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On the entropy for actions of some class of groups and its calculation

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The dynamical entropy for actions of $\oplus Z_k$, $k = 2, 3, \dots$, on C^*-algebras which is studied in this work, is a generalization of Connes–Narnhofer–Thirring entropy for actions of the torsion groups on C^*-algebras. The properties of such entropy are investigated and a formula for quantum dynamical entropy of the Bogoliubov action of $\oplus Z_k$, $k = 2, 3, \dots$, on the CAR-algebra is obtained. It is proved that the part of action corresponding to the singular spectrum gives zero contribution to the entropy.

1. In late 50-th Kolmogorov and Sinai introduced the notion of entropy in ergodic theory. That made it possible to solve a number of important problems in this theory. In particular, the existence of nonisomorphic ergodic systems with the same continuous spectrum was established using the entropy theory of Bernoulli automorphisms (this is in a sharp contrast to the case of discret spectrum). Thus the problem of extending this notion onto automorphisms of von Neumann algebras and C^*-algebras arose in a natural way. In 1975 Connes and Størmer presented a solution of this problem for some class of von Neumann algebras (II_1 factors), and in 1987 Connes, Narnhofer and Thirring extended the definition of the entropy to the case of automorphisms of C^*-algebras which preserve a given state [1]. Recently this entropy has been studied extensively from different points of view. In [2] Størmer and Voiculescu computed the dynamical entropy for a Bogoliubov automorphism on the algebra of canonical anticommutation relations (CAR-algebra) and showed that only absolutely continuous part of the unitary operator defining the Bogoliubov automorphism should be taken into account. Bezuglyi and Golodets [3] generalized those results to actions of free Abelian groups Z^n, $n > 1$, on the CAR-algebra.

The principal goal of this paper is to study the dynamical entropy for actions of the groups $\oplus Z_k$, $k = 2, 3, \ldots$, by automorphisms of C^*-algebras. We note that such groups are not free, and even in the classical case there is no good entropy theory for groups like those.

The structure of this paper is as follows: in Section 2 we introduce the dynamical entropy for actions of $Z/kZ \oplus Z/kZ \oplus \ldots$, $k = 2, 3, \ldots$, on C^*-algebras, list the properties of this entropy. In particular, it is shown that the entropy for the tensor product of group actions with respect to states on various C^*-algebras has as a lower bound the sum of entropies for each action of the group with respect to the related state. The analogue of the Kolmogorov-Sinai theorem is announced. A relationship between the entropy of a group and the entropy of a subgroup is studied. The formula for the entropy of a finite index subgroup is found. In Section 3 the entropy of a Bogoliubov state-preserving action corresponding to singular spectrum is shown to be zero. Next, the formula for the entropy of a Bogoliubov action corresponding to absolutely continuous spectrum is found and the entropy for some simple cases is calculated. Furthermore, the general case is studied. It is demonstrated that only absolutely continuous part of the unitary representation defining the Bogoliubov action should be taken into account.

2. Let us remind briefly some terminology and definitions (see [1, 2] more details). The term "unital" stands for containing or preserving the unit element. A completely positive unital map ϕ between two unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a positive unital map such that the map ϕ between $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ the $n \times n$ matrices with elements from $\mathcal{A}$ (respectively $\mathcal{B}$), $(\phi(a))_{ij} = \phi(a_{ij})$ is positive. In the vector space of linear maps the completely positive unital maps constitute a closed convex set. If $\mathcal{B} \subset \mathcal{A}$ a positive unital map with $\phi(b_1 a b_2) = b_1 \phi(a) b_2,\ b_i \in \mathcal{B}$, a $\in \mathcal{A}$, is called a unital conditional expectation. It is automatically completely positive.

Let $\mathcal{A}$ be a unital C^*-algebra, $C_1, \ldots, C_k$ finite dimensional C^*-algebras, and $\gamma_j : C_j \to \mathcal{A}$ unitial completely positive maps, $j = 1, \ldots, k$. Let ϕ be a state on $\mathcal{A}$ and P a unital completely positive map from $\mathcal{A}$ into a finite dimensional Abelian C^*-algebra $\mathcal{B}$ such that there is a state ψ on $\mathcal{B}$ for which $\psi \circ P = \phi$. If $p_1, \ldots, p_r$ are the minimal projections in $\mathcal{B}$, then there are states $\phi_i, i = 1, \ldots, r$, on $\mathcal{A}$ such that

$$P(x) = \sum_{i=1}^{r} \phi_i(x) p_i\,, \quad x \in \mathcal{A}, \quad \text{and} \quad \phi = \sum_{i=1}^{r} \psi(p_i) \phi_i\,.$$

We set up

$$\epsilon_\psi(P) = \sum_{i=1}^{r} S(\phi|\phi_i),$$

where $S(\phi|\phi_i)$ is the relative entropy of the ϕ and ϕ_i (see [1]). The entropy defect $s_\psi(P)$ is given by

$$s_\psi(P) = S(\psi) - \epsilon_\psi(P),$$

where $S(\psi) = -\sum_{i=1}^{r} \psi(p_i) log\psi(p_i)$ is the entropy of ψ.

Let B_j, $j = 1, \ldots, k$, be C^*-subalgebras of $\mathcal{B}$ and $E_j : \mathcal{B} \to B_j$ a ψ-preserving conditional expectation. Then $(\mathcal{B}, E_j, P, \psi)$ is called an abelian model for $(\mathcal{A}, \phi, \gamma_1, \ldots, \gamma_k)$. The entropy of such an Abelian model is defined to be

$$S(\psi|_{\bigvee_{j=1}^{k} B_j}) - \sum_{j=1}^{k} s_\psi(\rho_j),$$

where $\rho_j = E_j \circ P \circ \gamma_j : C_j \to B_j$. The supremum of entropies for all such abelian models is denoted by $H_\phi(\gamma_1, \ldots, \gamma_k)$. Properties of this function can be found in [1, Proposition III.6]. If α is a ϕ-preserving automorphism of $\mathcal{A}$ and $\gamma : C \to \mathcal{A}$ is a unital completely positive map of a finite dimensional C^*-algebra C, then we denote by

$$h_{\phi,\alpha}(\gamma) = \lim_{k\to\infty} \frac{1}{k} H_\phi(\gamma, \alpha \circ \gamma, \ldots, \alpha^{k-1} \circ \gamma).$$

The entropy of α with respect to ϕ is defined by the formula

$$h_\phi(\alpha) = \sup_\gamma h_{\phi,\alpha}(\gamma).$$

In the sequel we study the entropy for actions of $Z/kZ \oplus Z/kZ \oplus \ldots$, $k = 2, 3, \ldots$, on C^*-algebras. To simplify the exposition, we restrict our observations to $Z/2Z \oplus Z/2Z \oplus \ldots$-actions. It is clear that the general case may be considered in a similar way.

