Matematicheskaya fizika, analiz, geometriya
2001, v. 8, No. 3, p. 239–250

Almost periodic solutions of functional equations

V.V. Brytik

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail:vladimir@valbri.kharkov.ua

Received May 28, 2001
Communicated by I.V. Ostrovskii

We proved two theorems about almost periodic continuous solutions $w(z)$ of the equation $F(z,w)=0$, where $F(z,w)$ is an analytic function in $P=\{(z,w): z\in S, |w|\le c\}$ and almost periodic in $z\in S$.

1. Introduction

In 1938 H. Bohr, D.Flanders [1] and V.V. Brytik, S.Yu. Favorov [2] in 2000 proved that a continuous in the strip $S=\{z\in \mathrm{C}: a<\mathrm{Im} z<b\}$ (a can be $-\infty$ and b can be $+\infty$) solution $w(z)$ of equation

$$a_m(z)w^m + a_{m-1}(z)w^{m-1} + \cdots + a_1(z)w + a_0(z) = 0 \tag{1}$$

with analytic almost periodic coefficients $a_j(z)$, $j=0,\ldots,m$, in a strip S is almost periodic. However in [1] this result was obtained with following conditions: *a*) $a_m(z)\equiv 1$, *b*) the discriminant $D(z)$ of (1) does not vanish; this result was proved without any restrictions in [2].

Note that the assumption for $a_j(z)$ to be analytic in S is important. A. Walther [7] constructed a continuous almost periodic function $b(x)$, $x\in\mathbf{R}$, such that every continuous solution of the equation $w^2-b(x)=0$ is not almost periodic.

R.H. Cameron [3] obtains sufficient conditions under which any continuous solution $t(x)$, $x\in\mathbf{R}$, of the equation $F(x,t)=0$, where $F(x,t)$ is almost periodic function in x, $x\in\mathbf{R}$, is also almost periodic. The main condition is $F'_t(x,t)>0$ on the set $\{(x,t(x)): x\in\mathbf{R}\}$.

In the present paper we prove two theorems about almost periodic continuous solutions $w(z)$ of the equation $F(z,w)=0$, where $F(z,w)$ is an analytic function

Mathematics Subject Classififcations 2000: 43A60.
The research was supported by INTAS-99-00089.

in $P = \{(z,w) : z \in S, |w| \leq c\}$ and almost periodic in $z \in S$. In the first theorem we assume some restrictions on the function $F'_w(z,w)$ on the set $\{(z,w(z)) : z \in S\}$.(However this condition is essentially weaker than Cameron's). In the second theorem we assume that the zeros of $F(z,w)$ and $F'_w(z,w)$ are asymptotically coprime. These results generalize the result from [2].

Recall that a function $f(z)$ is said to be *almost periodic in the real axis* R if $f(z)$ belongs to the closure of the set of finite exponential sums

$$\sum a_n e^{i\lambda_n z}, \;\; a_n \in \mathrm{C}, \;\; \lambda_\mathrm{n} \in \mathrm{R}, \tag{2}$$

with respect to the topology of uniform convergence on R.

We write $S' \subset\subset S$ if $S' = \{z \in \mathrm{C} : a' < \mathrm{Im} z < b'\}$, $\quad a < a' < b' < b$. A function $f(z)$ is said to be *analytic almost periodic in a strip* S if $f(z)$ belongs to the closure of the set of sums (2) with respect to the topology of uniform convergence on every substrip $S' \subset\subset S$. The equivalent definitions are the following: the family $\{f(z+h)\}_{h\in\mathrm{R}}$ is a relative compact set with respect to the topology of uniform convergence on R (for almost periodic functions on the axis) or with respect to the topology of uniform convergence in every substrip $S' \subset\subset S$ (for analytic almost periodic functions).

By $AP(S)$ we denote the space of all analytic almost periodic functions in S equipped with the topology of uniform convergence in every substrip $S' \subset\subset S$; the zero set of a function $f \in AP(S)$ is denoted by $Z(f)$.

A function $F(z,w)$ is called *almost periodic in* $z \in S$*, uniformly in* $w \in \Omega \subset \mathbf{C}$, where Ω is bounded and closed set, if the family $\{F(z+h,w)\}$ is normal in $S' \times \Omega$ $S' \subset\subset S$ (for example, see [4]).

By $AP(S,\Omega)$ we denote the space of all analytic almost periodic functions on $S \times \Omega$ equipped with the topology of uniform convergence on every subset $S' \times \Omega$, $S' \subset\subset S$.

