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Guantum matrix ball: the Cauchy–Szegö kernel and the Shilov boundary

L. Vaksman

Mathematics Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkov, 61103, Ukraine
E-mail:vaksman@ilt.kharkov.ua

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This work produces a q-analogue of the Cauchy–Szegö integral representation that retrieves a holomorphic function in the matrix ball from its values on the Shilov boundary. Besides that, the Shilov boundary of the quantum matrix ball is described and the $U_q\mathfrak{su}_{m,n}$-covariance of the $U_q\mathfrak{s}(\mathfrak{u}_m \times \mathfrak{u}_n)$-invariant integral on this boundary is established. The latter result allows one to obtain a q-analogue for the principal degenerate series of unitary representations related to the Shilov boundary of the matrix ball.·

1. Introduction

Bounded symmetric domains form a favorite subject of research in geometry, function theory of several complex variable, and non-commutative harmonic analysis. The point is that they are simplest among non-compact homogeneous spaces of real semi-simple Lie groups.

Quantum analogues (q-analogues) of bounded symmetric domains were introduced in [9] via replacement of the ordinary Lie groups with their quantum analogues [2]. A simple example of classical bounded symmetric domain is the unit ball $\mathbb{U} = \{\mathbf{z} \in \mathrm{Mat}_n | \mathbf{z}\mathbf{z}^* < I\}$ in the space of complex $n \times n$ matrices. The present work studies a q-analogue of this matrix ball. The notions of Shilov boundary and Cauchy–Szegö kernel are associated to this subject. Just as in the

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case $q = 1$, a holomorphic function in the quantum matrix ball is determined unambiguously by its restriction onto the Shilov boundary and could be retrieved via the Cauchy–Szegö integral representation.

The proof of the latter result involves some properties of an invariant integral on the Shilov boundary. We also use these properties to produce the principal degenerate series of unitary representations of the quantum group $SU_{n,n}$. The appendix contains a discussion of generalizations of the main result onto the case of rectangular matrices.

In what follows we assume $\mathbb{C}$ to be the ground field, all the algebras are unital, and $q \in (0, 1)$.

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2. A construction of the Shilov boundary

Among the best known 'quantum' algebras one should mention the q-analogue $\mathbb{C}[\mathrm{Mat}_n]_q$ of the algebra of holomorphic polynomials on the space of matrices Mat_n. This algebra is given by its generators z_a^α, $a, \alpha = 1, 2, \dots, n$, and the following relations:

$$z_a^\alpha z_b^\beta - q z_b^\beta z_a^\alpha = 0, \qquad a = b \,\&\, \alpha < \beta \quad \text{or} \quad a < b \,\&\, \alpha = \beta, \tag{2.1}$$

$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha = 0, \qquad \alpha < \beta \,\&\, a > b, \tag{2.2}$$

$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha = (q - q^{-1}) z_a^\beta z_b^\alpha, \quad \alpha < \beta \,\&\, a < b. \tag{2.3}$$

The work [7] presents a definition of the $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$, which is a q-analogue of the polynomial algebra on the vector space Mat_n. Its generators are z_a^α, $(z_a^\alpha)^*$, $a, \alpha = 1, 2, \dots, n$, and the list of relations consists of (2.1), (2.2), (2.3), together with the commutation relations

$$(z_b^\beta)^* z_a^\alpha = q^2 \sum_{a', b', \alpha', \beta' = 1}^{n} R_{ba}^{b'a'} R_{\beta\alpha}^{\beta'\alpha'} z_{a'}^{\alpha'} (z_{b'}^{\beta'})^* + (1 - q^2) \delta_{ab} \delta^{\alpha\beta}, \tag{2.4}$$

where δ_{ab}, $\delta^{\alpha\beta}$ being the Kronecker symbols and

$$R_{ij}^{kl} = \begin{cases} q^{-1}, & i \neq j \,\&\, i = k \,\&\, j = l, \\ 1, & i = j = k = l, \\ -(q^{-2} - 1), & i = j \,\&\, k = l \,\&\, l > j, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the Shilov boundary of the matrix ball $\mathbb{U}$ is just the set $S(\mathbb{U})$ of all unitary matrices. Our intention is to produce a q-analogue of the Shilov boundary for the quantum matrix ball. Introduce the notation for the quantum minors of the matrix $\mathbf{z} = (z_a^\alpha)$:

$$(z^{\wedge k})_{\{a_1,a_2,\dots,a_k\}}^{\{\alpha_1,\alpha_2,\dots,\alpha_k\}} \stackrel{\text{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} z_{a_1}^{\alpha_{s(1)}} z_{a_2}^{\alpha_{s(2)}} \cdots z_{a_k}^{\alpha_{s(k)}},$$

with $\alpha_1 < \alpha_2 < \ldots < \alpha_k$, $a_1 < a_2 < \ldots < a_k$, and $l(s)$ being the number of inversion in $s \in S_k$.

It is well known that the quantum determinant

$$\det_q \mathbf{z} = (z^{\wedge n})_{\{1,2,\dots,n\}}^{\{1,2,\dots,n\}}$$

is in the center of $\mathbb{C}[\mathrm{Mat}_n]_q$. The localization of $\mathbb{C}[\mathrm{Mat}_n]_q$ with respect to the multiplicative system $(\det_q \mathbf{z})^{\mathbb{N}}$ is called the algebra of regular functions on the quantum GL_n and is denoted by $\mathbb{C}[GL_n]_q$.

Lemma 2.1. *i) There exists a unique involution $*$ in $\mathbb{C}[GL_n]_q$ such that*

$$(z_a^\alpha)^* = (-q)^{a+\alpha-2n} (\det_q \mathbf{z})^{-1} \det_q \mathbf{z}_a^\alpha,$$

with $\mathbf{z}_a^\alpha$ being the matrix derived from $\mathbf{z}$ via deleting the line α and the column a.
ii) $(\det_q \mathbf{z})(\det_q \mathbf{z})^* = (\det_q \mathbf{z})^*(\det_q \mathbf{z}) = q^{-n(n-1)}$.