Now let $\mathcal{A}$ be a unital C^*-algebra, $\Gamma = \bigoplus_{i=1}^{\infty} Z_i$, $\Gamma(n) = \oplus_{i=1}^{n} Z_i$, $\Gamma[n, m] = \bigoplus_{i=n+1}^{m} Z_i$, $\Gamma_n = \bigoplus_{i=n+1}^{\infty} Z_i$, where $Z_i = Z/2Z$. Let ϕ be a state on $\mathcal{A}$ and $\alpha : \Gamma \to Aut(\mathcal{A}, \phi)$ a ϕ-preserving action of Γ on $\mathcal{A}$ by $*$-automorphisms, i.e. α is an injective homomorphism from Γ into $Aut(\mathcal{A})$ such that $\alpha(\xi)$ is a ϕ-preserving $*$-automorphism of $\mathcal{A}$ for any $\xi \in \Gamma$. Let π be a representation of $\mathcal{A}$ corresponding to ϕ via the GNS-construction, $M = \pi(\mathcal{A})''$.

Define the dynamical entropy for the action α of Γ on $\mathcal{A}$.

According to the above definition of entropy, if $\gamma : C \to \mathcal{A}$ is a unital completely positive linear map of a finite dimensional C^*-algebra C, then the function $H_\phi(\alpha(\xi) \circ \gamma, \xi \in \Gamma(n))$ can be defined as a supremum of entropies for Abelian models of $(\mathcal{A}, \phi, \alpha(\xi) \circ \gamma, \quad \xi \in \Gamma(n))$.

Set up $H_n(\gamma) = H_\phi(\alpha(\xi) \circ \gamma, \xi \in \Gamma(n))$.

The properties of $H_n(\gamma)$ are as in [1, Proposition III.6]. We remind only some of them.

(a) Let $\theta_j : \mathcal{A}'_j \to \mathcal{A}_j$ be completely positive unital maps, then

$$H_\phi(\alpha(\xi_j) \circ \gamma \circ \theta_j, \xi_j \in \Gamma(n)) \leq H_\phi(\alpha(\xi) \circ \gamma, \xi \in \Gamma(n)).$$

Equality holds if $\mathcal{A}_j \subset \mathcal{A}'{}_j$ and θ_j is a conditional expectation for all $j = 1, \ldots, 2^n$.

(b) If $\theta : \mathcal{A} \to \mathcal{A}$ is a completely positive unital map with $\phi \circ \theta = \phi$, then

$$H_\phi(\theta \circ \alpha(\xi) \circ \gamma, \xi \in \Gamma(n)) \leq H_\phi(\alpha(\xi) \circ \gamma, \xi \in \Gamma(n)).$$

Equality holds if θ is an automorphism of $\mathcal{A}$.

(c) Let $L_1 = \{\alpha(\xi) \circ \gamma, \xi \in \Gamma(n)\}$ and L_2 be an another canal. Then

$$max\ \{H_\phi(L_1),\ H_\phi(L_2)\} \leq H_\phi(L_1 \bigcup L_2) \leq H_\phi(L_1)\ +\ H_\phi(L_2).$$

It follows from (c) that $H_{n+1}(\gamma) \leq 2H_n(\gamma)$.

Therefore, $\frac{1}{2^{n+1}} H_{n+1}(\gamma) \leq \frac{1}{2^n} H_n(\gamma)$, and there exists a limit

$$\lim_{n\to\infty} \frac{1}{2^n} H_n(\gamma) = h_{\phi,\alpha}(\gamma, \Gamma).$$

Definition 1. *The dynamical entropy of Γ-action α with respect to ϕ on a C^*-algebra $\mathcal{A}$ is*

$$h_\phi(\alpha, \Gamma) = \sup_\gamma h_{\phi,\alpha}(\gamma, \Gamma).$$

For reader's convenience sometime we will write $h_\phi(\alpha)$ instead of $h_\phi(\alpha, \Gamma)$. We formulate now the statements that generalize those from [1, 2] when $\Gamma = Z$. Our propositions can be proved by a similar method with Definition 1 being taken into account (see [4]).

Proposition 2. *Let ϕ be a pure state on a unital C^*-algebra $\mathcal{A}$, and α a ϕ-preserving Γ-action on $\mathcal{A}$. Then $h_\phi(\alpha, \Gamma) = 0$.*

Proposition 3. *Let ϕ be a state on a unital C^*-algebra $\mathcal{A}$, and α a ϕ-preserving Γ-action on $\mathcal{A}$. Suppose $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ such that there is an expectation $E : \mathcal{A} \to \mathcal{B}$ with $\phi \circ E = \phi$, and $\alpha(\xi)E = E\alpha(\xi)$, $\xi \in \Gamma$. Then $\alpha|_\mathcal{B}$ is an action of Γ on $\mathcal{B}$ and $h_\phi(\alpha|_\mathcal{B}, \Gamma) \leq h_\phi(\alpha, \Gamma)$.*

Proposition 4. *Let $\mathcal{A}$ be a C^*-algebra, ϕ a state, and α a ϕ-preserving action of Γ on $\mathcal{A}$. Let $\{\mathcal{A}_j\}_{j=1}^\infty$ be an increasing sequence of C^*-subalgebras such that the expectations $E_j : \mathcal{A} \to \mathcal{A}_j$ satisfy the following conditions:*