The function $\overline{F}(z,w)$ is called *limiting* for $F(z,w)$ $z \in S$, $w \in \Omega$ if there exists a sequence $\{h_n\}$, $h_n \in \mathbf{R}$ such that $F(z+h_n,w)$ converges to $\overline{F}(z,w)$ in $AP(S,\Omega)$.

By $B(0,r)$ we denote an open disk in $\mathbf{C}$ of radius r, centered at 0. By $\overline{B}(0,r)$ we denote a closed disk in $\mathbf{C}$ of radius r, centered at 0.

2. Lemma

We will need the following simple lemma on implicit function.

Lemma. *Let* $f(w) = 0$ *at a point* $w \in \Omega$. *Let* $\tilde{w}$ *be a point of the set* Ω. *Suppose that* $\sup\limits_{\xi \in [w,\tilde{w}]} |f''(\xi)| \leq B$. *If* $\varepsilon > 0$ *satisfies:*

1) $|f'(w)| \geq 4B\varepsilon$,

2) $|f(\tilde{w})| \leq 3B\varepsilon^2$.

Then either $|w-\tilde{w}| \le \varepsilon$ *or* $|w-\tilde{w}| \ge 3\varepsilon$.

P r o o f o f L e m m a. We have

$$0 = f(w) = (w-\tilde{w})f'(w) + \int_{\tilde{w}}^{w} [f'(\xi) - f'(w)]d\xi + f(\tilde{w}).$$

Therefore,

$$|w-\tilde{w}|4B\varepsilon \le |w-\tilde{w}| \max_{\xi\in[w,\tilde{w}]} |\int_{\xi}^{w} f''(\xi)d\xi| + 3B\varepsilon^2 . \qquad (3)$$

Let us consider the polynomial

$$BX^2 - 4B\varepsilon X + 3B\varepsilon^2 .$$

Let $X_1 = \varepsilon$,$X_2 = 3\varepsilon$ be the roots of this equation. From (3) we deduce that this polynomial is nonnegative as $X = |w-\tilde{w}|$. Hence, $|w-\tilde{w}|$ takes values off the segment $[\varepsilon, 3\varepsilon]$. This proves the lemma.

3. Theorem 1

Theorem 1. *Let* $w(z)$ *be an analytic and bounded* ($|w(z)| < \tilde{c}$) *function in* S. *Suppose* $w(s)$ *satisfies the equation* $F(z, w(z)) \equiv 0$. *Let* $F(z,w)$ *be an analytic function in* $P = \{(z,w) : |w| \le c, z \in S\}$, $c > \tilde{c}$ *with the following properties:*

1) $F(z,w)$ *is a.p. in* z *uniformly in* w *for all* w *in* $|w| \le c$*;*
2) $F'_w(z, w(z)) \in AP(S)$*;*
3) $F'_w(z, w(z)) \not\equiv 0$.

Then $w(z) \in AP(S)$.

P r o o f o f T h e o r e m 1. We set $\tau = \dfrac{c-\tilde{c}}{2}$. Since $F(z,w)$ is a. p. in z uniformly in w for all w in $|w| \le c$, $F(z,w)$ is bounded in P (for example, see [4, p. 52]).

It follows from properties of an analytic function that all derivatives are uniformly bounded in a smaller domain, in particular in $P_1 = \{(z,w) : |w| \le \tilde{c} + \frac{3}{4}\tau, z \in S_1\}$. It is easy to see that $F'_w(z,w)$ is uniformly continuous in w uniformly for all z in a smaller domain. We will show that for an arbitrary sequence $\{h_n\} \subset$ R there exists a subsequence $\{h_{n'}\}$ such that $w(z + h_{n'})$ is a Cauchy sequence in the space $AP(S)$. It is sufficient to check that these functions converge uniformly on each substrip $S_0 \subset\subset S_1 \subset\subset S$.

Without loss of generality, we may assume that the sequence $F(z+h_n,w)$ converges to the function $\overline{F}(z,w)$ uniformly in z and w in P_1 and $F'_w(z+h_n,w(z+h_n))$ converges to $\Phi(z)$ in the space $AP(S)$.

