Matematicheskaya fizika, analiz, geometriya
2001, v. 8, No. 3, p. 272–281

Noether theorems for higher rank functionals

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Received December 27, 2000
Communicated by V.I. Korobov

A generalization of Noether theorems in variational calculus is considered. We deal here with functional of higher rank, with one and several independent variables. ·

1. Introduction

It is well known in analitycal mechanics fact that if the integral of the action is invariant under a one-parameter family of transformation then there exists an integral of motion (a conservation law) which corresponds to the family of transformations. For example, if the integral of the action is invariant under the time-translations, then the energy of the system is conservated. The invariance under translations along x-axis induces the conservation law of the x-component of the momentum, and invariance under rotations around z-axis — conservation of the z-component of the angular momentum. In other words, continous symmetries of a physical system give conservation laws [1, 2]. It is known as the Noether theorem in analitycal mechanics (First Noether Theorem).

In the field theory the conservation laws also can be derived from the Noether theorems [3–7]. The kind of symmetry considered in the field theory very often is the gauge symmetry (it is an example of local symmetry).

For example, from the gauge symmetry in electrodynamics one can get the conservation law of the charge. In general, the consequation of a local symmetry fot the integral of the action are:

1) dependence between the Euler–Lagrange equations,
2) additional equations.

·Mathematics Subject Classififcations 2000: 49S05.

These properties are known as the Second Noether Theorem.

Explicite formulas for Noether theorems are known for functionals of the first rank. However Noether in the herself original paper [8] considered functionals of higher rank.

The aim of this paper is to give compact formulas for the both Noether theorems for functionals of an arbitrary rank. The paper is organised as follows: in the second section we deal with the fundamental variation of a functional depending on functions of the independent variable, and in the third with Noether theorems.

2. The fundamental variation

In this paper will be used the common notation: for a multiindex

$$\alpha = (\alpha_1, \dots, \alpha_M)$$

we put

$$|\alpha| = \sum_{r=1}^{M} \alpha_r\,, \quad \frac{Dy^{|\alpha|}}{Dx^\alpha} = \frac{\partial y^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_M^{\alpha_M}}\,, \quad y^\alpha = \frac{Dy^{|\alpha|}}{Dx^\alpha}\,,$$

and for another multiindex β we put the binomial Newton symbol

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_M}{\beta_M}$$

as the product of binomial Newton symbols over all components.

We will consider functionals of the form

$$I[\Omega, y] = \int\limits_{\Omega} F(x, y, \dots, y^\alpha, \dots,)\, d^M x\,, \tag{1}$$

where Ω is an open set in R^M, bounded by $\partial\Omega$,

$$x = (x_1, \dots, x_M) \in \Omega\,, \quad d^M x = dx_1, \dots, dx_M\,,$$

and

$$y = (y_1, \dots, y_N) : \tilde{\Omega} \longrightarrow R^N$$

is a function of the class C_{2n}, $\Omega \subset \tilde{\Omega}$. The function F depends on partial derivatives $y^{(\alpha)}$ of the rank not higher then n.

We also consider a family of transformations of the variables $x_1, \dots, x_M$, $y_1, \dots, y_N$, such that the domain Ω of the integral (Eq. 1) is carried on a domain contained in $\tilde{\Omega}$, and the function F is defined for new variables and their derivations. Moreover we assume that for the new variables the integral over the domain exists.

Let a smooth vector field A on $\partial\Omega$, be given. Consider a map

$$F_A : \partial\Omega \longrightarrow R^M, \quad f_A(p) = p + A(p)\,. \tag{2}$$

Let $\tilde{T_\Omega}$ be an open tubular neighbourhood of $\partial\Omega$, and let $\tilde{A}$ be an extention of A on $\tilde{T_\Omega}$ with a compact support. So there exists $t_1 > 0$ such that the image of the map

$$\tilde{f}_{t\tilde{A}} : \tilde{T_\Omega} \longrightarrow R^M, \quad \tilde{f}_{t\tilde{A}}(p) = p + t\tilde{A}(p)$$

is equal to $\tilde{T_\Omega}$. Since for small t the map $\tilde{f}_{t\tilde{A}}$ is close to identity in the topology $C_1(\tilde{T_\Omega}, R^M)$ (the support of $\tilde{A}$ is compact set, and $\tilde{A}$ is a smooth field) then there exists $t_0 > 0$, such that the corestriction

$$\tilde{f}_{t\tilde{A}}\,|^{\tilde{T}_\Omega} : \tilde{T}_\Omega \longrightarrow \tilde{T}_\Omega$$

is a diffeomorphism for $|t| < t_0$.