(i) $\alpha(\xi)E_j = E_j\alpha(\xi)$, $\xi \in \Gamma$, $j = 1, 2, \ldots$;

(ii) $E_{j+1}E_j = E_jE_{j+1} = E_j$, $j = 1, 2, \ldots$;

(iii) $E_j \to id_{\mathcal{A}}$ in the pointwise-norm topology.
Suppose that the norm closure of $\bigcup_j \mathcal{A}_j$ *is* $\mathcal{A}$. *Then* $\alpha|_{\mathcal{A}_j}$ *is an action on* $\mathcal{A}_j$ *for* $j \in \mathbf{N}$ *and*

$$h_\phi(\alpha, \Gamma) \le \liminf_j h_\phi(\alpha|_{\mathcal{A}_j}, \Gamma).$$

Moreover, if $\phi \circ E_j = \phi, j \in \mathbf{N}$, *then*

$$h_\phi(\alpha, \Gamma) = \lim_j h_\phi(\alpha|_{\mathcal{A}_j}, \Gamma).$$

Proposition 5. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be two* C^**-algebras with states* ϕ_1 *and* ϕ_2 *respectively. Let* α_i *be* Γ*-action on* $(\mathcal{A}_i, \phi_i)$ *such that* $\phi_i \circ \alpha_i = \phi_i$, $i = 1, 2$. *Then*

$$h_{\phi_1 \otimes \phi_2}(\alpha_1 \otimes \alpha_2, \Gamma) \ge h_{\phi_1}(\alpha_1, \Gamma) + h_{\phi_2}(\alpha_2, \Gamma).$$

Proposition 6. *Let* $\mathcal{A}$ *be a nuclear* C^**-algebra,* $\mathcal{A}_n$ *be finite dimensional* C^**-subalgebras,* α *and* ϕ *as above. Let* τ_n *be a sequence of completely positive unital maps* $\tau_n : \mathcal{A}_n \to \mathcal{A}$ *such that for suitable completely positive unital maps* $\sigma_n : \mathcal{A} \to \mathcal{A}_n$ *one has* $\tau_n \circ \sigma_n :\to id_{\mathcal{A}}$ *in the pointwise norm topology. Then*

$$\lim_{n \to \infty} h_{\phi,\alpha}(\tau_n, \Gamma) = h_\phi(\alpha, \Gamma).$$

Proposition 7. *Let* $\mathcal{A}$ *be a nuclear* C^**-algebra,* α *and* ϕ *as above. Let* $M = \pi_\phi(\mathcal{A})''$. *Then*

$$\sup_\gamma h_{\phi,\alpha}(\gamma, \Gamma) = \sup_N h_{\phi,\alpha}(N, \Gamma),$$

where N *runs through all the finite dimensional subalgebras of* M.

Proposition 8. *Let* M, ϕ, α *be as above. Let* N_k *be an ascending sequence of finite dimensional von Neumann algebras with* $\bigcup_k N_k$ *weakly dense in* M. *Then*

$$h_\phi(\alpha, \Gamma) = \lim_{k \to \infty} h_{\phi,\alpha}(N_k, \Gamma).$$

Proposition 9. *For all automorphisms* σ *of* $\mathcal{A}$ *(respectively* M*) we have*

$$h_\phi(\alpha, \Gamma) = h_{\phi \circ \sigma}(\sigma^{-1} \circ \alpha \circ \sigma, \Gamma),$$

where $\alpha \in Aut(\mathcal{A})$.

Proposition 10.

$$h_\phi(\alpha, \Gamma_n) = 2^n h_\phi(\alpha, \Gamma).$$

Proposition 11. *For all* $0 \le \lambda \le 1$ *we have*

$$h_{\lambda\phi_1 + (1-\lambda)\phi_2}(\alpha, \Gamma) = \lambda h_{\phi_1}(\alpha, \Gamma) + (1 - \lambda) h_{\phi_2}(\alpha, \Gamma).$$

Proposition 12. *Let $D \subset \Gamma$ be a subgroup of the group Γ, ϕ and α as above. Then*

$$h_\phi(\alpha, \Gamma) \le h_\phi(\alpha, D).$$

P r o o f. Let $D_n = D \bigcap \Gamma(n)$. Then $D_n \in \Gamma(n)$ is a subgroup of the group $\Gamma(n)$. Let $2^{k_n} = |\Gamma(n) : D_n|$ be an index of D_n in $\Gamma(n)$, and $\Gamma(n) = \bigcup_{i=1}^{2^{k_n}} g_i D_n, g_i \in \Gamma$ a decomposition into co-sets. If $\gamma : C \to \mathcal{A}$ be a unital completely positive linear unitar map of finite dimensional C^*-algebra C, then

$H_\phi(\alpha(\xi) \circ \gamma, \xi \in \Gamma(n)) = H_\phi(\alpha(\xi) \circ \gamma, \xi \in \bigcup_{i=1}^{2^{k_n}} g_i D_n) \le 2^{k_n} H_\phi(\alpha(\xi) \circ \gamma, \xi \in D_n)$.

The inequality is due to properties of $H_n(\gamma)$. Therefore

$$\frac{1}{2^n} H_\phi(\alpha(\xi) \circ \gamma, \xi \in \Gamma(n)) \le \frac{1}{2^{n-k_n}} H_\phi(\alpha(\xi) \circ \gamma, \xi \in D_n).$$

That is why

$$h_{\phi,\alpha}(\gamma, \Gamma) \le h_{\phi,\alpha}(\gamma, D)$$

and hence

$$h_\phi(\gamma, \Gamma) \le h_\phi(\gamma, D).$$

■

Proposition 13. *Let $D \subset \Gamma$ be a subgroup of Γ such that $\Gamma = D \times D'$, where D' is also a subgroup of Γ, and ϕ, α as above. If $h_\phi(\alpha, D) < \infty$ and $card D' = \infty$ then $h_\phi(\alpha, \Gamma) = 0$.*

P r o o f. Let D'_n be any subgroup of finite index in D' such that $|D' : D'_n| = 2^n$. Then $|\Gamma : \Gamma^n| = 2^n$, where $\Gamma^n = D \times D'_n$. In view of Proposition 10

$$h_\phi(\alpha, \Gamma^n) = 2^n h_\phi(\alpha, \Gamma).$$

As far as $D \subset \Gamma^n$ then by a virtue of Proposition 12 we have

$$h_\phi(\alpha, \Gamma^n) \le h_\phi(\alpha, D).$$

Therefore

$$2^n h_\phi(\alpha, \Gamma) \le h_\phi(\alpha, D)$$

that is $h_\phi(\alpha, \Gamma) \le \frac{1}{2^n} h_\phi(\alpha, D)$.

