. Math. Soc. (1934), v. 40, p. 895–904.

[2] *H. Bohr and D.A. Flanders*, Algebraic equation with almost-periodic coefficients. — Mat.-fysike Medd. (1937), v. 15, p. 1–49.

[3] *A. Walther*, Algebraic Funktionen von fastperiodischen Funktionen. — Monatshefte fur Mathematik und Physik (1933), Bd. 40, p. 444–457.

[4] *H. Bohr and D.A. Flanders*, Algebraic functions of almost-periodic functions. — Duke Math. J. (1938), v. 4, p. 779–787.

[5] *B.M. Levitan*, Almost-periodic functions. Gostekhizdat, Moscow (1953). (Russian)

[6] *B. Jessen and H. Torneave*, Mean motion and zeros of almost-periodic functions. — Acta Math. (1945), p. 137–279.

### Решение алгебраических уравнений с аналитическими почти периодическими коэффициентами

В.В. Бритик, С.Ю. Фаворов

Доказано, что непрерывные или мероморфные с малым числом полюсов решения алгебраических уравнений с аналитическими почти периодическими коэффициентами также почти периодичны.

### Рішення алгебраїчних рівнянь з аналітичними майже періодичними коефіцієнтами

В.В. Брітік, С.Ю. Фаворов

Доведено, що неперервні або мероморфні з малою кількістю полюсів рішення алгебраїчних рівнянь з аналітичними майже періодичними коефіцієнтами теж майже періодичні.