Consequently the map

$$f_{tA} : \partial\Omega \longrightarrow R^M$$

is a diffeomorphism on its image for $|t| < t_0$.

Let Ω_{tA} be the domain bounded by the hypersurface $\partial\Omega_{tA} := f_{tA}(\partial\Omega)$, and let $h : \tilde{\Omega} \longrightarrow R^M$ be a smooth function. We consider a family of functionals of the form

$$I[\Omega_{tA}, y + th] = \int\limits_{\Omega_{tA}} F(x, y + th, \dots, y^\alpha + th^\alpha, \dots)\, d^M x\,. \tag{3}$$

Definition. *The first fundamental variation of the functional (Eq. 1) for a given vector field $A = (A_1, \dots, A_M)$ on Ω, and $h = (h_1, \dots, h_N)$ — field of the class C_{2n} on Ω, is the next expression*

$$\delta^*_{[\Omega,y]} I[A, h] = \frac{d}{dt} I[\Omega_{tA}, y + th]\,|_{t=0}\ . \tag{4}$$

Lemma. *The following equalities*

$$\delta^* I = \sum_r \int\limits_{\partial\Omega} F A_r\, d\sigma_r + \sum_{i,\alpha} \int\limits_{\Omega} \frac{\partial F}{\partial y_i^{(\alpha)}} h_i^{(\alpha)}\, d^M x\,, \tag{5}$$

and

$$\delta^* I = \sum_r \int_{\partial\Omega} \left[F A_r + \sum_{\alpha \le \beta - e_r} (-1)^{|\alpha|} \sum_i \left(\frac{D^{|\alpha|}}{Dx^\alpha} \frac{\partial F}{\partial y_i^{(\beta)}} \right) h_i^{(\beta-\alpha-e_r)} \right] d\sigma_r$$
$$+ \sum_i \int_\Omega \left[\sum_\beta (-1)^{|\beta|} \left(\frac{D^{|\beta|}}{Dx^\beta} \frac{\partial F}{\partial y_i^{(\beta)}} \right) \right] h_i d^M x \,, \qquad (6)$$

are fulfilled, where

$$d\sigma_r = (-1)^{r-1} dx_1 \wedge \ldots \wedge \hat{dx_r} \wedge \ldots \wedge dx_M \,.$$

P r o o f. Let us consider the difference

$$I[\Omega_{tA}, y + th] - I[\Omega, y] \,. \qquad (7)$$

At first, let us divide the hypersurface $\partial\Omega$ on pieces $\Delta\sigma$, and calculate the oriented volumen of the tubular formed by $\partial\sigma$ and vectors tA:

$$\nu(\Delta\Omega) = \int_0^t \left[\int_{\Delta\sigma_r} < A, d\sigma > \, d\tau \right] ,$$

where

$$\Delta\tau_t = f_{\tau A}(\Delta\sigma) \,.$$

From the theorem about average value we have

$$\nu(\Delta\Omega) = t \int_{\Delta\sigma'_\tau} < A, d\sigma > \,.$$

Let us divide the difference (7) into two parts

$$I[\Omega_{tA}, y + th] - I[\Omega, y] = \delta_1 + \delta_2 \,,$$

where

$$\delta_1 = \int_{\Omega_{tA} \div \Omega} \tilde{F}(x) \nu(x) \, d^M x$$

for

$$\tilde{F}(x) = F(x, y(x) + th(x), \dots , y^{\alpha}(x) + th^{\alpha}(x))\,,$$

$$\nu(x) = \begin{cases} \ \ 1 & \text{for} \ \ x \in \Omega\backslash\Omega_{tA} \\ -1 & \text{for} \ \ x \in \Omega_{tA}\backslash\Omega \end{cases} \ ,$$

and

$$\delta_2 = \int\limits_{\Omega} [F(x, y + th, \dots , y^{\alpha} + th^{\alpha}) - F(x, y, \dots , y^{\alpha})]\, d^M x\,.$$