P r o o f. The uniqueness of the involution $*$ is obvious. To prove the existence, consider the $*$-algebra $\mathbb{C}[U_n]_q = (\mathbb{C}[GL_n]_q, \star)$ of regular functions on the quantum U_n (see [5]) and the automorphism $i : \mathbb{C}[GL_n]_q \to \mathbb{C}[GL_n]_q$ given by $i : z_a^\alpha \mapsto q^{\alpha-n} z_a^\alpha$, $a, \alpha = 1, 2, \dots, n$. Obviously, $i^{-1} \star i$ is an involution. What remains is to demonstrate that

$$i^{-1} \star i : z_a^\alpha \mapsto (-q)^{a+\alpha-2n} (\det_q \mathbf{z})^{-1} \det_q \mathbf{z}_a^\alpha.$$

This follows from the definition of $\star$ [5]:

$$(z_a^\alpha)^\star = (-q)^{a-\alpha} (\det_q \mathbf{z})^{-1} \det_q \mathbf{z}_a^\alpha.$$

The statement ii) of the lemma follows from a similar statement for the involution $\star$. ∎

The $*$-algebra $\mathrm{Pol}(S(\mathbb{U}))_q = (\mathbb{C}[GL_n]_q, *)$ is a q-analogue of the polynomial algebra on the Shilov boundary of the matrix ball $\mathbb{U}$, as one can see from

Theorem 2.2. *There exists a unique homomorphism of $*$-algebras $\psi : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(S(\mathbb{U}))_q$ such that $\psi : z_a^\alpha \mapsto z_a^\alpha$, $a, \alpha = 1, 2, \dots, n$.*

We premise the proof of the theorem with two remarks. Firstly, the homomorphism ψ is a q-analogue of the operator which restricts the polynomial onto the Shilov boundary. Secondly, we use in this work a purely algebraic approach to producing the Shilov boundary; we do not try to compare it to the analytic approach of the well known work by W. Arveson [1].

P r o o f. The uniqueness of the $*$-homomorphism ψ is obvious. Turn to the proof of its existence. To produce ψ, we need auxiliary $*$-algebras $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x}$ and $\mathcal{F}_x$, together with the embeddings of algebras $\mathrm{Pol}(\mathrm{Mat}_n)_q \hookrightarrow \mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x}$, $\mathrm{Pol}(S(\mathbb{U}))_q \hookrightarrow \mathcal{F}_x$. The crucial point in the proof is a construction of a homomorphism of $*$-algebras $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x} \to \mathcal{F}_x$ which leads to the commutative diagram

$$\begin{array}{ccc} \mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x} & \longrightarrow & \mathcal{F}_x \\ \uparrow & & \uparrow \\ \mathrm{Pol}(\mathrm{Mat}_n)_q & \longrightarrow & \mathrm{Pol}(S(\mathbb{U}))_q . \end{array}$$

To begin with, introduce the $*$-algebra $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x}$ 'of polynomials on the quantum Plücker manifold'.

Let $\mathbb{C}[\mathrm{Mat}_{2n}]_q$ be the $*$-algebra of functions on the quantum space of matrices Mat_{2n} determined by its generators $\{t_{ij}\}_{i,j=1,2,\dots,2n}$ and commutation relations similar to those listed in (2.1)–(2.3). Introduce the quantum minors

$$t_{IJ}^{\wedge n} = \sum_{s \in S_n} (-q)^{l(s)} t_{i_1 j_{s(1)}} t_{i_2 j_{s(2)}} \cdots t_{i_n j_{s(n)}}$$

determined by pairs of n-element subsets of the form

$$I = \{(i_1, i_2, \dots, i_n) |\, 1 \le i_1 < \ldots < i_n \le 2n\},$$

$$J = \{(j_1, j_2, \dots, j_n) |\, 1 \le j_1 < \ldots < j_n \le 2n\}.$$

Consider the subalgebra in $\mathbb{C}[\mathrm{Mat}_{2n}]_q$ generated by the elements $t_{\{1,2,\dots,n\}J}^{\wedge n}$, $t_{\{n+1,n+2,\dots,2n\}J}^{\wedge n}$, with $\mathrm{card}(J) = n$. It is easy to present a full list of relations between these generators. We follow [8] in equipping this algebra with the involution

$$\left(t_{\{1,2,\dots,n\}J}^{\wedge n}\right)^* = (-1)^{\mathrm{card}(\{1,2,\dots,n\} \cap J)} (-q)^{l(J,J^c)} t_{\{n+1,n+2,\dots,2n\}J^c}^{\wedge n}, \qquad (2.5)$$

where $J^c = \{1, 2, \dots, 2n\} \setminus J$ and $l(J, J^c) = \mathrm{card}\{(j', j'') \in J \times J^c | j' > j''\}$. The $*$-algebra arising this way is denoted by $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$. Let $t = t_{\{1,2,\dots,n\}\{n+1,n+2,\dots,2n\}}^{\wedge n}$ and $x = tt^*$. Obviously, x quasi-commutes with all the generators of $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$.

Let $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x}$ be the localization of the $*$-algebra $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$ with respect to the multiplicative system $x^{\mathbb{N}}$. The results of Section 2 of [8] can be used to prove the following statement intended to shed some light to the way the $*$-algebra $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$ works in our constructions.

Lemma 2.3. *Let* $J_{\alpha a} = \{a\} \cup \{n+1, n+2, \dots, 2n\} \setminus \{2n+1-\alpha\}$. *The map* $\mathcal{I} : z_a^\alpha \mapsto t^{-1} t^{\wedge n}_{\{1,2,\dots,n\}J_a^\alpha}$ *admits an extension up to an embedding of the* $*$*-algebras* $\mathcal{I} : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x}$

The next step in proving Theorem 2.2 is in producing an auxiliary $*$-algebra $\mathcal{F}$, together with a homomorphism of $*$-algebras $\varphi : \mathrm{Pol}(\mathrm{Pl}_{n,2n})_q \to \mathcal{F}$. Let $\mathbb{C}[\mathrm{Pl}_{n,2n}]_q \subset \mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$ be the subalgebra generated by the quantum minors $t^{\wedge n}_{\{1,2,\dots,n\}J}$, $\mathrm{card}(J) = n$, and $\mathbb{C}[\overline{\mathrm{Pl}_{n,2n}}]_q \subset \mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$ the subalgebra generated by the quantum minors $t^{\wedge n}_{\{n+1,n+2,\dots,2n\}J}$, $\mathrm{card}(J) = n$.

$\mathcal{F}$ appears to be an extension of $\mathbb{C}[\mathrm{Pl}_{n,2n}]_q$; it is given by adding the elements η, η^{-1} to the list of generators and the relations

$$\eta\eta^{-1} = \eta^{-1}\eta = 1, \qquad \eta t^{\wedge n}_{\{1,2,\dots,n\}J} = q^{-n} t^{\wedge n}_{\{1,2,\dots,n\}J}\eta, \qquad \mathrm{card}(J) = n$$

to the list of relations. (As it is well known, this list is exhausted by the commutation relations and q-analogues of Plücker relations).

Lemma 2.4. *The map*

$$\varphi : t^{\wedge n}_{\{1,2,\dots,n\}J} \mapsto t^{\wedge n}_{\{1,2,\dots,n\}J}, \tag{2.6}$$

$$\varphi : t^{\wedge n}_{\{n+1,n+2,\dots,2n\}J} \mapsto \eta t^{\wedge n}_{\{1,2,\dots,n\}J} \tag{2.7}$$

admits an extension up to a homomorphism of algebras $\varphi : \mathrm{Pol}(\mathrm{Pl}_{n,2n})_q \to \mathcal{F}$.