Since n it take arbitrary, $h_\phi(\alpha, \Gamma) = 0$. ■

3. We remind some definitions concerning the CAR-algebra. Let H be a Hilbert space. The CAR-algebra $\mathcal{A}(H)$ is a C^*-algebra with the property that there is a linear map $f \mapsto a(f)$ of H into $\mathcal{A}(H)$ whose range generates $\mathcal{A}(H)$ as a C^*-algebra, and satisfyies the canonical anticommutation relations

$$a(f)a(g)^* + a(g)^*a(f) = (f,g)I,$$

$$a(f)a(g) + a(g)a(f) = 0, \quad f,g \in H$$

where $(.,.)$ is the inner product on H, and I is the unit of $\mathcal{A}(H)$. Let $0 \leq A \leq I$ be an operator on H. The quasifree state ω_A on $\mathcal{A}(H)$ is defined by its values on products of the form $a(f_n)^* ... \ a(f_1)^* a(g_1) \ ... \ a(g_m)$, n, $m \in \mathbf{N}$, given by

$$\omega_A(a(f_n)^* ... \ a(f_1)^* a(g_1) \ ... \ a(g_m)) = \delta_{nm} \det((Ag_i, f_j)).$$

If U is a unitary representation of Γ in H such that $U(\xi)A = AU(\xi)$, $\xi \in \Gamma$, then U defines an action α of Γ on $\mathcal{A}(H)$ by

$$\alpha_{U(\xi)}(a(f)) = a(U(\xi)(f)), \ \xi \in \Gamma.$$

The action α_U is called a Bogoliubov action.

Let $\Gamma = Z/2Z \oplus Z/2Z \oplus \ldots, X$ be the dual group ($X = \hat{\Gamma}$), and μ a Haar measure on X. X can be realized as the space of infinite sequences $x = x_1 x_2 ...$, where $x_i = 0$ or 1, and μ a product measure on X that is $\mu = \prod_{i=1}^{\infty} \mu_i$ with μ_i being a measure on $X_i = \{0,1\}$ given by $\mu_i(0) = \mu_i(1) = \frac{1}{2}$.

Every element $\xi \in \Gamma$ is a sequence $\xi = (\xi_1, \xi_2, ...)$, with ξ_i zero or one, and only finitely many components ξ_i being non zero. Assosiate to every $\xi \in \Gamma$ a character χ_ξ of X:

$$\chi_\xi(x) = \prod_k (-1)^{x_k}, \ k \in \{t | \xi_t = 1\}.$$

If we have a unitary representation U of Γ in the Hilbert space H, then (see [5]) it can be disintegrated into irreducible one-dimensional representations such that $U = \int_X \oplus U_x d\nu(x)$, where ν is a Borel measure on X. Observe that we can consider ν as sum $\nu = \nu_a + \nu_s$, where ν_a is a Borel measure on X which is absolutely continuous with respect to the Haar measure μ on X, and ν_s is a singular measure. In accordance with this, the representation $\xi \to U(\xi)$ of Γ is a sum $U(\xi) = U_a(\xi) \oplus U_s(\xi)$, where U_a is the absolutely continuous part of U and U_s the singular part. Let us decompose the unitary representation U_a into irreducible one-dimensional representations such that $U_a = \int_X \oplus U_x d\nu_a(x)$. Then the Hilbert space H_a becomes the direct integral $H_a = \int_X \oplus H_x d\nu_a(x)$.

The function $m(U)(x) = dim H_x$ is cald the multiplicity function of the representation U_a.

Lemma 14. *Let $H = H_1 \oplus H_2$, A_i be an operator on H_i, $0 \le A_i \le I$, and U_i be a unitary representation of Γ in H_i, i = 1, 2. Suppose $A_i U_i(\xi) = U_i(\xi) A_i$, $\xi \in \Gamma$, $i = 1, 2$. Then we have*

$$h_{w_{A_1 \oplus A_2}}(\alpha_{U_1 \oplus U_2}, \Gamma) \ge h_{w_{A_1}}(\alpha_{U_1}, \Gamma).$$

P r o o f. Let $E : \mathcal{A}(H_1 \oplus H_2) \to \mathcal{A}(H_1) \otimes \mathcal{A}(H_2)_e$ be the expectation $E = \frac{1}{2}(id + \alpha_{I \oplus -I})$, where I denoting the identity on both H_1 and H_2, and $\mathcal{A}(H_2)_e$ the even CAR-algebra. Since $I \oplus -I$ commutes with $A_1 \oplus A_2$, $\alpha_{I \oplus -I}$ is $w_{A_1 \oplus A_2}-$preserving, as is E. Thus by Propositions 3 and 5 we have

$$\begin{aligned} h_{w_{A_1 \oplus A_2}}(\alpha_{U_1 \oplus U_2}) &\ge h_{w_{A_1 \oplus A_2}}(\alpha_{U_1 \oplus U_2}|_{\mathcal{A}(H_1) \otimes \mathcal{A}(H_2)_e}) \\ &= h_{w_{A_1} \otimes w_{A_2}}(\alpha_{U_1} \otimes \alpha_{U_2}|_{\mathcal{A}(H_1) \otimes \mathcal{A}(H_2)_e}) \\ &\ge h_{w_{A_1}}(\alpha_{U_1}|_{\mathcal{A}(H_1)}) + h_{w_{A_2}}(\alpha_{U_2}|_{\mathcal{A}(H_2)_e}) \\ &\ge h_{w_{A_1}}(\alpha_{U_1}|_{\mathcal{A}(H_1)}). \end{aligned}$$

■

Lemma 15. *Let U and V be unitary representations of Γ and $\lambda \in [0, 1]$. Then we have, identifying λ and λI,*

(i) If there is a unitary operator W such that $V(\xi) = WU(\xi)W^{-1}$, $\xi \in \Gamma$, then $h_{w_\lambda}(\alpha_U, \Gamma) = h_{w_\lambda}(\alpha_V, \Gamma)$.