Using anew the theorem about average value, we have

$$\delta_1 = \sum_{\Delta\sigma} \int\limits_{\Delta\Omega} \tilde{F}(x)\nu(x)\, d^M x = \sum_{\Delta\sigma} \nu(\Delta\Omega)\tilde{F}(x'_{\Delta\Omega}) = t \sum_{\Delta\sigma} \tilde{F}(x'_{\Delta\Omega}) \int\limits_{\Delta\sigma'_{\tau}} < A, d\sigma > \ .$$

So

$$\lim_{t\to 0} \frac{1}{t}\delta_1 = \int\limits_{\partial\Omega} F < A, d\sigma > \ . \tag{8}$$

The formula

$$\lim_{t\to 0} \frac{1}{t}\delta_2 = \int\limits_{\Omega} \frac{\partial F}{\partial y_i^{(\alpha)}} h_i^{(\alpha)}\, d^M x \tag{9}$$

is clear. From the Equations (8), (9) we obtain the Equation (5). The Equation (6) we can get using the n-dimensional theorem about integration by parts

$$\int\limits_{\Omega} g \frac{\partial f}{\partial x_r}\, d^M x = \int\limits_{\partial\Omega} gf\, d\sigma_r - \int\limits_{\Omega} f \frac{\partial g}{\partial x_r}\, d^M x\,.$$

■

Let N_y be an open neighbourhood of the graph of the function $y : \Omega \longrightarrow R^N$. Let us consider a smooth one parameter family of transformation of the independent and dependent variables

$$\Psi : N_y \times I \longrightarrow R^{n+m}, \quad \Psi(x, y, u) = (x^*, y^*)\,, \tag{10}$$

where $I = (-a, a)$ for an $a > 0$, such that

$$\Psi(x, y, 0) = (x, y)\,.$$

In the coordinates the transformation (10) can be expressed by formulas

$$x_r^*(x, y, u) = \phi_r(x, y, u), \quad y_i^*(x, y, u) = \psi_i(x, y, u)\,,$$

or shorter

$$x^*(x, y, u) = \phi_r(x, y, u), \quad y^*(x, y, u) = \psi(x, y, u)\,.$$

Now let us consider the transformation

$$\phi_u : \tilde{\Omega} \longrightarrow R^M$$

connected with the transformation Φ and the function y, defined by the formula

$$\phi_u(x) = \phi(x, y(x), u)\,,$$

and let

$$\Omega_u = \phi_u(\Omega)\,.$$

Under the transformation Ψ the function y goes into the function

$$y_u : \Omega_u \longrightarrow R^N\,,$$

satisfying the relation

$$y_u(\phi(x, y(x), u)) = \psi(x, y(x), u)\,. \tag{11}$$

So

$$\frac{d}{du} y_u(x) \mid_{u=0} = \left(\frac{\partial \psi}{\partial u} - \sum_r \frac{\partial y}{\partial x_r} \frac{\partial \psi}{\partial u} \right) \mid_{u=0}\ . \tag{12}$$

3. Noether Theorems

Let us start from the following definition.

Definition. *The functional $I[\Omega, y]$ is invariant under the family of transformation (10) if for any admissible $y \longrightarrow R^N$ exists an $\epsilon > 0$ such that*

$$I[\Omega_u, y_u] = I[\Omega, y]$$

for $|u| < \epsilon$.

First Noether Theorem. *Let the functional I be invariant under a one-parameter smooth family of transformations:*

$$x^*(x,y,u) = \phi(x,y,u), \quad y^*(x,y,u) = \psi(x,y,u)\,.$$

Let for $u=0$ *the transformations be the identity. Then such an equality is fulfilled*

$$\sum_r \frac{\partial}{\partial x_r}\Bigg[F\frac{\partial \phi_r}{\partial u} + \sum_i$$

$$\times \sum_{\alpha\le\beta-e_r} (-1)^{|\alpha|}\left(\frac{D^{|\alpha|}}{Dx^\alpha}\frac{\partial F}{\partial y_i^{(\beta)}}\right)\frac{D^{|\beta-\alpha-e_r|}}{Dx^{\beta-\alpha-e_r}}\left(\frac{\partial\psi_i}{\partial u} - \frac{\partial y_i}{\partial x_s}\frac{\partial \phi_s}{\partial u}\right)\Bigg] = 0 \qquad (13)$$

for any solution of the system of Euler–Lagrange equations.

