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Existence of a holomorphic function with given indicator in the described variable

P. Agranovich

Mathematics Division, B. Verkin Institute for Low Temperature Physics and Engineering National Academy of Sciences of Ukraine 47 Lenin Ave., Kharkov, 61103, Ukraine

E-mail:agranovich@ilt.kharkov.ua

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We establish characteristic properties of indicators of holomorphic functions of proximate order in a distinguished variable in domains of the form $G \times Y$ where G is a relatively compact pseudoconvex set in $\mathbf{C^n}$ and Y is an angle in $\mathbf{C}$.

1. Introduction

Let a set D be a domain in the space $\mathbb{C}^n, n \geq 1$, and

$$Y_{\alpha,\beta} = \{w : w \in \mathbb{C}, 0 < |w| < \infty, \alpha < \arg w < \beta\}.$$

Let $PSH(D)$ and $PSHC^{\infty}(D)$ be classes of plurisubharmonic functions in D and infinitely differentiable plurisubharmonic functions in D correspondingly.

The next definitions were introduced in [3, 4].

Definition 1. *We will say that $\varphi(z,w) \in PSH(D \times Y_{\alpha,\beta})$ is a function of not more than proximate order $\rho(r)$ in the variable w if there exists such $t_o > 0$ that $\forall D' \Subset D$ and $\forall \alpha', \beta', \alpha < \alpha' < \beta' < \beta$,*

$$\limsup_{t\to\infty} \frac{M(D',t)}{t^{\rho(t)}} < \infty,$$

where $M(D',t) = \max\limits_{z\in D', t_o<|w|<t, w\in Y_{\alpha',\beta'}} \varphi^{+}(z,w).$

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In particular, the domain D may be equal to $\mathbb{C}^n$.

Recall that a function $\rho(r)$ with the conditions

$$\lim_{r\to\infty}\rho(r)=\rho, \quad \lim_{r\to\infty} r\rho'(r)\ln(r)=0$$

is called a proximate order.

Further we will denote by $PSH(D)[\rho(r)]$ the set of plurisubharmonic functions of not more than proximate order $\rho(r)$.

Definition 2. *A holomorphic function $f(z,w), z\in D, w\in Y_{\alpha,\beta}$, is said to be a function of not more than proximate order $\rho(r)$ if the function $\ln|f(z,w)|$ satisfies Definition 1.*

Definition 3. *The function*

$$h^*_\varphi(z,w)=\limsup_{(z',w')\to(z,w)}\ \limsup_{t\to\infty}\frac{\varphi(z',tw')}{t^{\rho(t)}}, \quad (z',w')\in D\times Y_{\alpha',\beta'},$$

is called the indicator of the function $\varphi(z,w)\in PSH(D\times Y_{\alpha,\beta})[\rho(r)]$, and the indicator of the function $\ln|f(z,w)|$ is the indicator $h^*_f(z,w)$ of the holomorphic function $f(z,w)$.

It is easy to see that $h^*_\varphi(z,w)(h^*_f(z,w))$ is a plurisubharmonic function in $D\times Y_{\alpha,\beta}$ which is positively homogeneous of degree ρ ($\rho(r)\to\rho$ as $r\to\infty$) in the variable w.

G. Pólya [11], V. Berenstein [5], B. Levin [7] and V. Logvinenko [8] have proved that ρ-trigonometrically convexity is a characteristic property of indicator when $z\in\mathbb{C}^n$ is fixed and the angle $Y_{\alpha,\beta}$ coincides with the plane $\mathbb{C}$.

The case of functions of several variables is far more difficult. For functions of first order this problem was investigated by C. Kiselman [6]; A. Martineau in [9, 10] had considered functions of arbitrary orders. Functions of several complex variables of proximate order were considered in [1]. Note that in that paper it was studied holomorphic functions of some class of proximate orders in cones of special kind. Further the author succeeded in taking down many of these restrictions, but the complete result was given only in the thesis [2].

In this work we show that plurisubharmonicity and positive homogeneity with respect to a distinguished variable are characteristic properties of indicators of functions of proximate order in the distinguished variable in $\mathbb{C}^n\times Y_{\alpha,\beta}$.

Theorem 1. *Let* $\varphi(z,w) \in PSH(\mathbb{C}^n \times Y_{\alpha,\beta})$ *be a positively homogeneous function of degree* $\rho > 0$ *in the variable* w. *Then for any proximate order* $\rho(r), \rho(r) \to \rho, r \to \infty$, *and any pseudoconvex set* $G \Subset \mathbb{C}^n$ *there exists a holomorphic function* $f(z,w)$ *in* $G \times Y_{\alpha,\beta}$ *of the proximate order* $\rho(r)$ *with the indicator* $h_f^*(z,w)$ *that is equal to* $\varphi(z,w)$.

We prove this fact with the help of Martineau's scheme [9, 10] and use the following two results.