(ii) If U and V h ave the same singular parts and $m(U) \ge m(V)$, then $h_{w_\lambda}(\alpha_U, \Gamma) \ge h_{w_\lambda}(\alpha_V, \Gamma)$.

Proof is obvious. ■

Lemma 16. *Let (U_n) be a sequence of unitary representations and U a unitary representation of Γ, all with absolutely continuous spectrum. Suppose $m(U_n)$ is an increasing sequence with pointwise limit $m(U)$. Then $h_{w_\lambda}(\alpha_{U_n}, \Gamma)$ is an increasing sequence and*

$$h_{w_\lambda}(\alpha_U, \Gamma) = \lim_{n \to \infty} h_{w_\lambda}(\alpha_{U_n}, \Gamma).$$

P r o o f. Since the singular part of each unitary n question is zero the assumption on the multiplicity functions implies that we may assume U acts in a Hilbert space H and $U_n = U|_{H_n}$, where $H_n \subset H_{n+1} \subset H$, H_n are invariant subspaces of H with respect to U and $H = \bigcup_{n=1}^{\infty} H_n$, $n \in \mathbf{N}$. Thus the Lemma follows from Propositions 3 and 5 and the fact that the projections onto the H_n define expectations on $\mathcal{A}(H)$ satisfying the conditions in the propositions. ■

We first study the case when the unitary representation U of Γ has nontrivial singular part U_s, then we turn the case $U = U_a$, and finally reach the utmost generality in considering $U = U_a \oplus U_s$.

Proposition 17. *Let $U = U_s$ be a unitary representation of Γ in H. Assume that P is a finite rank orthogonal projection onto a subspace of H and let $\epsilon > 0$ be given. Then there is $k_0 \in \mathbf{N}$ such that for each $k \geq k_0$ there exists a finite rank projection Q_k with the properties:*
(i) $\|(I - Q_k)U(\xi)P\| < \epsilon, \ \xi \in \Gamma(k)$;
(ii) $dim Q_k \ \leq 2^k \epsilon$.

Theorem 18. *Let $U = U_s$ be a unitary representation of Γ in H. If ϕ is a state on $\mathcal{A}(H)$ such that α_U is ϕ-preserving, then*

$$h_\phi(\alpha_U, \Gamma) = 0.$$

The proof of this theorem (see [4]) is based on Proposition 17 and the method involved is just the same as that used in the proof of theorem from [2].

Now we calculate the entropy for some special cases.

Let $H = L_2(X, \mu)$, $f_\xi(x) = \chi_\xi(x)$, where $\xi \in \Gamma$ and $\chi_\xi(x)$ being a character of X. The representation of Γ in the space H is denoted by

$$U_\zeta f_\xi(x) = \chi_\zeta(x) f_\xi(x) = f_{\xi+\zeta}(x),$$

where $\xi, \ \zeta \in \Gamma$. Then this representation determines the Bogoliubov action α of Γ :

$$\alpha_\zeta(a(f)) = a(U_\zeta f), \quad a(f) \in \mathcal{A}(H).$$

Let A =cI be an operator on H, where $0 < c < 1$, and w_A the assosiated quasifree state on $\mathcal{A}(H)$. It is clear that the action α_ζ, $\zeta \in \Gamma$, of Γ preserves this state.

Lemma 19. *Let H, $\mathcal{A}(H)$, α and A be as above. Then*

$$h_{w_A}(\alpha, \Gamma) = S(w_c) = \eta(c) + \eta(1 - c),$$

where $\eta(c) = -c \ log \ c$. Furthermore, the same formula holds for the restrictions of w_c and α on $\mathcal{A}(H)_e$.

Given $\beta \in \Gamma(n)$, we consider $H_\beta = L_2(X(\beta), \mu|_{X(\beta)}) \subset H$. Let U_β be the restriction of U onto H_β. Then we have the representation $\xi \to U_\beta(\xi)$ of Γ in H_β which determines the action α_β of Γ on $\mathcal{A}(H_\beta)$.

Lemma 20. *Let H_β, U_β, α_β and $A = cI$ be as above. Then*

$$h_{w_c}(\alpha_\beta,\ \Gamma) = \frac{1}{2^n} S(w_c) = \frac{1}{2^n}(\eta(c) + \eta(1-c)),$$

where $\eta(c) = -c \log c$. Furthermore, the same formula holds for the restrictions of w_c and α_β on $\mathcal{A}(H_\beta)_e$.

Now we calculate the entropy for actions of Γ on CAR-algebra in more intricate cases.

Lemma 21. *Let H_i be an infinite dimensional separable Hilbert space with I_i being the identity map and U_i a unitary representation of Γ in H_i such that for each $i = 1, 2, \ldots, r$ there are $n_i \in \mathbf{N}$ with U_i being unitarily equivalent to U_{β_i} (see Lemma 20) for some $\beta_i \in \Gamma(n_i)$. Let $U = \bigoplus_{i=1}^r U_i$ and $A = \bigoplus_{i=1}^r c_i I_i$, where $0 < c_i < 1$. Then we have the formula*

$$h_{w_A}(\alpha,\ \Gamma) = \sum_{i=1}^r \frac{1}{2^{n_i}} S(w_{c_i}) = \sum_{i=1}^r \frac{1}{2^{n_i}}(\eta(c_i) + \eta(1-c_i)).$$

Furthermore, the same formula holds for the restrictions of w_c and α on $\mathcal{A}(H)_e$, where $H = \oplus_{i=1}^r H_i$.