Since the function $\Phi(z)$ belongs to $AP(S)$, the number of its zeros inside a rectangle $\{z \in S_1 : |\mathrm{Re} z - t| < 1\}$ is bounded by a number K independent of $t \in \mathbf{R}$ (see [5]). We denote by $Z(\Phi)$ the zero set of $\Phi(z)$ in S. Let U_r be the r-neighborhood of the set $Z(\Phi)\bigcap S_1$. We claim that for sufficiently small r there exist closed rectangles $\Pi_l = \{z : c_l \le \mathrm{Re} z \le c'_l,\ d_l \le \mathrm{Im} z \le d'_l\}$ such that $S_0 \subset \bigcup\limits_l \Pi_l \subset S_1$ and $\partial\Pi_l$ disjoint from U_r for all $l \in \mathrm{N}$ (see, for example, [2]).

It follows from the properties of analytic almost periodic functions (see [5]) that $|\Phi(z)| > m$ for z in $M = \{z : z \in S_1 \setminus U_r\}$, where m is a strictly positive constant. Hence for $n \ge N$ and $z \in S_1 \setminus U_r$, we have $|F'_w(z+h_n,w(z+h_n))| \ge m$. In other words, $|F'_w(z,w(z))| \ge m$ in $z \in \{M+h_n; n > N\}$.

Suppose $z_0 \in \bigcup\limits_l \partial\Pi_l$ and $|\Phi(z_0)| > m$. Then we get $|F'_w(z_0+h_n,w(z_0+h_n))| > m$ for large enough n. Since $w(z)$ is bounded, we can assume without loss of the generality that the sequence $\{w(z_0+h_n)\}$ converges.

Let $B = \sup|F''_{ww}(z,w)|$ for $(z,w) \in P_1$. We show that for arbitrary $\varepsilon < \dfrac{m}{4B}$ there exists N_1 such that for every $n,k > N_1$:

$$|w(z+h_n) - w(z+h_k)| < \varepsilon \quad for \quad z \in S_0.$$

For n,k large enough we have:

$$|w(z_0+h_n) - w(z_0+h_k)| < \varepsilon, \tag{4}$$

$$|F(z+h_n,w) - F(z+h_k,w)| < 3B\varepsilon^2. \tag{5}$$

Since $F(z+h_k,w(z+h_k)) = 0$, we see that $|F(z+h_n,w(z+h_k))| < 3B\varepsilon^2$. Using Lemma 2, we have two possibilities at every point of $\bigcup\limits_l \partial\Pi_l$:

$$|w(z+h_n) - w(z+h_k)| < \varepsilon \tag{6}$$

or

$$|w(z+h_n) - w(z+h_k)| > 3\varepsilon\,. \tag{7}$$

Inequality (6) holds for $z = z_0$. Since the set $\bigcup\limits_l \partial\Pi_l$ is connected and the function $w(z)$ is continuous, we see that (6) holds on this set. Using the maximum principle, we obtain that (6) is true for all $z \in \bigcup_l \Pi_l \supset S_0$. Since the inequality is true for large enough n,k, the proof is complete.

Corollary. *Let $w(z)$ be an analytic and bounded ($|w(z)| < \tilde{c}$) function in S. Let $F(z,w)$ be an analytic function in $P = \{(z,w) : |w| \le c, z \in S\}$, $c > \tilde{c}$ with the following properties:*

1) *$F(z,w)$ is a.p. in z uniformly in w for all w in $|w| \le c$;*

2) *$F(z,w(z)) \in AP(S)$;*

3) *$F'_w(z,w(z)) \in AP(S)$;*

4) *$F'_w(z,w(z)) \not\equiv 0$.*

Then $w(z) \in AP(S)$.

P r o o f o f C o r o l l a r y. Let $F(z,w(z)) = f(z)$. Denote $F_1(z,w) = F(z,w) - f(z)$, then ${F_1}'_w(z,w) = F'_w(z,w)$. We see that $F_1(z,w)$ satisfies conditions of Theorem 1. Therefore, $w(z)$ is almost periodic in S. This completes the proof.

R e m a r k. Note that the conditions of the Theorem 1 are necessary in certain sense for $w(z)$ to be almost periodic in S. In fact, if $F(z,w)$ is almost periodic in z uniformly in w for all w and $w(z)$ is also almost periodic in S, then $F(z,w(z))$ is almost periodic in S. Therefore, if $w(z) \in AP(S)$ is a solution of the equation $F(z,w) = 0$, then $F(z,w(z)) \equiv 0$ and $F^{(n)}_{w^n}(z,w(z)) \in AP(S)$ for all n. Let k be a maximum of such n that $F^{(n)}_{w^n}(z,w(z)) \equiv 0$ for all $n \le k$. Then conditions of the theorem hold for $F^{(k)}_{w^k}(z,w)$. The case $F^{(n)}_{w^n}(z,w(z)) \equiv 0$ for all n is impossible if $F(z,w) \not\equiv 0$. Indeed, since $F^{(n)}_{w^n}(z_0,w)|_{w(z_0)} = 0$, using uniqueness theorem we obtain $F(z_0,w) \equiv 0$. Consequently, $F(z,w) \equiv 0$ in P_1.