P r o o f. The definitions imply the existence of a homomorphism $\mathbb{C}[\mathrm{Pl}_{n,2n}]_q \to \mathcal{F}$ which satisfies (2.6) and a homomorphism $\mathbb{C}[\overline{\mathrm{Pl}}_{n,2n}]_q \to \mathcal{F}$ which satisfies (2.7). The rest of the relations between the generators of $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$ are commutation relations. What remains is to establish that the same commutation relations are also valid for images of the elements $t^{\wedge n}_{\{1,2,\dots,n\}J}$, $t^{\wedge n}_{\{n+1,n+2,\dots,2n\}I}$ with respect to φ. For that, we use the R-matrix form of commutation relations between the matrix elements of the corresponding fundamental representation of the Hopf algebra $U_q\mathfrak{sl}_{2n}$.

$$\mathrm{const}'(q,n) t^{\wedge n}_{\{n+1,n+2,\dots,2n\}I} t^{\wedge n}_{\{1,2,\dots,n\}J}$$
$$= \sum_{\{I',J' | \mathrm{card}(I') = \mathrm{card}(J') = \mathrm{n}\}} R^{I'J'}_{IJ} t^{\wedge n}_{\{1,2,\dots,n\}J'} t^{\wedge n}_{\{n+1,n+2,\dots,2n\}I'},$$

$$\mathrm{const}''(q,n)t^{\wedge n}_{\{1,2,\ldots,n\}I}t^{\wedge n}_{\{1,2,\ldots,n\}J}$$
$$=\sum_{\{I',J'\,|\,\mathrm{card}(I')=\mathrm{card}(J')=\mathrm{n}\}} R^{I'J'}_{IJ}t^{\wedge n}_{\{1,2,\ldots,n\}J'}t^{\wedge n}_{\{1,2,\ldots,n\}I'}.$$

Here $(R^{I'J'}_{IJ})$ is the R-matrix from the right hand side of the well known relation $RTT=TTR$ [2], and the constants $\mathrm{const}'(q,n)$, $\mathrm{const}''(q,n)$ describe the action of the R-matrix in the left hand side of that relation. We are to prove now that $\mathrm{const}''(q,n)=q^{-n}\mathrm{const}'(q,n)$. This follows from

$$t^{\wedge n}_{JI}t^{\wedge n}_{IJ}=t^{\wedge n}_{IJ}t^{\wedge n}_{JI},\qquad t^{\wedge n}_{II}t^{\wedge n}_{IJ}=q^n t^{\wedge n}_{IJ}t^{\wedge n}_{II},$$

with $I=\{1,2,\ldots,n\}$, $J=\{n+1,n+2,\ldots,2n\}$. ■

Equip $\mathcal{F}$ with an involution.

Lemma 2.5. *i) There exists a unique involution* $*$ *in* $\mathcal{F}$ *such that*

$$\eta^*=q^{-n(n-1)}\eta^{-1},$$
$$\left(t^{\wedge n}_{\{1,2,\ldots,n\}J}\right)^*=(-1)^{\mathrm{card}(\{1,2,\ldots,n\}\cap J)}(-q)^{l(J,J^c)}\eta t^{\wedge n}_{\{1,2,\ldots,n\}J^c}.$$

ii) The above homomorphism $\varphi:\mathrm{Pol}(\mathrm{Pl}_{n,2n})_q\to\mathcal{F}$ *is a homomorphism of* $*$*-algebras.*

R e m a r k. The motives to deduce the latter equality are as follows:

$$\begin{aligned}\left(t^{\wedge n}_{\{1,2,\ldots,n\}J}\right)^* &= \left(\varphi\left(t^{\wedge n}_{\{1,2,\ldots,n\}J}\right)\right)^*=\varphi\left(\left(t^{\wedge n}_{\{1,2,\ldots,n\}J}\right)^*\right)\\ &= (-1)^{\mathrm{card}(\{1,2,\ldots,n\}\cap J)}(-q)^{l(J,J^c)}\varphi\left(t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}J^c}\right)\\ &= (-1)^{\mathrm{card}(\{1,2,\ldots,n\}\cap J)}(-q)^{l(J,J^c)}\eta\cdot t^{\wedge n}_{\{1,2,\ldots,n\}J^c}.\end{aligned}$$

P r o o f. The uniqueness of the involution $*$ is straightforward, and its existence follows from (2.5). More precisely, (2.5) and the isomorphism $\mathbb{C}[\mathrm{Pl}_{n,2n}]\tilde{\to}\mathbb{C}[\overline{\mathrm{Pl}_{n,2n}}]$, $t^{\wedge n}_{\{1,2,\ldots,n\}J}\mapsto t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}J}$, $\mathrm{card}(J)=n$, imply the existence of an antilinear antiautomorphism $*:\mathcal{F}\to\mathcal{F}$ with the required properties. The property $**=\mathrm{id}$ is to be verified separately:

$$\left(t^{\wedge n}_{\{1,2,\ldots,n\}J}\right)^{**}=q^{l(J,J^c)+l(J^c,J)}\eta t^{\wedge n}_{\{1,2,\ldots,n\}J}\eta^*=q^{n^2-n}t^{\wedge n}_{\{1,2,\ldots,n\}J}\eta\eta^*=t^{\wedge n}_{\{1,2,\ldots,n\}J}.$$

Now the definition of the involution and (2.5) imply that φ is a $*$-homomorphism. ■

Note that φ admits a unique extension up to a homomorphism $\varphi_x : \mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x} \to \mathcal{F}_x$ of the localizations of $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$ and $\mathcal{F}$ with respect to the multiplicative system $x^{\mathbb{N}}$, $x = tt^*$.

Here is the last element of our construction.

Lemma 2.6. *The map* $z_a^\alpha \mapsto t^{-1} t^{\wedge n}_{\{1,2,\dots,n\}J_{a\alpha}}$, $a, \alpha = 1, 2, \dots, n$, *admits a unique extension up to an embedding of* $*$*-algebras* $\mathrm{Pol}(S(\mathbb{U}))_q \to \mathcal{F}_x$.