Theorem 2. *Let* $\varphi(z,w) \in PSH(\mathbb{C}^n \times Y_{\alpha,\beta})$ *be a function that is positively homogeneous of degree* $\rho > 0$ *in the variable* w; $\rho(r)$ *be an arbitrary proximate order,* $\rho(r) \to \rho$ *as* $r \to \infty$. *Then in any set* $G \Subset \mathbb{C}^n$ *there exists a function* $u(z,w) \in PSH(G \times Y_{\alpha,\beta})[\rho(r)]$ *such that its indicator* $h_u^*(z,w)$ *is equal to* $\varphi(z,w)$.

Theorem 3. *Let* $\varphi(z,w) \in PSH(\mathbb{C}^n \times Y_{\alpha,\beta})$ *be a positively homogeneous of degree* $\rho > 0$ *in the variable* w; $\rho(r)$ *be an arbitrary proximate order,* $\rho(r) \to \rho$ *as* $r \to \infty$, *and* G *be any pseudoconvex set. Then for every point* $(z^0, w^0) \in \mathbb{C}^n \times Y_{\alpha,\beta}$ *there exists a holomorphic function* $f_{(z^0,w^0)}(z,w)$ *of the proximate order* $\rho(r)$ *in the variable* w *such that*

$$h^*_{f_{(z^0,w^0)}}(z,w) \le \varphi(z,w), \quad \forall (z,w) \in G \times Y_{\alpha,\beta},$$

and

$$h^*_{f_{(z^0,w^0)}}(z^0,w^0) = \varphi(z^0,w^0).$$

2. Auxiliary statements

Lemma 1. *For every proximate order* $\rho(r) > 0, \rho(r) \to \rho$ *as* $r \to \infty$, *there exists a proximate order* $\tilde\rho(r)$ *such that*

$$\tilde\rho(r) \to \rho, \quad r \to \infty,$$

$$r^{\tilde\rho(r)-\rho(r)} \to 1, \quad r \to \infty, \tag{1}$$

and

$$r^2 \tilde\rho''(r) \ln r \to 0, \quad r \to \infty. \tag{2}$$

P r o o f. Assume

$$\tilde\rho(r) = \rho + \frac{\ln l(r)}{\ln r},$$

where

$$l(r) = \frac{1}{r}\int_0^r t^{\rho(t)-\rho}\,dt.$$

Using well-known properties of proximate orders ([7]) and carring out not complicated calculation we obtain that the proximate order $\tilde{\rho}(r)$ satisfies (1) and (2). ■

Proofs of the following lemmas are similar to the proofs of corresponding facts of Martineau (see [12]), so we only formulate them here.

Lemma 2. (see [12, p. 200]) *For any non-negative function $\gamma(r), 0 < r < \infty$, which tends to zero as $r \to \infty$ and any proximate order $\rho(r)$ which satisfies the property* (2) *there exists a function $\Phi(r)$ such that*

$$\Phi(r) \geq \gamma(r) r^{\rho(r)},$$

$$\lim_{r\to\infty} \frac{\Phi(r)}{r^{\rho(r)}} = 0$$

and the function $\Phi(|z|), z \in \mathbb{C}^n$, is infinitely differentiable and strictly plurisubharmonic.

Lemma 3. (see [12, p. 178]) *If $h_u^*(z,w)$ is the indicator of a function $u(z,w) \in PSH(\mathbb{C}^n \times Y_{\alpha,\beta})[\rho(r)]$ then for every point $(z^0, w^0) \in \mathbb{C}^n \times Y_{\alpha,\beta}, |w^0| = 1$, and any number $A > h_u^*(z^0,w^0)$ there exist an angle $Y \subset Y_{\alpha,\beta}$ with apex at the origin and a neighborhood ω_{z^0} of the point z^0 in $\mathbb{C}^n$ such that $(z^0,w^0) \in \omega_{z^0} \times Y$ and for all $w \in Y$, expect perhaps points w in some bounded set K,*

$$u(z,w) \leq A|w|^{\rho(r)}, \quad z \in \omega_{z^0}, w \in Y \setminus K.$$

Lemma 4. (see [12, p. 178]) *The indicator $h_u^*(z,w)$ of a function $u(z,w) \in PSH(\mathbb{C}^n \times Y_{\alpha,\beta})[\rho(r)]$ is independent of the choice of the origin in the plane $\mathbb{C}$.*

3. Proof of Theorem 2

Without loss of generality by virtue of Lemma 1 we will assume further that our proximate order $\rho(r)$ satisfies the condition (2).