Remark 22. Suppose we are in the conditions of Lemma 21. Then since

$$\mu(X(\beta_i)) = \frac{1}{2^{n_i}},$$

the formula can be written as follows

$$h_{w_A}(\alpha,\ \Gamma) = \sum_{i=1}^r \mu(X(\beta_i))(\eta(c_i) + \eta(1-c_i)).$$

Lemma 23. *Let $\xi \to U(\xi)$ be a unitary representation of Γ in H with Lebesgue spectrum consisting of nonintersect subset $F_i \subset X$ such that $F_i = \bigcup_{\beta \in \Gamma(n_i)} X(\beta_i)$, where $i \in J \subset \mathbf{N}$ and $n_i \in \mathbf{N}$, $n_1 \leq n_2$. Let $H_i = L_2(F_i,\ \mu|_{F_i})$ considered as a subspace of $L_2(X,\ \mu)$, and write $U = \bigoplus_{i \in J} U_i$ with $U_i = U|_{H_i}$. Suppose U_i has constant finite multiplicity m_i, and let $0 \leq A_i \leq I$ act on H_i and commute with U_i. Writing $U_j = V_j \oplus \ldots \oplus V_j$ (m_j times) we assume $A_j = \bigoplus_{k=1}^{m_j} c_{jk} I_j$, where I_j is the identity on the space on which representation V_j acts. Let B_j denote the diagonal matrix*

$$B_j = \begin{pmatrix} c_{j1} & & 0 \\ & \ddots & \\ 0 & & c_{jm_j} \end{pmatrix}.$$

Then $A_j = B_j \otimes I_j$, and we have the formula

$$h_{w_A}(\alpha_U,\ \Gamma) = \sum_{j\in J} \mu(F_j) Tr_{m_j}(\eta(B_j) + \eta(I - B_j)),$$

where Tr_{m_j} is the usual trace on $M_{m_j}(\mathbf{C})$, $A = \bigoplus_j A_j$. Furthermore, the same formula holds for the restrictions of w_A on $\mathcal{A}(H)_e$.

P r o o f. We first assume J is finite, say J = 1, 2, ..., r. We may write

$$U = \underbrace{(V_1 \oplus ... \oplus V_1)}_{m_1} \oplus ... \oplus \underbrace{(V_r \oplus ... \oplus V_r)}_{m_r},$$

$$A = (c_{11}I_1 \oplus ... \oplus c_{1m_1}I_1) \oplus ... \oplus (c_{r1}I_r \oplus ... \oplus c_{rm_r}I_r).$$

In view of Remark 22 we have

$$h_{w_A}(\alpha_U) = \sum_{j=1}^{r} \mu(F_j) \sum_{i=1}^{m_j} (\eta(c_{ij}) + \eta(1 - c_{ij}))$$

$$= \sum_{j=1}^{r} \mu(F_j) Tr_{m_j}(\eta(B_j) + \eta(I - B_j)).$$

If J is infinite we may assume $J = \mathbf{N}$. Let $W_r = \bigoplus_{i=1}^{r} U_i$. Then $W_r A = A W_r$, so if Q_r is the orthogonal projection of H onto $\bigoplus_{j=1}^{r} H_j$ then the expectations E_r of $\mathcal{A}(H)$ onto $\mathcal{A}(\bigoplus_{j=1}^{r} H_j)$ defined by Q_r satiafying the conditions of Proposition 4. Thus by Proposition 4 the proof is complete. ■

Now we consider the case when a singular part U_s of U is absent.

Let $m(U)$ denotes the multiplicity function of U_a, that is $m(U) = m(U_a)$. Let $A = \lambda I$, $0 \le \lambda \le 1$, and $w_\lambda = w_{\lambda I}$.

Theorem 24. *If $U = U_a$ be a unitary representation of Γ in H and $0 \le \lambda \le 1$, then*

$$h_{w_\lambda}(\alpha_U, \Gamma) = (\eta(\lambda) + \eta(1-\lambda)) \int_X m(U)(x) d\mu(x),$$

where $\eta(l) = -l\ log\ l$, $0 \le l \le 1$.

P r o o f. The cases $\alpha = 0$ and $\alpha = 1$ can be treated as in the proof of [2, Theorem 6.1]. Let $0 < \lambda < 1$. We consider the entropy $h_{w_\lambda}(\alpha_U)$ as a map which assigns to the multiplicity $m(U)$ a quantity $t(m(U)) = \frac{1}{S(w_\lambda)} h_{w_\lambda}(\alpha_U(m(U)))$, where $S(w_\lambda) = \eta(\lambda) + \eta(1-\lambda)$. The remainder of the proof follows from

Lemma 25. *Let $\mathcal{S}$ be the additive semigroup of functions $f : X \to \mathbf{N}\bigcup 0$ which are measurable with respect to Haar measure on X, $\mathbf{1}$ be a constant function equal to 1 on X, and $T_n : \mathcal{S} \to \mathcal{S}$, $n \in \mathbf{N}$, given by $(T_n f)(x) = \sum_{g\in\Gamma(n)} f(g+x)$, $x \in X$. Suppose the map $t : \mathcal{S} \to \mathbf{R}^+$ satisfies the following conditions:*

(i) $t(n\mathbf{1}) = n$,

(ii) $f \le g \Rightarrow t(f) \le t(g)$,

(iii) $f_j \nearrow f \Rightarrow t(f_j) \nearrow t(f)$, $j \in \mathbf{N}$,

$(i\vee)$ $t(T_n f) = 2^n t(f)$,

$(\vee)$ $t(f) = t(g)$ *if* $f = g(mod\ 0)$ *with respect to the Haar measure.*

Then

$$t(f) = \int\limits_X f(x)d\mu(x).$$

If $\mathcal{S}$ consists of the multiplicity functions $f(x) = m(U)(x)$, one can prove that $t(m(U))$ satiafies the conditions (i) — $(\vee)$. For example, the condition $(i\vee)$ is the corollary of Proposition 10. Thus the formula holds when $U = U_a$. ■

Now we consider the general case.