P r o o f. Note first that $l(J_{\alpha a}, J^c_{\alpha a}) = a + \alpha + (n-1)^2 - 2$. Hence

$$\begin{aligned}
&\left(t^{-1} t^{\wedge n}_{\{1,2,\dots,n\}J_{a\alpha}}\right)^* \\
&\quad = -(-q)^{a+\alpha+(n-1)^2-2} \eta t^{\wedge n}_{\{1,2,\dots,n\}J^c_{\alpha a}} \left((-q)^{n^2} \eta t^{\wedge n}_{\{1,2,\dots,n\}\{1,2,\dots,n\}}\right)^{-1} \\
&\qquad = -(-q)^{a+\alpha-2n-1} t^{\wedge n}_{\{1,2,\dots,n\}J^c_{\alpha a}} \left(t^{\wedge n}_{\{1,2,\dots,n\}\{1,2,\dots,n\}}\right)^{-1} \\
&\qquad\quad = (-q)^{a+\alpha-2n} \left(t^{\wedge n}_{\{1,2,\dots,n\}\{1,2,\dots,n\}}\right)^{-1} t^{\wedge n}_{\{1,2,\dots,n\}J^c_{\alpha a}} \\
&\qquad\qquad = (-q)^{a+\alpha-2n} \left(t^{-1} t^{\wedge n}_{\{1,2,\dots,n\}\{1,2,\dots,n\}}\right)^{-1} \left(t^{-1} t^{\wedge n}_{\{1,2,\dots,n\}J^c_{\alpha a}}\right).
\end{aligned}$$

On the other hand, in the algebra $\mathbb{C}[\mathrm{Pl}_{n,2n}]_{q,t}$ defined as a localization of $\mathbb{C}[\mathrm{Pl}_{n,2n}]_q$ with respect to the multiplicative system $t^{\mathbb{N}}$, one has the following relations, established in [8]:

$$t^{-1} t^{\wedge n}_{\{1,2,\dots,n\}\{1,2,\dots,n\}} = \det_q \mathbf{z},$$
$$t^{-1} t^{\wedge n}_{\{1,2,\dots,n\}J^c_{a\alpha}} = \det_q \mathbf{z}^\alpha_a.$$

What remains is to use the natural embedding $\mathbb{C}[\mathrm{Pl}_{n,2n}]_{q,t} \to \mathcal{F}_x$. ■

To complete the proof of Theorem 2.2, one has to observe that the $*$-homomorphism $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x} \to \mathcal{F}_x$ takes the images of $z_a^\alpha \in \mathrm{Pol}(\mathrm{Mat}_n)_q$ with respect to the embedding $\mathrm{Pol}(\mathrm{Mat}_n)_q \hookrightarrow \mathrm{Pol}(\mathrm{Pl}_{n,2n})_q$ to the images of the corresponding elements $z_a^\alpha \in \mathrm{Pol}(S(\mathbb{U}))_q$ with respect to the embedding $\mathrm{Pol}(S(\mathbb{U}))_q \hookrightarrow \mathcal{F}_x$. Now Theorem 2.2 is proved. ■

Proposition 2.7. *i)* $\mathrm{Im}\psi = \mathrm{Pol}(S(\mathbb{U}))_q$.
ii) $\ker \psi \subset \mathrm{Pol}(\mathrm{Mat}_n)_q$ *constitutes a* $U_q\mathfrak{sl}_{2n}$*-submodule.*

P r o o f. The first statement follows from $(\det_q \mathbf{z})^{-1} = q^{n(n-1)}(\det_q \mathbf{z})^*$, since $z_a^\alpha \in \mathrm{Im}\psi$ for all $a, \alpha = 1, 2, \dots, n$ and $(\det_q \mathbf{z})^* \in \mathrm{Im}\psi$. The second statement involves the structures of $U_q\mathfrak{sl}_{2n}$-module algebras in $\mathrm{Pol}(\mathrm{Mat}_n)_{q,x}$, $\mathcal{F}_x$, which are imposed in an obvious way (the elements $\eta, \eta^{-1} \in \mathcal{F}_x$ are $U_q\mathfrak{sl}_{2n}$-invariants). The statement to be proved follows now from the fact that the homomorphisms $\mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x}$, $\mathrm{Pol}(\mathrm{Pl}_{n,2n})_{q,x} \to \mathcal{F}_x$ are morphisms of $U_q\mathfrak{sl}_{2n}$-modules. ∎

Recall the notation E_i, F_i, $K_i^{\pm 1}$ for the standard generators of the Hopf algebra $U_q\mathfrak{sl}_{2n}$ and the notation $U_q\mathfrak{su}_{n,n}$ for the Hopf $*$-algebra $(U_q\mathfrak{sl}_{2n}, *)$, with

$$E_n^* = -K_nF_n, \qquad F_n^* = -E_nK_n^{-1}, \qquad (K_n^{\pm 1})^* = K_n^{\pm 1},$$
$$E_j^* = K_jF_j, \qquad F_j^* = E_jK_j^{-1}, \qquad (K_j^{\pm 1})^* = K_j^{\pm 1}, \qquad \text{for} \quad j \neq n.$$

Equip $\mathrm{Pol}(S(\mathbb{U}))_q$ with a structure of $U_q\mathfrak{su}_{n,n}$-module algebra via the canonical isomorphism $\mathrm{Pol}(S(\mathbb{U}))_q \simeq \mathrm{Pol}(\mathrm{Mat}_n)_q/\ker\psi$. Thus, $S(\mathbb{U})_q$ is a homogeneous space of the quantum group $SU_{n,n}$.

To conclude, introduce one more $U_q\mathfrak{su}_{n,n}$-module algebra.

It was noted earlier that z_a^α, a, $\alpha = 1, 2, \ldots, n$, generate the $*$-algebra $\mathrm{Pol}(S(\mathbb{U}))_q$. Consider its extension in the class of $*$-algebras given by adding generator t, together with the additional relations

$$tt^* = t^*t, \quad tz_a^\alpha = q^{-1}z_a^\alpha t, \quad t^*z_a^\alpha = q^{-1}z_a^\alpha t^*, \quad a, \alpha = 1, 2, \ldots, n.$$

This algebra will be denoted by $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$. Our intention is to extend the structure of $U_q\mathfrak{su}_{n,n}$-module algebra from $\mathrm{Pol}(S(\mathbb{U}))_q$ onto $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$. This is accessible via embedding the $*$-algebra $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ into the $*$-algebra $\mathcal{F}_x$: $i : t \mapsto t$, $i : z_a^\alpha \mapsto t^{-1}t^{\wedge n}_{\{1,2,\ldots,n\}J_{\alpha a}}$, $a, \alpha = 1, 2, \ldots, n$. In fact, the $*$-subalgebra $i(\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q)$ contains all the elements t, $t^{-1}t^{\wedge n}_{\{1,2,\ldots,n\}J}$, $\mathrm{card}(J) = n$, and hence is a $U_q\mathfrak{su}_{n,n}$-module subalgebra. What remains is to transfer this structure onto $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$. It follows, as, in [7], that

$$E_jt = F_jt = (K_j^{\pm 1} - 1)t = 0, \qquad j \neq n,$$
$$F_nt = (K_n^{\pm 1} - 1)t = 0, \qquad E_nt = q^{-1/2}tz_n^n.$$

The $U_q\mathfrak{su}_{n,n}$-module algebra $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ will be an essential tool in producing the principal degenerate series of representations of the quantum group $SU_{n,n}$.