Let $\eta(\xi)$ be an infinitely differentiable function that depends on $|\xi|$ only and such that $\eta(\xi) \geq 0,\ \xi \in \mathbb{C}^n,\ \eta(\xi) = 0$ when $|\xi| > 1$ and

$$\int \eta(\xi) d\xi = 1.$$

Let $\kappa(\zeta)$ be an infinitely differentiable non-negative function that depends on $|\zeta|$ and such that $\kappa(\zeta) = 0$ when $|\zeta| > 1$ and $|\zeta| < \frac{1}{4}$ and

$$\int \kappa(\zeta) d\zeta = 1.$$

Now consider the function

$$\varphi_{\eta,\kappa}(z,w) = \int \varphi\left(z + \frac{\xi}{\zeta}, \zeta\right) \eta(\xi)\kappa(w - \zeta - a) d\xi d\zeta, \tag{4}$$

where the vector a is chosen in such a way that

$$\bigcup_{\zeta \in \overline{Y_{\alpha,\beta}} - a} \{w : |w - \zeta| < \frac{3}{2}\} \Subset Y_{\alpha,\beta}.$$

It is easy to see that the function $\varphi_{\eta,\kappa}(z,w) \in C^\infty(\mathbb{C}^n \times Y_{\alpha,\beta})$ and what is more this function is plurisubharmonic in the set

$$D = (\mathbb{C}^n \times Y_{\alpha,\beta}) \setminus (\mathbb{C}^n \times \{w : |w| < 1\}).$$

Denote

$$\psi(z,w) = \varphi_{\eta,\kappa}(z, w + b),$$

where b is chosen in such a way that $Y_{\alpha,\beta} + b \subset Y_{\alpha,\beta} \setminus \{w : |w| < 1\}$, $\psi(z,w) \in PSHC^\infty(\mathbb{C}^n \times Y_{\alpha,\beta})[\rho(r)]$ and $h^*_\psi(z,w) = h^*_{\varphi_{\eta,\kappa}}(z,w)$. From Lemma 4 it is clear that such choice of b is possible.

As indicated earlier, for every point $z^0 \in \mathbb{C}^n$ the function $\varphi(z^0, w)$ is continuous in the variable w in the angle $Y_{\alpha,\beta}$. Besides, from the well-known properties of ρ-trigonometrically convex functions [7] it follows that in any fixed point $z^0 \in \mathbb{C}^n$ for any θ such that $re^{i\theta} \in Y_{\alpha,\beta}$ the function $\varphi(z^0, re^{i\theta})$ has a derivative from the left that is continuous from the left and a derivative from the right that is continuous from the right. Moreover,

$$\varphi'_{\theta,+}(z^0, re^{i\theta}) \geq \varphi'_{\theta,-}(z^0, re^{i\theta}).$$

Hence by virtue of the positive homogeneity of the function $\varphi(z,w)$ in the variable w we obtain that

$$|\varphi_{\eta,\kappa}(z,w)| = \left| \int \varphi\left(z + \frac{\xi}{w - \zeta - a}, w - \zeta - a\right) \eta(\xi)\kappa(\zeta) d\xi d\zeta \right|$$

$$= |w|^\rho \left| \int \varphi\left(z + \frac{\xi}{w - \zeta - a}, \frac{w - \zeta - a}{|w - \zeta - a|}\right) \left|1 - \frac{\zeta - a}{w}\right|^\rho \eta(\xi)\kappa(\zeta) d\xi d\zeta \right|$$

$$= |w|^\rho \left| \int \left|1 - \frac{\zeta - a}{w}\right|^\rho \kappa(\zeta) d\zeta \int \varphi\left(z + \frac{\xi}{w - \zeta - a}, \frac{w - \zeta - a}{|w - \zeta - a|}\right) \eta(\xi) d\xi \right|.$$

It follows that for any $\alpha', \beta', \alpha < \alpha' < \beta' < \beta$, and $G \Subset \mathbb{C}^n$ there exists such $R > 0$ that on the set

$$D_{G,R,\alpha',\beta'} := \{(z,w) : z \in G, w \in Y_{\alpha,\beta}, R < |w| < \infty, \alpha' \leq \arg w \leq \beta'\}$$

the inequality

$$|\psi(z,w)| \le C_{(G,\alpha',\beta')}|w|^{\rho} \tag{5}$$

is valid.

In a similar manner we can show that on the set $D_{G,R,\alpha',\beta'}$ the inequalities

$$\left|\frac{\partial\psi(z,w)}{\partial w}\right| \le C_{(G,\alpha',\beta')}|w|^{\rho-1}, \tag{6_1}$$

$$\left|\frac{\partial\psi(z,w)}{\partial \overline{w}}\right| \le C_{(G,\alpha',\beta')}|w|^{\rho-1} \tag{6_2}$$

are true.