4. Theorem 2

Theorem 2. *Let $w(z)$ be an analytic and bounded ($|w(z)| < \tilde{c}$) function in S. Suppose $w(s)$ satisfies the equation $F(z,w(z)) \equiv 0$. Let $F(z,w)$ be an analytic function in $P = \{(z,w) : |w| \le c, z \in S\}$, $c > \tilde{c}$ with the following properties:*

1) *$F(z,w)$ is a.p. in z uniformly in w for all $w \in \overline{B}(0,c)$;*

2) *the zero sets of $F(z,w)$ and $F'_w(z,w)$ are coprime. And the same is true for all limiting functions $\overline{F}(z,w)$.**

Then $w(z) \in AP(S)$.

* This condition means that the map $(F(z,w), F'_w(z,w)) : S \times \{w : |w| < c\} \to \mathbf{C}^2$ is regular in the sense of [6].

P r o o f o f T h e o r e m 2. As above, we set $\tau = \dfrac{c - \tilde{c}}{2}$. We show that for an arbitrary sequence $\{h_n\} \subset \mathbf{R}$ there exists a subsequence $\{h_{n'}\}$ such that the functions $w(z + h_{n'})$ form a Cauchy sequence in the space $AP(S)$. It is sufficient to check that the sequence converges uniformly on each substrip $S_0 \subset\subset S_1 \subset\subset S_2 \subset\subset S$. We may assume that the sequence $F(z + h_n, w)$ converges to $\overline{F}(z, w)$ in $AP(S, B(0, c))$. Hence, the sequence $F'_w(z + h_n, w)$ converges to $\overline{F}'_w(z, w)$ in $AP(S, B(0, c))$.

It is easy to see that $F(z, w)$ and $F'_w(z, w)$, $\overline{F}(z, w)$ and $\overline{F}'_w(z, w)$ are uniformly continuous in $P_2 = \{(z, w) : z \in S_2, |w| \in B(0, \tilde{c} + \frac{3}{4}\tau)\}$. First note that at every point $z = z_0 \in S$ the equation $\overline{F}(z, w) = 0$ has a finite number of the roots in $w \in B(0, \tilde{c} + \frac{3}{4}\tau)$. Otherwise we would have $\overline{F}(z_0, w) \equiv \overline{F}'_w(z_0, w) \equiv 0$. This contradicts condition (2) of the theorem.

We call a point (z_0, w_0) to be "exceptional" if the following conditions hold:

1) $\overline{F}(z_0, w_0) = 0$;

2) $\overline{F}'_w(z_0, w_0) = 0$.

Let us check that the projections of the "exceptional" points from the set $P_3 = \{(z, w) : z \in S, |w| \in \overline{B}(0, \tilde{c} + \frac{3}{4}\tau)\}$ onto the plane $w = 0$ have no accumulation points in the interior of the strip S.

Suppose z_0 is an accumulation point. Let $\{w_i\}_{i=1}^k$ be the roots of the equation $\overline{F}(z_0, w) = 0$. In small enough neighborhoods of these points $U_{z_0, w_i} = \{(z, w) \in P : |z - z_0| < \delta_{w_i}, |w - w_i| < \mu_i\}$ we have the following representations, which is given by Weierstrass Preparation Theorem:

$$\begin{gathered}\overline{F}(z, w) = ((w - w_i)^l + c_1^i(z)(w - w_i)^{l-1} + \cdots + c_l^i(z))\Phi_i(z, w) \\ c_j^i(z_0) = 0 \;\; j = \overline{1...l}; \;\; \Phi_i(z, w) \neq 0 \;\; in \;\; U_{z_0, w_i}\,,\end{gathered} \tag{8}$$

$$\begin{gathered}\overline{F}'_w(z, w) = (l(w - w_i)^{l-1} + (l-1)c_1^i(z)(w - w_i)^{l-2} + \cdots + c_{l-1}^i(z))\Phi_i(z, w) \\ +((w - w_i)^l + c_1^i(z)(w - w_i)^{l-1} + \cdots + c_l^i(z))\Phi_i{}'_w(z, w)\,.\end{gathered} \tag{9}$$