3. A $\mathbf{U_q\mathfrak{sl}_{2n}}$-invariant integral

There exists a unique invariant integral on the quantum group U_n normalized in such a way that $\int\limits_{(U_n)_q} 1d\nu = 1$. Use the natural isomorphism of $*$-algebras

$\mathrm{Pol}(S(\mathbb{U}))_q \to \mathbb{C}[U_n]_q$, $z_a^\alpha \mapsto q^{\alpha-n} z_a^\alpha$, $a, \alpha = 1, 2, \dots, n$, to transfer this integral onto $\mathrm{Pol}(S(\mathbb{U}))_q$. It is easy to demonstrate that the linear functional we get this way

$$\mathrm{Pol}(S(\mathbb{U}))_q \to \mathbb{C}, \qquad f \mapsto \int\limits_{S(\mathbb{U})_q} f d\nu,$$

is not $U_q\mathfrak{sl}_{2n}$-invariant provided $\mathrm{Pol}(S(\mathbb{U}))_q$ is equipped with the ordinary structure of $U_q\mathfrak{sl}_{2n}$-module. This section is intended to prove the $U_q\mathfrak{sl}_{2n}$-invariance of this integral with respect to a 'twisted' structure of $U_q\mathfrak{sl}_{2n}$-module in $\mathrm{Pol}(S(\mathbb{U}))_q$. It is custom to say 'these are not functions that should be integrated, but highest order differential forms'. The change of the structure of $U_q\mathfrak{sl}_{2n}$-module in $\mathrm{Pol}(S(\mathbb{U}))_q$ is nothing more than a passage from functions to highest order forms.

Recall the notation E_i, F_i, $K_i^{\pm 1}$ for the standard generators of the Hopf algebra $U_q\mathfrak{sl}_{2n}$, and $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$ stands for its subalgebra, generated by $K_n^{\pm 1}$, E_j, F_j, $K_j^{\pm 1}$, $j \neq n$. It is easy to prove the $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariance of the above integral $f \mapsto \int\limits_{S(\mathbb{U})_q} f d\nu$.

We use the notation $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n) = (U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n), *)$

Proposition 3.1. *There exists such a structure of $U_q\mathfrak{sl}_{2n}$-module in $\mathrm{Pol}(S(\mathbb{U}))_q$ that*

i) its restriction to the subalgebra $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$ coincides with the restriction of the standard structure of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module in $\mathrm{Pol}(S(\mathbb{U}))_q$,

ii) the integral $\mathrm{Pol}(S(\mathbb{U}))_q \to \mathbb{C}$, $f \mapsto \int\limits_{S(\mathbb{U})_q} f d\nu$, is a morphism of $U_q\mathfrak{sl}_{2n}$-modules.

P r o o f. We first produce some structure of $U_q\mathfrak{sl}_{2n}$-module in $\mathrm{Pol}(S(\mathbb{U}))_q$, and then prove that it satisfies i) and ii).

Consider the $U_q\mathfrak{su}_{n,n}$-module $*$-algebra $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ introduced in the previous section. The construction implies that the $*$-algebra $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$ is generated by t, t^*, and the elements of its $U_q\mathfrak{su}_{n,n}$-module subalgebra $\mathrm{Pol}(S(\mathbb{U}))_q$. Consult [7] for explicit formulae which describe the action of the standard generators E_j, F_j, $K_j^{\pm 1}$, $j = 1, 2, \dots, 2n-1$; the action on the conjugate elements is easily derivable from those formulae since

$$(\xi f)^* = (S(\xi))^* f^*, \qquad \xi \in U_q\mathfrak{su}_{n,n}, \qquad f \in \mathrm{Pol}(\widehat{S}(\mathbb{U}))_q.$$

The elements t, t^*, x quasi-commute with all the generators t, z_a^α, $a, \alpha = 1, 2, \dots, n$, of the $*$-algebra $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_q$. This allows one to consider the localization $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_{q,x}$ of this $*$-algebra with respect to the multiplicative system

$x^{\mathbb{N}}$ and then to extend the structure of $U_q\mathfrak{su}_{n,n}$-module algebra onto it. Equip $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_{q,x}$ with a bigrading:

$$\deg t = (1,0), \ \ \deg t^* = (0,1), \ \ \deg(z_a^\alpha) = \deg(z_a^\alpha)^* = (0,0), \ \ a, \alpha = 1, 2, \dots, n.$$

Obviously, the homogeneous components

$$\mathrm{Pol}(\widehat{S}(\mathbb{U}))_{q,x}^{(i,j)} = \{f \in \mathrm{Pol}(\widehat{S}(\mathbb{U}))_{q,x} |\ \deg f = (i,j)\} = t^{*i} \cdot \mathrm{Pol}(S(\mathbb{U}))_q \cdot t^j$$

form submodules of the $U_q\mathfrak{sl}_{2n}$-module $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_{q,x}$. Equip $\mathrm{Pol}(S(\mathbb{U}))_q$ with a structure of $U_q\mathfrak{su}_{n,n}$-module via the vector space isomorphism

$$\mathrm{Pol}(S(\mathbb{U}))_q \to \mathrm{Pol}(\widehat{S}(\mathbb{U}))_{q,x}^{(-n,-n)}, \qquad f \mapsto (t^*)^{-n} f t^{-n}.$$

It follows from the $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariance of t, t^* that the new $U_q\mathfrak{sl}_{2n}$-module structure in $\mathrm{Pol}(S(\mathbb{U}))_q$ coincides with the previous one on the subalgebra $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n) \subset U_q\mathfrak{sl}_{2n}$. So, our integral is again $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant. What remains is to prove that

$$\int\limits_{S(\mathbb{U})_q} (t^*)^n F_n((t^*)^{-n} f t^{-n}) t^n d\nu = 0,$$

$$\int\limits_{S(\mathbb{U})_q} (t^*)^n E_n((t^*)^{-n} f t^{-n}) t^n d\nu = 0, \qquad\qquad f \in \mathrm{Pol}(S(\mathbb{U}))_q.$$

Observe that the integral in question is a real linear functional $\int\limits_{S(\mathbb{U})_q} f^* d\nu = \overline{\int\limits_{S(\mathbb{U})_q} f d\nu}$, and $\mathrm{Pol}(\widehat{S}(\mathbb{U}))_{q,x}$ is a $U_q\mathfrak{su}_{n,n}$-module $*$-algebra. Thus, it suffices to prove the following