Define the function

$$\gamma(t) = \max\left\{\left|\frac{6|[\rho(t)-\rho]+t\rho'(t)\ln t|^{1/2}}{[\rho(t)+t\rho'(t)\ln t]^2}\right|,\right.$$

$$\left.\left|\frac{3|\{[\rho(t)-\rho]+t\rho'(t)\ln t\}^2+t^2\rho''(t)\ln t+2t\rho'(t)-2t\rho'(t)\ln t|^{1/2}}{[\rho(t)+t\rho'(t)\ln t]^2+3t^2\rho''(t)\ln t+6t\rho'(t)-6t\rho'(t)\ln t}\right|\right\}.$$

It is easy to see that

$$\gamma(t)\to 0, \quad t\to\infty,$$

$$\gamma(t)\ge 0, \quad 0<t<\infty.$$

Let $\chi(t)$ be a convex infinitely differentiable function * such that $\chi(t)\to 0$ as $t\to\infty, \chi(t)\ge\gamma(\sqrt{t}),\ t\in(0,\infty)$, and

$$t\chi'(t)+\frac{(\rho-\nu)^2\chi(t)}{12(\rho+2)}\ge 0,$$

where $\rho-\nu>0$ and $\nu>0$.

Then put

$$\hat{\psi}(z,w) = \psi(z,w)|w|^{\rho(|w|)-\rho} + \chi(|w|^2)|w|^{\rho(|w|)}.$$

Now we will establish that for any $G\Subset\mathbb{C}^n$ and any interval $(\alpha',\beta')\Subset(\alpha,\beta)$ there exists such a number $R>0$ that the function $\hat{\psi}(z,w)$ is plurisubharmonic in the set $D_{G,R,\alpha',\beta'}$.

* Existence of such a function is evident from the construction of a corresponding polygon line and its smoothing.

Indeed, consider the Levi form $L(\hat{\psi};\xi,\zeta)$ of the function $\hat{\psi}(z,w)$:

$$\begin{aligned} L(\hat{\psi};\xi,\zeta) &= |w|^{\rho(|w|)-\rho}\Big\{\sum_{i,l=1}^{n}\frac{\partial^2\psi}{\partial z_i\partial\overline{z}_l}\xi_i\overline{\xi}_l+\sum_{i=1}^{n}\frac{\partial^2\psi}{\partial z_i\partial\overline{w}}\xi_i\overline{\zeta} \\ &+ \sum_{i=1}^{n}\frac{\partial^2\psi}{\partial\overline{z}_i\partial w}\overline{\xi}_i\zeta+\frac{\partial^2\psi}{\partial w\partial\overline{w}}|\zeta|^2\Big\}+\sum_{i=1}^{n}\frac{\partial\psi}{\partial z_i}\frac{\partial|w|^{\rho(|w|)-\rho}}{\partial\overline{w}}\xi_i\overline{\zeta} \\ &+ \sum_{i=1}^{n}\frac{\partial\psi}{\partial\overline{z}_i}\frac{\partial|w|^{\rho(|w|)-\rho}}{\partial w}\overline{\xi}_i\zeta+\frac{\partial\psi}{\partial w}\frac{\partial|w|^{\rho(|w|)-\rho}}{\partial\overline{w}}|\zeta|^2 \\ &+ \frac{\partial\psi}{\partial\overline{w}}\frac{\partial|w|^{\rho(|w|)-\rho}}{\partial w}|\zeta|^2+\psi(z,w)\frac{\partial^2|w|^{\rho(|w|)-\rho}}{\partial w\partial\overline{w}}|\zeta|^2 \\ &+ \frac{\partial^2\chi(|w|^2)|w|^{\rho(|w|)}}{\partial w\partial\overline{w}}|\zeta|^2. \end{aligned}$$

Let us rewrite $L(\hat{\psi};\xi,\zeta)$ in the following way:

$$L(\hat{\psi};\xi,\zeta)=|w|^{\rho(|w|)-\rho}\sum_{k=1}^{4}I_k,$$

where

$$I_1=\sum_{i,l=1}^{n}\frac{\partial^2\psi}{\partial z_i\partial\overline{z}_l}\xi_i\overline{\xi}_l+\sum_{i=1}^{n}\frac{\partial^2\psi}{\partial z_i\partial\overline{w}}\xi_i\overline{\zeta}+\sum_{i=1}^{n}\frac{\partial^2\psi}{\partial\overline{z}_i\partial w}\overline{\xi}_i\zeta+\frac{\partial^2\psi}{\partial w\partial\overline{w}}|\zeta|^2;$$

$$\begin{aligned} I_2 &= \frac{\psi}{4|w|^2}|\zeta|^2\Big(\{[\rho(|w|)-\rho]+|w|\rho'(|w|)\ln|w|\}^2 \\ &+ |w|^2\rho''(|w|)\ln|w|+2|w|\rho'(|w|)-2|w|\rho'(|w|)\ln|w|\Big) \\ &+ \frac{\chi(|w|^2)}{4}|w|^{\rho-2}\Big(\frac{[\rho(|w|)+|w|\rho'(|w|)\ln|w|]^2}{3} \\ &+ |w|^2\rho''(|w|)\ln|w|+2|w|\rho'(|w|)-2|w|\rho'(|w|)\ln|w|\Big); \end{aligned}$$