Since $\overline{F}(z, w) = 0$, we see that the second term in (9) is equal to zero. It is readily seen that "exceptional" points in the neighborhood U_{z_0, w_i} are the points such that the following conditions are satisfied:

$$((w - w_i)^l + c_1^i(z)(w - w_i)^{l-1} + \cdots + c_l^i(z))) = 0. \tag{10}$$

$$(l(w - w_i)^{l-1} + (l-1)c_1^i(z)(w - w_i)^{l-2} + \cdots + c_{l-1}^i(z)) = 0. \tag{11}$$

This means that the projections of the "exceptional" points onto the plane $\{w = 0\}$ can be only at the points where the discriminant of the polynomial (10) is equal to zero. The number of these points inside $U^z_{z_0,w_i} = \{z : (z,w) \in U_{z_0,w_i}\}$ is finite, otherwise there would be an accumulation point. Hence discriminant is identically equal to zero and the zero sets of $\overline{F}(z,w)$ and $\overline{F}'_w(z,w)$ are not coprime. That is impossible.

Suppose $\overline{F}(z_0,w') \neq 0$ at a point (z_0,w'), $w' \in \overline{B}(0,\tilde{c}+\frac{3}{4}\tau)$. Then there exists a neighborhood $U_{z_0,w'} = \{(z,w) \in D : |z-z_0| < \delta_{z_0,w'}, |w-w'| < \mu_{z_0,w'}\}$ such that $\overline{F}(z,w) \neq 0$ in this neighborhood. In the same way we can consider the case $\overline{F}'_w(z_0,w\prime) \neq 0$ at a point $(z_0,w\prime)$. We can choose a finite covering $\{U_{z_0,w^i}\}$ from the covering $\overline{B}(0,\tilde{c}+\frac{3}{4}\tau)$ by the neighborhoods $U^w_{z_0,w_i} = \{w : (z,w) \in U_{z_0,w_i}\}$ and $U^w_{z_0,w'} = \{w : (z,w) \in U_{z_0,w'}\}$. If $\delta = \min \delta_{w^i}$, then there exists only a finite number of "exceptional" points in a $\delta-$neighborhood of the point z_0. This leads us to a contradiction with the assumption about z_0.

Suppose $S_1 \subset\subset S_2$. We show that the number of "exceptional" points of $\overline{F}(z,w)$ is bounded inside the rectangle $\{z \in S_1, w \in \overline{B}(0,\tilde{c}+\frac{1}{2}\tau) : |Rez - t| < 1\}$ by a constant P independent of t.

Indeed, if it were not true, we would have a sequence $\{x_n\}$ such that the number of "exceptional" points of $\overline{F}(z,w)$ in the set $\{z \in S_1, w \in \overline{B}(0,\tilde{c}+\frac{1}{2}\tau) : |x_n - t| < 1\}$ is at least n.

Consider the sequence $\overline{F}(z+x_n,w)$. Without loss of generality we can assume that $\overline{F}(z+x_n,w)$ converges to a function $\overline{\overline{F}}(z,w)$ in $AP(S,B(0,c))$. Therefore $\overline{F}'_w(z+x_n,w)$ converges to $\overline{\overline{F}}'_w(z,w)$ in $AP(S,B(0,c))$.

It is easy to see that there exists a sequence $\{h_k\}$ such that $\overline{\overline{F}}(z,w)$ is limiting for $F(z,w)$. Consequently, zero sets of $\overline{\overline{F}}(z,w)$ and $\overline{\overline{F}}'_w(z,w)$ are coprime. For the same reason, function $\overline{\overline{F}}(z,w)$, as well as $\overline{F}(z,w)$, has a finite number of the "exceptional" points in the set $C = \{(z,w) : z \in \overline{S}_1, |Rez| < 1, w \in \overline{B}(0,\tilde{c}+\frac{1}{2}\tau)\}$.

Let us show that this contradicts the choice of the sequence $\{x_k\}$. We consider points of the compact C of the following three types according to whether $\overline{\overline{F}}(z_0,w_0) \neq 0$, $\overline{\overline{F}}'_w(z_0,w_0) \neq 0$, or both $\overline{\overline{F}}(z_0,w_0) = 0$, $\overline{\overline{F}}'_w(z_0,w_0) = 0$.

Suppose $\overline{\overline{F}}(z_0,w_0) \neq 0$ or $\overline{\overline{F}}'_w(z_0,w_0) \neq 0$. It is not hard to prove that for n large enough there exists the neighborhood $U_{z_0,w_0} \subset \{(z,w) : z \in S_2, w \in B(0,\tilde{c}+\frac{3}{4}\tau)\}$ such that $\overline{F}(z+x_n,w)$ has no "exceptional" points in this neighborhood.