Lemma 3.2. *For all* $f \in \mathrm{Pol}(S(\mathbb{U}))_q \simeq \mathrm{Pol}(\widehat{S}(\mathbb{U}))_{q,x}^{(0,0)}$ *one has*

$$\int\limits_{S(\mathbb{U})_q} (t^*)^n F_n((t^*)^{-n} f t^{-n}) t^n d\nu = 0. \tag{3.1}$$

P r o o f. Identify the subalgebra $\mathrm{Pol}(S(\mathbb{U}))_q$ with its image under the embedding into $\mathcal{F}_x$ to note that $F_n t = F_n \eta = 0$, $t^* = (-q)^{n^2} \eta t \det_q \mathbf{z}$. Hence (3.1) is equivalent to

$$\int\limits_{S(\mathbb{U})_q} (\det_q \mathbf{z})^n F_n((\det_q \mathbf{z})^{-n} f) d\nu = 0. \tag{3.2}$$

On the other hand, $K_n \det_q \mathbf{z} = q^2 \det_q \mathbf{z}$, as one can see from an explicit form for $K_n z_a^\alpha$, $a, \alpha = 1, 2, \ldots, n$. Hence, by a virtue of the $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$-invariance of the integral in question (3.2) is also equivalent to

$$\int\limits_{S(\mathbb{U})_q} (\det_q \mathbf{z})^n \widetilde{F}_n((\det_q \mathbf{z})^{-n} f) d\nu = 0, \tag{3.3}$$

with $\widetilde{F}_n = K_n F_n$.

We start with proving (3.2), (3.3) in the special case $f = z_n^n$. In view of the relation $F_n z_n^n = q^{1/2}$ one has

$$(\det_q \mathbf{z})^n F_n((\det_q \mathbf{z})^{-n} z_n^n) = (\det_q \mathbf{z})^n F_n((\det_q \mathbf{z})^{-n} K_n^{-1} z_n^n) + q^{1/2}.$$

Apply the explicit form for $F_n z_a^\alpha$, $K_n^{\pm 1} z_a^\alpha$, $a, \alpha = 1, 2, \ldots, n$, found in [7] to establish that $K_n^{-1} z_n^n = q^{-2} z_n^n$,

$$(\det_q \mathbf{z})^n F_n((\det_q \mathbf{z})^{-n}) = q^{1/2} \frac{q^{2n} - 1}{q^{-2} - 1} (\det_q \mathbf{z})^{-1} \det_q(\mathbf{z}_n^n).$$

Hence

$$(\det_q \mathbf{z})^n F_n((\det_q \mathbf{z})^{-n} z_n^n) = q^{1/2} \frac{q^{2n} - 1}{q^{-2} - 1} (\det_q \mathbf{z})^{-1} \det_q(\mathbf{z}_n^n) q^{-2} z_n^n + q^{1/2}.$$

On the other hand, $(\det_q \mathbf{z})^{-1} \det_q(\mathbf{z}_n^n) = (z_n^n)^*$. Thus to prove (3.2) in the special case $f = z_n^n$, we need only to verify that

$$\int\limits_{S(\mathbb{U})_q} (z_n^n)^* z_n^n d\nu = \frac{1 - q^2}{1 - q^{2n}}.$$

For that, it suffices to elaborate the following relations:

$$\int\limits_{S(\mathbb{U}_n)_q} (z_n^n)^* z_n^n d\nu = \int\limits_{(U_n)_q} (z_n^n)^* z_n^n d\nu,$$

$$\int\limits_{(U_n)_q} (z_n^n)^* z_n^n d\nu = \frac{q^{-(n-1)}}{q^{n-1} + q^{n-3} + \ldots + q^{-(n-3)} + q^{-(n-1)}} = \frac{1 - q^2}{1 - q^{2n}}.$$

The first relation follows from the explicit form for the natural isomorphism of $*$-algebras $\mathrm{Pol}(S(\mathbb{U}))_q \simeq \mathbb{C}[U_n]_q$, and the second one is a consequence of the orthogonality relations for the quantum group SU_n (see [2, p. 457]).

In the special case $f = z_n^n$, (3.2), and hence (3.3) are proved.

Turn to the general case $f \in \mathrm{Pol}(S(\mathbb{U}))_q$.

Normally we work with admissible (see [7, 8]) modules over quantum enveloping algebras. We also use the standard basis $H_1, \dots, H_{2n-1}$ of the Cartan subalgebra of $\mathfrak{sl}_{2n}$.

Let $H_0 = \sum\limits_{j=1}^{n-1} jH_j + nH_n + \sum\limits_{j=1}^{n-1} jH_{2n-j}$ be the element of the Cartan subalgebra of the Lie algebra $\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n) \subset \mathfrak{sl}_{2n}$, which is in its center and is normalized so that $[H_0, E_n] = 2E_n$, $[H_0, F_n] = -2F_n$. Associate to every admissible simple finite dimensional $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module V a pair $(\boldsymbol{\lambda}, k)$, with $\boldsymbol{\lambda}$ being the highest weight of the corresponding $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_n$-module, and the number k is determined by $H_0 v = 2kv$, $v \in V$. It is easy to obtain the following decomposition of the $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module $\mathrm{Pol}(S(\mathbb{U}))_q$ into a sum of simple finite dimensional pairwise non-isomorphic $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-modules:

$$\mathrm{Pol}(S(\mathbb{U}))_q = \bigoplus_{(\boldsymbol{\lambda},k)} V_{(\boldsymbol{\lambda},k)}.$$

Assume that for some $f \in \mathrm{Pol}(S(\mathbb{U}))_q$ (3.3) fails. Our immediate intention is to prove that f belongs to some component $V_{(\boldsymbol{\lambda}',k')}$, and to find the corresponding pair $(\boldsymbol{\lambda}', k')$.

The presence of the structure of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module in $\mathrm{Pol}(S(\mathbb{U}))_q$ leads to a structure of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module in $\mathrm{End}_{\mathbb{C}}\mathrm{Pol}(S(\mathbb{U}))_q$ and a morphism of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module algebras $U_q\mathfrak{sl}_{2n} \to \mathrm{End}_{\mathbb{C}}\mathrm{Pol}(S(\mathbb{U}))_q$. Consider the linear operator in $\mathrm{Pol}(S(\mathbb{U}))_q$:

$$L : \psi \mapsto (\det\nolimits_q \mathbf{z})^n \widetilde{F}_n((\det\nolimits_q \mathbf{z})^{-n}\psi).$$

Prove that the $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-submodule of the $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module $\mathrm{End}_{\mathbb{C}}\mathrm{Pol}(S(\mathbb{U}))_q$ generated by L is isomorphic to $V_{(\boldsymbol{\lambda}'',k'')}$, with $\boldsymbol{\lambda}'' = (0, \dots, 0, 1) \times (1, 0, \dots, 0)$, $k'' = -1$. In fact, $\widetilde{F}_n$ generates a finite dimensional simple $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-submodule of the $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module $U_q\mathfrak{sl}_{2n}$ in view of the Serre relations for $F_1, F_2, \dots, F_{2n-1}$ and the relation

$$\mathrm{ad}_q(E_j)\widetilde{F}_n = 0, \qquad j \neq n.$$

What remains is to apply the commutation relations between $K_j^{\pm 1}$, $j = 11, 2, \dots, 2n-1$, and $\widetilde{F}_n$.