$$\begin{aligned} I_3 &= \Big(\sum_{i=1}^{n}\frac{\partial\psi}{\partial z_i}\xi_i+\sum_{i=1}^{n}\frac{\partial\psi}{\partial\overline{z}_i}\overline{\xi}_i\Big)|\zeta|^2\Big(\frac{[\rho(|w|)-\rho]+|w|\rho'(|w|)\ln|w|}{2|w|^2}\Big) \\ &+ \frac{\chi(|w|^2)}{12}|w|^{\rho-2}(\rho(|w|)+|w|\rho'(|w|)\ln|w|)^2; \end{aligned}$$

$$\begin{aligned} I_4 &= \frac{\chi(|w|^2)}{12}|w|^{\rho-2}|\zeta|^2(\rho(|w|)+|w|\rho'(|w|)\ln|w|)^2 \\ &+ \chi'(|w|^2)|w|^\rho\{\rho+1+|w|\rho'(|w|)\ln|w|\}+\chi''(|w|^2)|w|^{\rho+2}|\zeta|^2. \end{aligned}$$

Since ψ is plurisubharmonic function in $\mathbb{C}^n \times Y_{\alpha,\beta}$, the expression I_1 is non-negative in this domain.

From the inequalities (5), (6_1)–(6_2), the properties of the proximate order $\rho(t)$, the functions $\gamma(t)$ and $\chi(t)$ it follows that for any $G \Subset \mathbb{C}^n$ and any $\alpha', \beta', \alpha < \alpha' < \beta' < \beta$, there exists such a number $R > 0$ that the values $I_i(z,w) > 0$, i = 2, 3, 4, if

$$(z,w) \in \{(z,w) : (z,w) \in G \times Y_{\alpha,\beta}, R < |w| < \infty, \alpha' \arg w < \beta'\}.$$

Now let us "correct" the function $\hat{\psi}(z,w)$ in such a way that the new function will become plurisubharmonic in the set $G \times Y_{\alpha,\beta}$ with the same asymptotic properties as the function $\hat{\psi}(z,w)$.

For this consider the family of functions

$$\tau_A(w) = \max(-\ln\delta(w) + \frac{1}{2}\ln(1+|w|^2), A) - A,$$

where $\delta(w)$ is the Euclidean distance from the point w to the boundary of the angle $Y; A \in (0,\infty)$.

Let $Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset \ldots$ be a sequence of angles which are contained in the angle $Y_{\alpha,\beta}$ with tops in the top of the angle $Y_{\alpha,\beta}$ and $\bigcup\limits_{j\geq 1} Y_j = Y_{\alpha,\beta}$. Let further $\{A_i\}_{i=1}^{\infty}$ be a monotone increasing sequence of numbers, $\lim\limits_{j\to\infty} A_j = +\infty$; $\tau_i(w) := \tau_{A_i}(w),\ i = 1, 2, \ldots\ ; \{\Omega_j\}_{j=1}^{\infty}$ be the sequence of sets such that

$$\Omega_j = \{w : w \in Y_{\alpha,\beta}, \tau_j(w) = 0\}.$$

and $\bigcup\limits_{j\geq 1} \Omega_j = Y_{\alpha,\beta}$.

Let ω be some neighborhood of the point 0 in the plane $\mathbb{C}$. Put

$$u(z,w) = \hat{\psi}(z, w+\tilde{w}) + \sum_{j=1}^{\infty} C_j\tau_j(w), \tag{7}$$

where the vector $\tilde{w}$ is chosen so that the function $\hat{\psi}(z, w+\tilde{w})$ is plurisubharmonic in the set $(G \times Y_2)\bigcup(G \times \omega)$. We will define the positive constants C_j below.

As follows from the properties of the function $\hat{\psi}$, for any angle $Y_i, Y_i \subset Y_{\alpha,\beta}$ there exists such a number $R > 0$ that the function $\hat{\psi}(z,w)$ is plurisubharmonic in the set

$$D_i = \{(z,w) : z \in G, w \in Y_i \bigcap \{w : |w| > R\}\}.$$

We choose the angles Y_j and the numbers A_j so that the inclusions

$$D_j \subset G \times \Omega_j \subset G \times Y_{j+1}, \quad j = 1, 2, \dots ,$$

are valid. It is easy to see that such choice is possible.

We take the number C_1 so that the function $\hat{\psi}(z, w+\tilde{w})+C_1\tau_1(w) \in PSH(G\times Y_3)$. It can be done in the following way. The set $G \times ((Y_3 \setminus \omega) \setminus (Y_2 \bigcup (Y_3 \bigcap \{w : |w| > R\}))$ is a relatively compact subset of the set $G \times (Y_{\alpha,\beta} \setminus \Omega_1)$. Having $\Delta\tau_1(w) > 0$ in $Y_{\alpha,\beta} \setminus \Omega_1$, the inequality

$$\inf\{\Delta\tau_1(w), w \in Y_3 \setminus \omega \setminus (Y_2 \bigcup \{Y_3 \bigcap \{w : |w| > R\})\} > 0 \qquad (8)$$

is true.