P r o o f. From the hypothesis and from the Equation (12) we have $\delta^*_{\Omega,y}I[A,h]=0$ for

$$A = \frac{\partial\phi}{\partial u}\,|_{u=0,y=y(x)} \quad \text{and} \quad h_i(x) = \left(\frac{\partial\psi_i}{\partial u} - \sum_r \frac{\partial y_i}{\partial x_r}\frac{\partial\phi_r}{\partial u}\right)|_{u=0,y=y(x)}\,.$$

So from the Euler–Lagrange equation

$$\delta^* I = \sum_r \int_{\partial\Omega}\left[FA_r + \sum_i\sum_{\alpha\le\beta-e_r}(-1)^{|\alpha|}\left(\frac{D^{|\alpha|}}{Dx^\alpha}\frac{\partial F}{\partial y_i^{(\beta)}}\right)h_i^{(\beta-\alpha-e_r)}\right]d\sigma_r = 0\,. \qquad (14)$$

From the Gauss theorem we get the Equation (13). ■

Second Noether Theorem. *Let the functional I be invariant under a local smooth family of transformations:*

$$x^*(x,y,f(x),\dots,f^\alpha(x),\dots) = \phi(x,y,f(x),\dots,f^\alpha(x),\dots)\,,$$
$$y^*(x,y,f(x),\dots,f^\alpha(x),\dots) = \psi(x,y,f(x),\dots,f^\alpha(x),\dots)\,,$$

where $u : I \longrightarrow U \in R$ *runs functions of the class* C_{2n}. *Let for* $u\equiv 0$ *the transfrmation be the identity. Then there is a dependness between Euler–Lagrange equations*

$$\sum_{i,\beta}(-1)^{|\beta|}\frac{D^{|\beta|}}{Dx^\beta}\left\{\left[\sum_\alpha(-1)^{|\alpha|}\frac{D^{|\alpha|}}{Dx^\alpha}\frac{\partial F}{\partial y_i^{(\alpha)}}\right]E^i_\beta\right\} = 0\,, \qquad (15)$$

and moreover such equalities are fulfilled

$$\sum_r \left[\frac{\partial}{\partial x_r} \left(F \frac{\partial \phi_r}{\partial f^{(\lambda)}} \right) + F \frac{\partial \phi_r}{\partial f^{(\lambda - e_r)}} \right. \tag{16}$$

$$\left. + \sum_i \left(\sum_{\substack{\alpha \le \beta \\ \alpha_r = 0}} - \sum_{\substack{\alpha \le \beta \\ \alpha_r = \beta_r}} \right) (-1)^{|\alpha|} \left(\frac{D^{|\alpha|}}{Dx^\alpha} \frac{\partial F}{\partial y_i^{(\beta)}} \right) \sum_\gamma \binom{\beta - \alpha}{\gamma} \frac{D^{|\beta - \alpha - \gamma|}}{Dx^{\beta - \alpha - \gamma}} E^i_{\lambda - \gamma} \right] = 0,$$

for any solution of the system of Euler–Lagrange equations, where

$$E^i_\alpha = \frac{\partial \psi_i}{\partial f^{(\alpha)}} - \sum_r \frac{\partial y^i}{\partial x_r} \frac{\partial \phi_r}{\partial f^{(\alpha)}} .$$

P r o o f. For any establish smooth function $f : \tilde{\Omega} \longrightarrow R^M$ we can define the one-parameter family of transformations:

$$x^*_f(x, y, u) = \phi(x, y, uf(x), \dots, uf^\alpha(x), \dots),$$
$$y^*_f(x, y, u) = \psi(x, y, uf(x), \dots, uf^\alpha(x), \dots).$$

From the hipothesis we have, that $\delta^*_{[\Omega, y[} I[A, h] = 0$ for

$$A = \frac{\partial x^* f}{\partial u} \mid_{u=0} = \sum_\alpha \frac{\partial \phi}{\partial f^{(\alpha)}} f^{(\alpha)} \quad \text{and} \quad h_i = \sum_\alpha E^i_\alpha f^\alpha . \tag{17}$$