Let $\overline{\overline{F}}(z_0, w_0) = 0$ and $\overline{\overline{F}}'_w(z_0, w_0) = 0$. Then there exists a neighborhood $U_{z_0,w_0} \subset \{(z, w) : z \in S_2, w \in B(0, \tilde{c} + \frac{3}{4}\tau)\}$ such that we have

$$\begin{aligned}\overline{\overline{F}}(z, w) &= ((w - w_0)^l + c_1(z)(w - w_0)^{l-1} + \dots + c_l(z))\overline{\overline{\Phi}}(z, w) \\ &= \overline{\overline{P}}(z, w)\overline{\overline{\Phi}}(z, w),\ \overline{\overline{\Phi}}(z, w) \neq 0 \quad in \quad U_{z_0,w_0}. \end{aligned} \tag{12}$$

"Exceptional" points are roots of the discriminant of the (12). We recall the way we construct the coefficients of the polynomial (12). We can find r_0^w such that $\overline{\overline{F}}(z_0, w) \neq 0$ for $\{w : 0 < |w - w_0| \leq r_0^w\}$. Since $\overline{\overline{F}}(z, w)$ is continuous, there exists a circle $B(z_0, r_0^z)$ such that $\overline{\overline{F}}(z, w) \neq 0$ in the set $\{(z, w) : z \in B(z_0, r_0^z), |w - w_0| = r_0^w\}$. Suppose $z' \in B(z_0, r_0^z)$. Then the number of zeros of $\overline{\overline{F}}(z', w)$ in the set $z' \in B(w_0, r_0^w)$ is equal to

$$\frac{1}{2\pi i} \int\limits_{\partial B(w_0, r_0^w)} \frac{\overline{\overline{F}}'_w(z', w)}{\overline{\overline{F}}(z', w)} dw = l\,, \tag{13}$$
$$where \quad l \quad is \quad independent \quad of \quad z' \in B(z_0, r_0^z).$$

Since $\overline{F}(z + x_k, w)$ and $\overline{F}'_w(z + x_k, w)$ converge to $\overline{\overline{F}}(z, w)$ and $\overline{\overline{F}}'_w(z, w)$ respectively and $\overline{\overline{F}}(z', w) \neq 0$ on $\{(z', w) : w \in \partial B(w_0, r_0^w)\}$, we see that for k large enough:

$$\frac{1}{2\pi i} \int\limits_{\partial B(w_0, r_0^w)} \frac{\overline{F}'_w(z' + x_k, w)}{\overline{F}(z' + x_k, w)} dw = l\,,$$
$$where \quad l \quad is \quad independent \quad of \quad z' \in B(z_0, r_0^z).$$

In other words, for each $z' \in B(z_0, r_0^z)$ for k large enough the functions $\overline{F}(z' + x_k, w)$ and $\overline{\overline{F}}'_w(z', w)$ have the same number of zeros in the set $B(w_0, r_0^w)$.

We fix a point $z' \in B(z_0, r_0^z)$. Let

$$\begin{aligned} w_k^\nu &= w_k^\nu(z'), \qquad \nu = \overline{1 \dots l}, \\ w^\nu &= w^\nu(z'), \qquad \nu = \overline{1 \dots l}, \end{aligned}$$

be roots of $\overline{F}(z' + x_k, w)$ and $\overline{\overline{F}}'_w(z', w)$ respectively in $B(w_0, r_0^w)$. We consider the polynomials:

$$P_n(z, w) = \prod_{\nu=1}^{l} (w - w_n^\nu(z')) = ((w - w_0)^l + c_1^n(z')(w - w_0)^{l-1} + \dots + c_l^n(z')), \tag{14}$$

$$P(z,w)=\prod_{\nu=1}^{l}(w-w^{\nu}(z'))=((w-w_0)^l+c_1(z')(w-w_0)^{l-1}+\cdots+c_l(z')). \tag{15}$$

It easy to prove that the coefficients of these polynomials are analytic in the set $B(z_0,r_0^z)$. Using the argument principle for analytic function $\beta(w)$, we obtain:

$$\sum_{\nu=1}^{l}\beta(w_n^{\nu}(z'))=\frac{1}{2\pi i}\int\limits_{\partial B(w_0,r_0^w)}\beta(w)\frac{\overline{F}'_w(z'+x_k,w)}{\overline{F}(z'+x_k,w)}dw, \tag{16}$$

$$\sum_{\nu=1}^{l}\beta(w^{\nu}(z'))=\frac{1}{2\pi i}\int\limits_{\partial B(w_0,r_0^w)}\beta(w)\frac{\overline{\overline{F}}'_w(z',w)}{\overline{\overline{F}}(z',w)}dw. \tag{17}$$

Since $\overline{\overline{F}}(z,w)$ and $\overline{F}(z+x_n,w)$ are not equal to zero when $w\in\partial B(w_0,r_0^w)$, $z\in B(z_0,r_0^z)$, we see that the sums in the left part of the equalities (16) and (17) are analytic in $B(z_0,r_0^z)$.