By virtue of the choice of the vector $\tilde{w}$ and inequality (8) we obtain that for sufficiently large C_1 the function $\hat{\psi}(z, w + \tilde{w}) + C_1\tau_1(w) \in PSH(G \times Y_3)$. Repeating these reasonings we choose the numbers $C_2, C_3, \dots , C_n, \dots$, so that the corresponding functions $\hat{\psi}(z, w + \tilde{w}) + C_1\tau_1(w) + C_2\tau_2(w), \dots , \hat{\psi}(z, w + \tilde{w}) + \sum_{j=1}^{n} C_j\tau_j(w), \dots$ become plurisubharmonic in the sets $G \times Y_4, \dots , G \times Y_{n+2}, \dots$, correspondingly.

Moreover, having $\tau_j(w) = 0$ in $\Omega_i, \Omega_i \subset \Omega_{i+1}$ and $\bigcup \Omega_i = Y_{\alpha,\beta}$ the series (7) contains only finite number of non-zero terms in the neighborhood of each point of the angle $Y_{\alpha,\beta}$ and hence it converges.

Thus, the function $u(z, w)$ defined by formula (7) belongs to the set $PSH(G\times Y_{\alpha,\beta})$. For the completion of the proof of this theorem we must establish that

$$h^*_u(z, w) = \varphi(z, w).$$

First and foremost we note that as follows from the construction of the function $u(z, w)$,

$$h^*_{\hat{\psi}}(z, w) = \limsup_{(z',w')\to(z,w)} \limsup_{t\to\infty} \frac{\psi(z', tw')}{t^\rho}$$

and each next transformation does not change the value of this limit, hence $h^*_u(z, w) = h^*_{\hat{\psi}}(z, w)$. In turn it is easy to see that

$$\limsup_{(z',w')\to(z,w)} \limsup_{t\to\infty} \frac{\psi(z', tw')}{t^\rho} = \limsup_{(z',w')\to(z,w)} \limsup_{t\to\infty} \frac{\varphi_{\eta,\kappa}(z', tw')}{t^\rho}.$$

So it is sufficient to show that

$$H(z, w) := \limsup_{(z',w')\to(z,w)} \limsup_{t\to\infty} \frac{\varphi_{\eta,\kappa}(z', tw')}{t^\rho} = \varphi(z, w).$$

Indeed, from (4) it follows:

$$\begin{aligned}\varphi_{\eta,\kappa}(z,w) &= \int\limits_{1/4}^{1} d\nu \int\limits_{0}^{1} dt \int\limits_{S(0,t)} \eta(\xi)d\xi \times \int\limits_{S(0,\nu)} \varphi\left(z+\frac{\xi}{\zeta}, w-\zeta-a\right)\kappa(\zeta)d\zeta \\ &= const \int\limits_{0}^{1} \eta(t)dt \int\limits_{1/4}^{1} \kappa(\nu)\mathbf{M}\nu^2 t^{2n-1} d\nu \geq \varphi(z,w),\end{aligned}$$

where $S(0,t)$ is the sphere with the center at the origin of the corresponding space and the radius t, $\mathbf{M}$ is the average of the function $\varphi(z,w)$ on $S(0,t)\times S(0,\nu)$. Here and later on we will denote by the mark *const* some absolute constant which depends on the dimension of the space only.

Thus,

$$H(z,w) \geq \varphi(z,w).$$

Now let us prove the inverse inequality. Let $g(z,w)$ be a continuous positively homogeneous function of the order ρ in the variable w in $G\times Y_{\alpha,\beta}$, majorizing the function $\varphi(z,w)$. From Lemma 3 it follows that for any fixed number $\varepsilon>0$ each point $(z^0,w^0), |w^0|=1$, there is a neighborhood $\omega_{z^0}\times\Omega_{w^0}$ such that for $z\in\omega_{z^0}$ $\frac{w}{|w^0|}\in\Omega_{w^0}$ and some constant $C(z^0,w^0,\varepsilon)$ the inequality

$$\varphi(z,w-a) \leq (g(z,w)+\varepsilon|w|^{\rho})^{\rho(r)-\rho} + C(z^0,w^0,\varepsilon).$$

is valid.