Now we are in a position to finish the proof that $f \in V_{(\boldsymbol{\lambda}',k')}$ and to find $\boldsymbol{\lambda}'$, k'. Use the natural embedding $\mathrm{End}_{\mathbb{C}}\mathrm{Pol}(S(\mathbb{U}))_q \otimes \mathrm{Pol}(S(\mathbb{U}))_q \to \mathrm{Pol}(S(\mathbb{U}))_q$ and the fact that $L \in V_{(\boldsymbol{\lambda}'',k'')}$, $Lf \in \mathbb{C}1$, to get a non-zero morphism of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-modules $V_{(\boldsymbol{\lambda}'',k'')} \otimes \left(\bigoplus\limits_{(\boldsymbol{\lambda},k)} V_{(\boldsymbol{\lambda},k)} \right) \to \mathbb{C}$. This means that $k' = -k''$ and $\boldsymbol{\lambda}'$ is the highest weight of the dual representation of $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_n$. Finally,

$\boldsymbol{\lambda}' = (1, 0, \dots, 0) \times (0, \dots, 0, 1)$, $k' = 1$. Furthermore, f is the lowest weight vector in $V_{(\boldsymbol{\lambda}', k')}$ since L is the highest weight vector in $V_{(\boldsymbol{\lambda}'', k'')}$. It follows that $f = \mathrm{const} \cdot z_n^n$, which contradicts to the relation (3.3) proved earlier for z_n^n. ∎

4. On certain irreducible $*$-representation of $U_q\mathfrak{su}_{n,n}$

Consider the embedding $\mathcal{I} : \mathbb{C}[\mathrm{Mat}_n]_q \hookrightarrow \mathbb{C}[\mathrm{Pl}_{n,2n}]_{q,t}$ described in Lemma 2.3 and another embedding

$$\mathcal{J} : \mathbb{C}[\mathrm{Mat}_n]_q \to \mathbb{C}[\mathrm{Pl}_{n,2n}]_{q,t}, \qquad \mathcal{J} : f \mapsto (\mathcal{I}f)t^{-n}.$$

It is easy to verify that $\mathcal{J}\mathbb{C}[\mathrm{Mat}_n]_q$ is a submodule of the $U_q\mathfrak{sl}_{2n}$-module $\mathbb{C}[\mathrm{Pl}_{n,2n}]_{q,t}$. Hence, there exists a unique representation π of $U_q\mathfrak{sl}_{2n}$ in the vector space $\mathbb{C}[\mathrm{Mat}_n]_q$ such that

$$\pi(\xi)f = \mathcal{J}^{-1}\xi(\mathcal{J}f), \qquad f \in \mathbb{C}[\mathrm{Mat}_n]_q, \quad \xi \in U_q\mathfrak{sl}_{2n}.$$

We refer to the previous sections for constructions of a $*$-homomorphism

$$\psi : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(S(\mathbb{U}))_q, \qquad \psi : f \mapsto f|_{S(\mathbb{U})_q},$$

and an invariant integral

$$\mathrm{Pol}(S(\mathbb{U}))_q \to \mathbb{C}, \qquad f \mapsto \int\limits_{S(\mathbb{U})_q} f d\nu.$$

These are to be used to equip the vector space $\mathbb{C}[\mathrm{Mat}_n]_q$ with a Hermitian scalar product

$$(f_1, f_2) \stackrel{\mathrm{def}}{=} \int\limits_{S(\mathbb{U})_q} (f_2|_{S(\mathbb{U})_q})^* f_1|_{S(\mathbb{U})_q} d\nu.$$

The following lemma is a consequence of the definitions of $U_q\mathfrak{sl}_{2n}$-module structures and invariance of the integral.

Lemma 4.1. *For all $\xi \in U_q\mathfrak{su}_{n,n}$, $f_1, f_2 \in \mathbb{C}[\mathrm{Mat}_n]_q$ one has*

$$(\pi(\xi)f_1, f_2) = (f_1, \pi(\xi^*)f_2).$$

To rephrase, π is a $*$-representation of $U_q\mathfrak{su}_{n,n}$ in the pre-Hilbert space $\mathbb{C}[\mathrm{Mat}_n]_q$. The proof of irreducibility for this representation uses the following

Lemma 4.2. *If for some $f \in \mathbb{C}[\mathrm{Mat}_n]_q$ and some $m_1, m_2, \dots, m_{2n-1} \in \mathbb{Z}$ one has*

$$F_j f = 0, \qquad K_j^{\pm 1} f = q^{\pm m_j} f, \qquad j = 1, 2, \dots, 2n-1, \tag{4.1}$$

then $f \in \mathbb{C}1$.

P r o o f. Due to $F_i f = 0$, $i \neq n$, $K_j^{\pm 1} f = q^{\pm m_j} f$, $j = 1, 2, \ldots, 2n-1$, it follows that, up to a constant complex multiple

$$f = (z_n^n)^{k_n} \left(\det_q \mathbf{z}^{\wedge 2\{n-1,n\}}_{\{n-1,n\}}\right)^{k_{n-1}} \left(\det_q \mathbf{z}^{\wedge 3\{n-2,n-1,n\}}_{\{n-2,n-1,n\}}\right)^{k_{n-2}} \cdots (\det_q \mathbf{z})^{k_1}$$

for some $k_1, k_2, \ldots, k_n \in \mathbb{Z}_+$. Prove that $F_n f = 0$ implies $k_1 = k_2 = \ldots = k_n = 0$. Let J be the two-sided ideal of $\mathbb{C}[\mathrm{Mat}_n]_q$, determined by the 'non-diagonal' generators z_a^α, $a \neq \alpha$. Obviously, $K_n^{-1} J \subset J$, $F_n J \subset J$. This allows one to arrange computations modulo the ideal J:

$$F_n f = \mathrm{const} \prod_{a=1}^{n-1} (z_a^\alpha)^{\sum\limits_{i=1}^{a} k_i} (z_n^n)^{\sum\limits_{i=1}^{n} k_i - 1} \qquad (\mathrm{mod}\ J)$$

$$\mathrm{const} = q^{1/2} \sum_{j=1}^{n} q^{-2 \sum\limits_{i=j+1}^{n} k_i} \cdot \frac{q^{-2k_j} - 1}{q^{-2} - 1}.$$