Substituting w^{ν} for $\beta(w)$ $\nu=\overline{1\ldots l}$ in (16)and (17), we obtain that the sums of the $\nu-$degrees of the roots of (14) and (15) are analytic in $B(z_0,r_0^z)$. The coefficients of (14) and (15) are the polynomials of these sums. Since $\overline{\overline{F}}'_w(z',w)\neq 0$ in the set $w\in\partial B(w_0,r_0^w)$, from (16) and (17) we get that the sums of the $\nu-$degrees of the roots of (14) converge to $\nu-$degrees of the roots of (15) uniformly on $B(z_0,r_0^z)$. Therefore, $c_i^n(z)$ converges to $c_i(z)$ uniformly on $B(z_0,r_0^z)$. It can be easily be checked that functions

$$\overline{\overline{\Phi}}(z,w)=\frac{\overline{\overline{F}}(z,w)}{P(z,w)}\quad \textit{and}\quad \overline{\Phi}_n(z,w)=\frac{\overline{F}(z+x_n,w)}{P_n(z,w)}$$

are analytic in the polydisc $U_{z_0,w_0}=\{(z,w):|z-z_0|\le r_0^z,|w-w_0|\le w_0^w\}$ and are not equal to zero in this polydisc. From convergence of $c_i^n(z)$ to $c_i(z)$ it follows that the sequence of discriminants $D_n(z)$ in (14) converges to the discriminant $D(z)$ in (15) uniformly on $B(z_0,r_0^z)$. This implies that for large enough n $D_n(z)$ has the same number of roots as $D(z)$. Hence, the number of "exceptional" points of $\overline{F}(z+x_k,w)$ in U_{z_0,w_0} is uniformly bounded. Choosing a finite covering from the covering of the compact C by $U_{z,w}$, we obtain that the number of the "exceptional" points of $\overline{F}(z+x_k,w)$ in C is bounded. It obviously contradicts the assumption.

We show that there exists $m=m(r)$ such that for every z in $M=\{z\in S_1, w\in\overline{B}(0,\tilde{c}+\frac{1}{2}\tau),\rho(projection(z,w),projection\text{"}exceptional\text{"}points)>r\}$ the quality $\overline{F}(z,w)=0$ implies the inequality $|\overline{F}'_w(z,w)|>m$.

Assume for contradiction that there exists a sequence $(z_i, w_i) \in M$ such that the following conditions hold

$$\overline{F}(z_i, w_i) = 0,$$
$$|\overline{F}'_w(z_i, w_i)| < \frac{1}{2^i}.$$

We can assume without loss of generality that $y_i \to \tilde{y}$ and $w_i \to \tilde{w}$, $\overline{F}(z + x_i, w)$ and $\overline{F}'_w(z + x_i, w)$ converge to $\tilde{F}(z, w)$ and $\tilde{F}'_w(z, w)$ respectively. Since $\overline{F}(z, w)$ and $\overline{F}'_w(z, w)$ are uniformly continuous in P_2, we have

$$\tilde{F}(iy, \tilde{w}) = \lim_{i\to\infty} \overline{F}(iy_i + x_i, w_i) = 0,$$
$$\tilde{F}'_w(iy, \tilde{w}) = \lim_{i\to\infty} \overline{F}'_w(iy_i + x_i, w_i) = 0.$$

As above, we obtain that the functions $\overline{F}(z + x_k, w)$ for large enough k have "exceptional" points in the $\frac{r}{3}-$neighborhood of the point $(iy, \tilde{w})$. Since $y_k \to \tilde{y}$ and $w_k \to \tilde{w}$, we see that points (iy_k, w_k) are in the $\frac{r}{3}-$neighborhood of the point (iy_k, w_k). This contradicts the definition of M.