The following lemma is a refinement of Lemma 21.

Lemma 26. *Let H_i be an infinite dimensional separable Hilbert space with I_i being the identity map and U_i unitary representation of Γ in H_i such that for each $i = 1, 2, \ldots, r$ there are $n_i \in \mathbf{N}$ that U_i is unitarily equivalent to $U_{\beta_i} \oplus ... \oplus U_{\beta_i}$ (m_i times) where U_{β_i} from Lemma 20 for some $\beta_i \in \Gamma(n_i)$. Let $U = \bigoplus_{i=1}^r U_i$ and $A = \bigoplus_{i=1}^r c_i I_i$, where $0 < c_i < 1$. Then we have the formula*

$$h_{w_A}(\alpha,\ \Gamma) = \sum_{i=1}^r \frac{m_i}{2^{n_i}} S(w_{c_i}) = \sum_{i=1}^r \frac{m_i}{2^{n_i}} (\eta(c_i) + \eta(1 - c_i)).$$

Furthermore, the same formula holds for the restrictions of w_c and α on $\mathcal{A}(H)_e$ where $H = \oplus_{i=1}^r H_i$.

Theorem 27. *Let U be a unitary representation of Γ in H with absolutely continuous part U_a acting in $H_a \subset H$ and singular part U_s acting in $H_s \subset H$. Let $A = A_a \oplus A_s$ commute with $U(\xi) = U_a(\xi) \oplus U_s(\xi)$, $\xi \in \Gamma$, $0 \le A \le I$. Assume A_a and U_a are as in Lemma 23. Then we have*

$$h_{w_A}(\alpha_U, \Gamma) \ = \ h_{w_{A_a}}(\alpha_{U_a}, \Gamma)$$

and $h_{w_{A_a}}(\alpha_{U_a}, \Gamma)$ is given by the formula in Lemma 23. Furthermore the same hold for the restrictions to $\mathcal{A}(H)_e$.

P r o o f. We may restrict our attention to the case when U_a consists of a finite number of disjoint cylinder subset of X. Futhermore, if the multiplicity of U_a on one of the cylinder subsets is infinite then both sides of the formula are infinite, hence we may assume each multiplicity to be finite.

If we can prove the lemma for U_a and A_a by the argument of Lemma 26, then the general case follows. So we assume $U_a = \bigoplus_{i=1}^r U_i$ and $A_a = \bigoplus_{i=1}^r c_i I_i$.

Assume $f_{i,\xi},\ \xi \in \Gamma$, is an orthogonal basis of H_i such that

$$(U_{i\zeta} f_{i,\xi})(x) = \pm f_{i,\xi}(x),\ \zeta \in \Gamma(n_i),$$

$$(U_{i\zeta} f_{i,\xi})(x) = f_{i,\xi+\zeta}(x),\ \zeta \in \Gamma_{n_i}.$$

Let

$$F = [f_{i,\xi}(x),\ \xi \in \Gamma[n_i, n_r],\ i = 1,\ 2, \ldots, r]$$

be a subspace of H.

For $n \in \mathbf{N}$ let $F_n = \bigvee_{\xi \in \Gamma(n)} U_a(\xi) F$. Then $\bigcup_{n=0}^{\infty} F_n$ is dense in H_a. Choose an increasing sequence (G_n) of finite dimensional subspaces of H_s with union dense in H_s. Then $\bigcup_n F_n \oplus G_n$ is dense in H, so by Proposition 8

$$h_{w_A}(\alpha_U) \ = \ \lim_n h_{w_A, \alpha_U}(\mathcal{A}(F_n \oplus G_n)).$$

Let

$$j_{F_n} : F_n \to H_a, \qquad\qquad j_{G_n} : G_n \to H_s,$$

$$j_n \ = \ j_{F_n} \oplus j_{G_n} : F_n \oplus G_n \to H$$

be the inclusion maps, and let

$$\alpha_{j_{F_n}} : \mathcal{A}(F_n) \to \mathcal{A}(H_a) \subset \mathcal{A}(H)$$

be the inclusion maps of the corresponding algebras. Fix $n \in \mathbf{N}$ and let P_n be the orthogonal projection onto G_n. Let $\epsilon > 0$, $k_0 \in \mathbf{N}$ and $Q_{k'}$ for $k' \geq k_0$ be as in Proposition 17. Let $k = n + k'$. Denote by *pol* $(Q_k U_s(\delta) j_{G_n})$ the partial isometry appearing in the polar decomposition of $Q_k U_s(\delta) j_{G_n}$. Then

$$||U_a(\delta) j_{F_n} \oplus U_s(\delta) j_{G_n} \ - \ U_a(\delta) j_{F_n} \oplus pol\ (Q_k U_s(\delta) j_{G_n})||$$

is smal, where $\delta \in \Gamma(k)$. Therefore, we can by [1, Proposition IV.3] assume

$$H_{w_A}(\alpha_{U(\delta)} \circ \alpha_{j_n},\ \delta \in \Gamma(k))$$

$$\leq 2^k\epsilon \ + \ H_{w_A}(\alpha_{U_a(\delta)j_{F_n}\oplus pol(Q_kU_s(\delta)j_{G_n})}, \ \delta \in \Gamma(k)).$$

Let $v : Q_k(H_s) \to H_s$ be the inclusion, and let $i_n : F_n \to F_n$ be the identity map. Then we have

$$\alpha_{U_a(\delta)j_{F_n}\oplus pol(Q_kU_s(\delta)j_{G_n})} \ = \ \alpha_{U_a(\delta)\oplus v} \circ \alpha_{i_n\oplus pol(Q_kU_s(\delta)j_{G_n})}.$$