Let $f^\alpha \mid_\Omega \equiv 0$ for any multiindex α. Then for any function $y(x)$ we have

$$\sum_i \int_\Omega \left[\sum_\alpha (-1)^{|\alpha|} \left(\frac{D^{|\alpha|}}{Dx^\alpha} \frac{\partial F}{\partial y_i^{(\alpha)}} \right) \right] \sum_\beta E^i_\beta f^{(\beta)} \, d^M x = 0 .$$

Because

$$\sum_{\alpha, \beta} \int_\Omega \left[(-1)^{|\alpha|} \left(\frac{D^{|\alpha|}}{Dx^\alpha} \frac{\partial F}{\partial y_i^{(\alpha)}} \right) \right] \sum_\beta E^i_\beta f^{(\beta)} \, d^M x$$
$$= \sum_{\alpha, \beta} \int_\Omega (-1)^{|\beta|} \frac{D^{|\beta|}}{Dx^\beta} \left[(-1)^{|\alpha|} \left(\frac{D^{|\alpha|}}{Dx^\alpha} \frac{\partial F}{\partial y_i^{(\alpha)}} \right) E^i_\beta \right] f \, d^M x$$

we get the Equation (15).

Now let us assume, that Euler–Lagrange equations are satisfied, but function f and its derivations are not equal to zero on the bound $\partial\Omega$. So $\delta^* I$ is given by (14) for (17).

The Equation (16) one can obtain using the formula

$$\frac{\partial}{\partial x_r}\left[\sum_{\alpha\le\beta-e_r}(-1)^{|\alpha|}\left(\frac{D^{|\alpha|}}{Dx^\alpha}\frac{\partial F}{\partial y_i^{(\beta)}}\right)h_i^{(\beta-\alpha-e_r)}\right]$$
$$=\left(\sum_{\substack{\alpha\le\beta\\ \alpha_r=0}}-\sum_{\substack{\alpha\le\beta\\ \alpha_r=\beta_r}}\right)(-1)^{|\alpha|}\left(\frac{D^{|\alpha|}}{Dx^\alpha}\frac{\partial F}{\partial y_i^{(\beta)}}\right)h_i^{(\beta-\alpha)}$$

and the fact that partial derivatives $f^{(\alpha)}$ are mutually independent. ■

Remark. In the case of the functional depending on the functions of one independent variable the analogs of the Formulas (13), (15), (16) are:

$$F\frac{\partial\phi}{\partial u}+\sum_i\sum_{i<k}(-1)^l\left(\frac{d^l}{dx^l}\frac{\partial F}{\partial y_i^{(k)}}\right)\frac{d^{k-l-1}}{dx^{k-l-1}}\left(\frac{\partial\psi^i}{\partial u}-y_i'\frac{\partial\phi}{\partial u}\right)=const\,, \tag{18}$$

$$\sum_i\sum_{l=0}^{n}(-1)^l\frac{d^l}{dx^l}\left\{\sum_{k=0}^{n}(-1)^k\left(\frac{d^k}{dx^k}\frac{\partial F}{\partial y_i^{(k)}}\right)E_l^i\right\}=0\,, \tag{19}$$

$$F\frac{\partial\phi}{\partial f^{(m)}}+\sum_i\sum_{l<k}(-1)^l\left(\frac{d^l}{dx^l}\frac{\partial F}{\partial y_i^{(k)}}\right)\sum_j\binom{k-l-1}{j}\frac{d^{k-l-1-j}}{dx^{k-l-1-j}}E_{m-j}^i=0,$$
$$g=0,1,\dots\,, \tag{20}$$

respectively.

The formula (18) effects the fact that a smooth family of transformations gives a first integral of the system of Euler–Lagrange equations.

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Теоремы Нётера для функционалов высокого ранга

Ян Милевски

В работе рассматриваются обобщения теорем Нётера в вариационном исчислении. Мы рассматриваем функционалы высокого ранга с одной или несколькими независимыми переменными.

Теореми Ньотера для функціоналів високого рангу

Ян Мілевські

У роботі розглянуто узагальнення теорем Ньотера у варіаційному обчисленні. Ми розглядаємо функціонали високого рангу з однією та кількома незалежними змінними.