Since the number of the "exceptional" points of $\overline{F}(z, w)$ inside the rectangle $\{z \in S_1 : |\mathrm{Re} z - t| < 1\}$ is bounded by a number K independent of $t \in \mathbf{R}$, we claim that for sufficiently small r there exist closed rectangles $\Pi_l = \{z : c_l \le \mathrm{Re} z \le c'_l,\ d_l \le \mathrm{Im} z \le d'_l\}$ such that $S_0 \subset \bigcup_l \Pi_l \subset S_1$ and $\partial\Pi_l$ are disjoint from $r-$neighborhoods of the "exceptional" points for all $l \in N$ (for details see [2]).

Furthermore, since $F(z, w(z)) = 0$ and $\bigcup_l \partial\Pi_l \subset M$, we have an inequality $|F'_w(z, w(z))| > m$ at every point of $\bigcup_l \partial\Pi_l$.

From conditions of the theorem, we see that the function $w(z)$ is bounded. Therefore, without loss of generality it can be assumed that sequence $w(z + h_n)$ converges uniformly on every compact set to an analytic function $\overline{w}(z)$. Since $F(z + h_n, w)$ converges to $\overline{F}(z, w)$ in $AP(S, \Omega)$ and since the functions $F(z, w)$ and $\overline{F}(z, w)$ are uniformly continuous, we see that $\overline{F}(z, \overline{w}(z)) \equiv 0$.

We choose $z_0 \in S_0$ such that $|\overline{F}'_w(z_0, \overline{w}(z_0))| > m$. Then $|F'_w(z_0 + h_n, w(z_0 + h_n))| > m$ for n large enough .

Let us note that $|F'_w(z + h_n, w(z + h_n))| > m$ at every point of $\bigcup_l \partial\Pi_l - h_n$. To be definite, assume $z_0 \in \Pi_{l_{(n)}} - h_n$. Using the maximum principle for $F'_w(z + h_n, w(z + h_n))$, we see that the connected component of the set $\{z : |F'_w(z + h_n, w(z + h_n))| > m\}$, containing z_0, intersects $\partial\Pi_{l_n} - h_n$. Denote this component by E_n. It is clear that $\bigcup_l \partial\Pi_l - h_n \subset E_n$.

Let $B = \sup |F''_{ww}(z,w)|$ for $(z,w) \in P_2$. Arguing as at the end of the Theorem 1, we get that for n, k large enough inequality

$$|w(z + h_n) - w(z + h_k)| < \varepsilon, \tag{18}$$

holds for all $z \in E_n$. Using the maximum principle, we obtain that (18) is true for all $z \in S_0$. Since the inequality is true for large enough n, m. The proof is complete.

The author is grateful to S.Yu. Favorov for his guidance and constant attention to this work.

References

[1] *H. Bohr and D.A. Flanders*, Algebraic functions of almost-periodic functions. — *Duke Math* (1938), v. 4, p. 779–787.

[2] *V.V. Brytik and S.Yu. Favorov*, Solution of algebraic equations with almost-periodic coefficients. — *Mat. fiz., analiz, geom.* (2000), v. 7, No. 4, p. 380–386.

[3] *R.H. Cameron*, Implicit functions of almost periodic functions. — *Bull. Am. Math. Soc.* (1934), v. 40, p. 895–904.

[4] *C. Corduneanu*, Almost-periodic functions. — Interscience publishers, a division of John Wiley & Sons, New York, London, Sydney, Toronto. P. 1–236.

[5] *B.M. Levitan*, Almost-periodic functions. Gostechtheoretizdat, Moscow (1953) (Russian).

[6] *L.I. Ronkin*, Theorem of Jessen for analytic almost periodic mappings. — *Ukr. Math. J.* (1990), v. 42, p. 1094–1107 (Russian).

[7] *A. Walther*, Algebraische Funktionen von fastperiodischen Funktionen. — *Monatshefte fur Math. und Phys.* (1933), Bd. 40, p. 444–457.

Почти периодические решения функциональных уравнений

В.В. Бритик

Доказаны две теоремы о почти периодических непрерывных решениях $w(z)$ уравнения $F(z, w) = 0$, где $F(z, w)$ — аналитическая функция в $P = \{(z, w) : z \in S, |w| \leq c\}$ и почти периодическая по $z \in S$.

Майже періодичні розв'язки функціональних рівнянь

В.В. Бритік

Доведено дві теореми про майже періодичні неперервні розв'язки $w(z)$ функціональних рівнянь $F(z, w) = 0$, де $F(z, w)$ — голоморфна функція у $P = \{(z, w) : z \in S, |w| \leq c\}$ і також майже періодична по $z \in S$.