It follows from properties of $H_n(\gamma)$ that

$$H_{w_A}(\alpha_{U_a(\delta)j_{F_n}\oplus pol(Q_kU_s(\delta)j_{G_n})}, \ \delta \in \Gamma(k))$$

$$\leq H_{w_A}(\alpha_{U_a(\delta)j_{F_n}\oplus v})$$

$$= H_{w_A}(\alpha_{U_a(\delta)\oplus I_s} \circ \alpha_{j_{F_n}\oplus v}, \ \delta \in \Gamma(k)),$$

where I_s is the identity on H_s. We may as in [1] identity $\alpha_{j_{F_n}\oplus v}$ with $M_n = \mathcal{A}(F_n \oplus Q_k(H_s))$. Then the last expression in above inequality becomes

$$H_{w_A}(\alpha_{U_a(\delta)\oplus I_s}(M_n), \ \delta \in \Gamma(k)).$$

Let $Z = \bigoplus_{\delta\in\Gamma(k)} U_a(\delta)F_n$. Then $\mathcal{A}(Z)$ is a factor. Moreover, subspace Z is invariant with respect to A. Therefore Z has an orthonormal basis of eigenvectors for A and we have

$$w_A \ = \ w|_{\mathcal{A}(Z)} \otimes w|_{\mathcal{A}(Z)^o},$$

where $\mathcal{A}(Z)^o$ is a relative commutant $\mathcal{A}(Z)$ in $\mathcal{A}(H)$. Let $M = \mathcal{A}(Z \oplus Q_k(H_s))$ then $\mathcal{A}(Z)^o \bigcap M \simeq \mathcal{A}(Q_k(H_s))$. Since $\mathcal{A}(Z)$ is the tensor product with w_A a product state 2^k copies of $\mathcal{A}(F)$ we have, since $dim \ \mathcal{A}(Q_k(H_s)) \leq 2^{2^k\epsilon}$,

$$\begin{aligned} S(w_A|_M) \ &= \ 2^kS(w_A|_{\mathcal{A}(F)}) \ + \ S(w_A|_{\mathcal{A}(Z)^o\bigcap M}) \\ &\leq \ 2^kS(w_A|_{\mathcal{A}(F)}) \ + \ 2^k\epsilon \ log \ 2. \end{aligned}$$

Since M contains $\alpha_{U_a(\delta)\oplus I_s}(M_n), \ \delta \in \Gamma(k)$, as subalgebras,

$$H_{w_A}(\alpha_{U_a(\delta)\oplus I_s}(M_n), \ \delta \in \Gamma(k))$$

$$\leq 2^kS(w_A|_{\mathcal{A}(F)}) \ + \ 2^k\epsilon \ log \ 2.$$

We thus have, going back in the proof

$$\frac{1}{2^k}H_{w_A}(\alpha_{U(\delta)} \circ \alpha_{j_n}, \ \delta \in \Gamma(k))$$

$$\leq \epsilon \ + \ S(w_A|_{\mathcal{A}(F)}) \ + \ \epsilon \ log \ 2.$$

We therefore conclude that

$$h_{w_A,\alpha_U}(\alpha_{j_n}) \leq S(w_A|_{\mathcal{A}(F)}),$$

whence

$$h_{w_A}(\alpha_U) \leq S(w_A|_{\mathcal{A}(F)}).$$

But view of Lemma 26 (or Lemma 21) we have

$$h_{w_{A_a}}(\alpha_{U_a}) \ = \ S(w_A|_{\mathcal{A}(F)}).$$

By Lemma 14, $h_{w_{A_a}}(\alpha_{U_a}) \leq h_{w_A}(\alpha_U)$, hence they are equel and

$$h_{w_A}(\alpha_U) = S(w_A|_{\mathcal{A}(F)}) = h_{w_{A_a}}(\alpha_{U_a}).$$

To see that the entropy is the same for the restriction to the even algebra $\mathcal{A}(H)_e$ we know by Lemma 14 that

$$h_{w_{A_a}|_{\mathcal{A}(H)_e}}(\alpha_{U_a}|_{\mathcal{A}(H)_e}) \leq h_{w_A|_{\mathcal{A}(H)_e}}(\alpha_U|_{\mathcal{A}(H)_e})$$
$$\leq h_{w_A}(\alpha_U) = h_{w_{A_a}}(\alpha_{U_a}).$$

But by Lemma 23 (for restriction) we have

$$h_{w_{A_a}|_{\mathcal{A}(H)_e}}(\alpha_{U_a}|_{\mathcal{A}(H)_e}) = h_{w_{A_a}}(\alpha_{U_a}).$$

Thus the proof is complete. ■

Theorem 28. *With $U = U_a \oplus U_s$ a unitary representation of Γ in the Hilbert space $H = H_a \oplus H_s$ and $0 \leq \lambda \leq 1$, we have*

$$h_{w_\lambda}(\alpha_U, \ \Gamma) \ = \ (\eta(\lambda) \ + \ \eta(1-\lambda)) \int_X m(U)(x) d\mu(x),$$

where $m(U) = m(U_a)$.

The proof of theorem is cumbersome. We skip it but the idea is as in [2].

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Энтропия действий некоторого класса групп и ее вычисление

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Изучено динамическую энтропию действий групп $\oplus Z_k$, $k = 2, 3, \dots$, на C^*-алгебрах, которая является обобщением энтропии Конна–Нарнхофер–Тирринга действий групп, не являющихся свободными на C^*-алгебре. Исследованы свойства такой энтропии и получена формула для квантовой динамической энтропии боголюбовского действия групп $\oplus Z_k$, $k = 2, 3, \dots$, на CAR-алгебре. Доказано, что часть действия, соответствующая сингулярному спектру, дает нулевой вклад в энтропию.

Ентропія дій деякого класу груп та її обчислення

В.М. Олексенко

Вивчено динамічну ентропію дій груп $\oplus Z_k$, $k = 2, 3, \dots$, на C^*-алгебрах, яка є узагальненням ентропії Кона–Нарнхофер–Тіррінга дій груп, що не є вільними на C^*-алгебрі. Досліджено властивості такої ентропії і одержано формулу для квантової динамічної ентропії боголюбовської дії груп $\oplus Z_k$, $k = 2, 3, \dots$, на CAR-алгебрі. Доведено, що частина дії, яка відповідає сингулярному спектру, дає нульовий внесок в ентропію.