Hence for such (z,w) by virtue of the properties of the proximate order $\rho(r)$ the function

$$\Phi(z,w) := \sup_{|\xi|<1,|\zeta|<1} \varphi(z+\xi,w+\zeta)$$

satisfies the inequality

$$\Phi(z,w) \leq C(\varepsilon)+\varepsilon(|w|+|\zeta|)^{\rho(|w|+|\zeta|)} + \sup_{|\xi|<1,|\zeta|<1} g(z+\xi,w+\zeta)\frac{(|w|+|\zeta|)^{\rho(|w|+|\zeta|)}}{(|w|+|\zeta|)^{\rho}}.$$

From here for sufficiently large $|w|$ it follows that

$$\Phi^*(z,w) \leq C(\varepsilon)+2^{\rho+1}\varepsilon|w|^{\rho(|w|)} + (1+\varepsilon)|w|^{\rho(|w|)} \sup_{|\xi|<1,|\zeta|<1} g\left(z+\xi,\frac{w+\zeta}{|w|}\right).$$

Using this inequality and the continuity of the function $g(z,w)$, we conclude that

$$H(z,w) \leq g(z,w).$$

Since the function $\varphi(z, w)$ is clearly the limit of functions $g(z, w)$ of the above type, then in view of Lemma 4 we obtain that

$$H(z, w) \le \varphi(z, w). \qquad \blacksquare$$

R e m a r k. Simultaneously, we have proved that the convolutions with smooth functions of the type η and κ do not change the indicators of the functions.

4. Proof of Theorem 3

First of all we take the series of auxiliary statements as well.

Lemma 5. (see [12, p. 200]). *Let a holomorphic function $f(z, w), (z, w) \in G \times Y_{\alpha,\beta}$ satisfy the inequality*

$$\int |f(z, w)|^2 e^{-2\varphi(z,w-a)} dz dw < \infty$$

where $\varphi(z, w)$ is a plurisubharmonic function of the proximate order $\rho(r)$ in the variable w and the number a is chosen in the proof of Theorem 2. Denote that here and later on we use the same notations as in the proof of Theorem 2. Then

$$h_f^*(z, w) \le h_\varphi^*(z, w).$$

Before proceeding to the last lemma, we introduce some notations. Let $\mathbb{C}^{(j)}$ be the subspace of $\mathbb{C}^n$ consisting of the points $(z^j) = (z_1, \dots, z_j, 0, \dots, 0)$, and dz^j be the volume element of the space $\mathbb{C}^{(j)}$ in the metric of $\mathbb{C}^n$; $D^{(j)} = D \bigcap \mathbb{C}^{(j)}$. Further given a function $\psi(z, w)$, set $\psi_0(z, w) \equiv \psi(z, w - a)$. We denote $\psi_j'(z, w), j = 1, \dots, n$, the regularization of the function

$$\sup_{|\xi_j|<2} \psi_{j-1}\left(z_j, \dots, z_{j-1}, z_j + \frac{\xi_j}{w}, z_{j+1}, \dots, z_n, w\right).$$

Consider the function

$$\tilde{\psi}_j(z, w) = \int \psi_j'(\xi, \zeta)\eta(\xi\zeta - z\zeta)\kappa(w - \zeta)|\zeta|^2 d\xi d\zeta.$$

We transform this function as in the proof of Theorem 2. As a result of this we obtain the plurisubharmonic function $\psi_j(z, w)$ in $G \times Y_{\alpha,\beta}, G \Subset \mathbb{C}^n$.

Lemma 6. (see [12, p. 202]) *Let the function $\psi(z, w)$ be plurisubharmonic in $G \times Y_{\alpha,\beta}$ where G is a pseudoconvex set in $\mathbb{C}^n$, and the function $f(z^{j-1}, w)$ be holomorphic in $G^{(j-1)} \times Y_{\alpha,\beta}$ and such that*

$$\int |f(z^{j-1}, w)|^2 \exp\{-\psi_{j-1}(z^{j-1}, w)\}(1 + |z^{j-1}|^2)^{-3(j-1)+1} dz^{j-1} dw < \infty.$$

Then in $G^{(j)} \times Y_{\alpha,\beta}$ there exists a holomorphic function $g(z^j, w)$ such that $g(z^{j-1}, w) = f(z^{j-1}, w)$ and

$$\int |g(z^j, w)|^2 \exp\{-\psi_j(z^j, w)\}(1 + |z^j|^2)^{-3j+1} dz^j dw < \infty.$$

Now we can prove Theorem 3.

Suppose G and $\tilde{G}$, $G \Subset \tilde{G}$, be pseudoconvex sets in $\mathbb{C}^n$, and $\tilde{G}$ be sufficiently large. Let $u(z, w)$ denote the function constructed by the function $\varphi(z, w)$ in $\tilde{G} \times Y_{\alpha,\beta}$ in Theorem 2. Without loss of generality we will suppose that $z^0 = 0$. Let $f(0, w)$ be a holomorphic function in the angle $Y_{\alpha,\beta}$ of the proximate order $\rho(r)$ whose indicator is equal to $\varphi(0, w)^\star$.

Let $D_j^{(1)} = \{w : w \in Y_j, |w| > R_j\}$. Denote

$$\gamma(r) = \sup_{|w|=1, w \in \cup_1^\infty D_j^{(1)}} \{\max[r^{-\rho(r)}[\ln |f(0, rw)| - u(0, rw)], 0]\}.$$

Let $\Phi(r)$ be the function constructed by $\gamma(r)$ in Lemma 2 and

$$\psi(z, w) = 2(u(z, w) + \Phi(w) + \ln(1 + |z|^2 + |w|^2)).$$

As is easy to see, this function is infinitely differentiable, plurisubharmonic, and such that

$$h^*_\psi(z, w) = 2h^*_u(z, w) = 2\varphi(z, w) \tag{9}$$

and

$$\int |f(0, w)|^2 \exp\{-\psi(0, w) + \ln(1 + |w|^2)\} dw < \infty.$$

Applying Lemma 6 n times, we get a holomorphic function $g(z, w)$ in $\tilde{G} \times Y_{\alpha,\beta}$ such that

$$g(0, w) = f(0, w)$$

and

$$\int |g(z, w)|^2 \exp\{-\psi_n(z, w)\}(1 + |z|^2)^{-3n+1}(1 + |w|^2) dz dw < \infty.$$

By virtue of (9)

$$h^*_{\psi_n}(z, w) = h^*_\psi(z, w) = 2\varphi(z, w).$$

By the mean value theorem for harmonic functions

$$g(z, w) = \int g\left(z + \frac{\xi}{w - \zeta - a}, w - \zeta - a\right) \eta(\xi)\kappa(\zeta) d\xi d\zeta.$$

* The proof of the existence of such function is in [2].

From here and the Schwartz–Bunjakovskii inequality it follows that

$$|g(z,w)| = \left|\int g\left(z+\frac{\xi}{w-\zeta-a}, w-\zeta-a\right)\right.$$

$$\times \exp\left\{-1/2(\psi_n\left(z+\frac{\xi}{w-\zeta-a}, w-\zeta-a\right)\right\}\eta(\xi)\kappa(\zeta)$$

$$\left.\times \exp\left\{1/2\psi_n\left(z+\frac{\xi}{w-\zeta-a}, w-\zeta-a\right)\right\}d\xi d\zeta\right|$$

$$\leq const \exp\left\{1/2 \sup_{|\xi|<1,|\zeta|<1} \psi_n\left(z+\frac{\xi}{w-\zeta-a}, w-\zeta-a\right)\right\}.$$

Hence

$$h_g^*(z,w) \leq \varphi(z,w) \quad \forall (z,w) \in G \times Y_{\alpha,\beta}$$

and

$$h_g^*(z^0,w^0) \geq h_f^*(z^0,w^0) = \varphi(z^0,w^0). \qquad \blacksquare$$

5. Proof of Theorem 1

Having Theorem 3, for the completion of the proof of Theorem 1 we must introduce the Fréchet space of holomorphic functions and repeat the reasoning of Martineau [10, 12].

Express the function $\varphi(z,w)$ as a limit

$$\varphi(z,w) = \lim_{j\to\infty} \varphi_j(z,w),$$

where $\varphi_j(z,w) \in PSHC^\infty(\mathbb{C}^n \times (Y_{\alpha,\beta}\setminus\{0\}))$ are positively homogeneous of degree ρ in the variable $w, j = 1,2,\ldots; \varphi_j(z,w) \geq \varphi_{j+1}(z,w)\ \forall (z,w) \in G \times Y_{\alpha,\beta}, G \Subset \mathbb{C}^n$.

Let E_φ be the class of all holomorphic functions $f(z,w)$ of the proximate order $\rho(r)$ in the variable w in $G \times Y_{\alpha,\beta}$ such that

$$h_f^*(z,w) \leq \varphi(z,w).$$

Topologize this class by the countable family of norms

$$||f||_\varphi^{(p)} = \max_{\{|z|<p\}\times Y_{\alpha,\beta}} |f(z,w)| \exp\left(-\frac{\varphi_p(z,w)}{|w|^\rho} - \frac{1}{p}\right)|w|^{\rho(|w|)}.$$

It is clear that E_φ is a complete and countably normed space in this topology. By the word for word repetition of the proof of Theorem 3.5.1 from [12] we establish the truth of Theorem 1.

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Существование голоморфной функции с заданным индикатором по выделенной переменной

П. Агранович

Устанавливаются характеристические свойства индикаторов голоморфных функций уточненного порядка по выделенной переменной в областях вида $G \times Y$, где G — относительно компактное псевдовыпуклое множество в $\mathbf{C^n}$ и Y — угол в $\mathbf{C}$.

Існування голоморфної функції з даним індикатором за виділеною змінною

П. Агранович

Встановлюються характеристичні властивості індикаторів голоморфних функцій уточненого порядку за виділеною змінною на областях вигляду $G \times Y$, де G — відносно компактна псевдовипукла множина на $\mathbf{C^n}$ та Y — кут у $\mathbf{C}$.