Now the isomorphism $\mathbb{C}[\mathrm{Mat}_n]_q / J \simeq \mathbb{C}[z_1^1, z_2^2, \ldots, z_n^n]$ implies the relation $k_1 = k_2 = \ldots = k_n = 0$, which is just our statement. ■

Proposition 4.3. *The representation π is irreducible.*

P r o o f. Suppose π is reducible. Then by Lemma 4.1 it is a sum of two nontrivial admissible subrepresentations in the subspaces L_1, L_2: $\mathbb{C}[\mathrm{Mat}_n]_q = L_1 \oplus L_2$. Each of those further decomposes as a sum of weight subspaces and, in particular, possesses a lowest weight vector. This vector $f \neq 0$ satisfies the relations

$$\pi(F_j) f = 0, \qquad \pi(K_j^{\pm 1}) f = q^{\pm m_j} f, \qquad j = 1, 2, \ldots, 2n-1.$$

On the other hand, in $\mathbb{C}[\mathrm{Pl}_{n,2n}]_{q,t}$ one has:

$$\pi(K_j^{\pm 1}) t = \begin{cases} q^{\mp 1} t, & j = n \\ t, & j \neq n \end{cases}, \qquad \pi(F_j) t = 0, \qquad j = 1, 2, \ldots, 2n-1.$$

We conclude that f is a solution of the equation system (4.1), so, by a virtue of Lemma 4.2, $f \in \mathbb{C}1$. Hence, $L_1 \supset \mathbb{C}1$, $L_2 \supset \mathbb{C}1$, $L_1 \cap L_2 \neq 0$. This is a contradiction coming from our assumption on reducibility of π. The proposition is proved. ■

5. The Cauchy–Szegö integral representation

Recall that $\mathbb{C}[\overline{\mathrm{Mat}}_n]_q \subset \mathrm{Pol}(\mathrm{Mat}_n)_q$ stands for the subalgebra generated by $(z_a^\alpha)^*$, $a, \alpha = 1, 2, \ldots, n$. We follow [8] in introducing the algebra

$$\mathbb{C}[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]_q = \mathbb{C}[\mathrm{Mat}_n]_q^{\mathrm{op}} \otimes \mathbb{C}[\overline{\mathrm{Mat}}_n]_q,$$

with op indicating the change of a multiplication law to the opposite one. This algebra is bigraded:

$$\deg(z_a^\alpha \otimes 1) = (1, 0), \qquad \deg(1 \otimes (z_a^\alpha)^*) = (0, 1).$$

Its completion with respect to this bigrading is denoted by $\mathbb{C}[[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]]_q$. The elements of $\mathbb{C}[[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]]_q$ are q-analogues for kernels of integral operators, while the elements of the subalgebra $\mathbb{C}[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]_q$ are q-analogues of polynomial kernels.

We refer to [8] for a definition of the pairwise commuting 'kernels'

$$\chi_k = \sum_{\substack{J', J'' \subset \{1, 2, \ldots, n\} \\ \mathrm{card}(J') = \mathrm{card}(J'') = k}} z^{\wedge k}{}^{J'}_{J''} \otimes \left(z^{\wedge k}{}^{J'}_{J''} \right)^* \in \mathbb{C}[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]_q$$

and a one-parameter family of the elements of $\mathbb{C}[[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]]_q$ which includes the kernel

$$C_q = \prod_{j=0}^{n-1} \left(1 + \sum_{k=1}^{n} (-q^{2j})^k \chi_k \right)^{-1}.$$

We write $C_q(\mathbf{z}, \boldsymbol{\zeta}^*)$ instead of C_q, $C_q(\mathbf{z}, \boldsymbol{\zeta}^*) \cdot f(\boldsymbol{\zeta})$ instead of $C_q \cdot (1 \otimes f)$, and $\int\limits_{S(\mathbb{U})_q} f(\boldsymbol{\zeta}) d\nu(\boldsymbol{\zeta})$ instead of $\int\limits_{S(\mathbb{U})_q} f|_{S(\mathbb{U})_q} d\nu$, as it is custom in the classical analysis. It is easy to demonstrate [8, Proposition 2.11] that in the formal passage to a limit $q \to 1$ one has $C_q(\mathbf{z}, \boldsymbol{\zeta}^*) \to \det(1 - \mathbf{z}\boldsymbol{\zeta}^*)^{-n}$.

We call $C_q \in \mathbb{C}[[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]]_q$ the Cauchy–Szegö kernel for the quantum matrix ball.

The following result provides a motivation for this definition.

Theorem 5.1. *For any element $f \in \mathbb{C}[\mathrm{Mat}_n]_q$ one has*

$$f(\mathbf{z}) = \int\limits_{S(\mathbb{U})_q} C_q(\mathbf{z}, \boldsymbol{\zeta}^*) f(\boldsymbol{\zeta}) d\nu(\boldsymbol{\zeta}).$$

P r o o f. Equip $\mathrm{Pol}(\mathrm{Mat}_n)_q$ with a grading: $\deg(z_a^\alpha)=1$, $\deg(z_a^\alpha)^*=-1$, $a,\alpha=1,2,\ldots,n$. It is easy to show that $\int\limits_{S(\mathbb{U})_q} f|_{S(\mathbb{U})_q} d\nu=0$ for all f with $\deg(f)\neq 0$. Hence the integral operator

$$T:\mathbb{C}[\mathrm{Mat}_n]_q\to\mathbb{C}[\mathrm{Mat}_n]_q,\qquad T: f(z)\mapsto\int\limits_{S(\mathbb{U})_q} C_q(\mathbf{z},\boldsymbol{\zeta}^*)f(\boldsymbol{\zeta})d\nu(\boldsymbol{\zeta})$$

is well defined. It follows from the invariance of the integral on the Shilov boundary and the results of [8, Section 8] that $\pi(\xi)T=T\pi(\xi)$ for all $\xi\in U_q\mathfrak{sl}_{2n}$. Furthermore, $T1=1$. Hence $Tf=f$ for all $f\in\mathbb{C}[\mathrm{Mat}_n]_q$ in view of Proposition 4.3. The theorem is proved. ∎

Note that there exists another proof of Theorem 5.1 which uses only the orthogonality relations for the quantum group U_n and one of the Milne's relations for Schur's functions [6]. We intend to publish this proof in a future work.

6. Appendix. Another description of the $*$-algebra $\mathrm{Pol}(S(\mathbb{U}))_q$ and its generalization onto the case of rectangular matrices

We produce a system of equations which distinguish the quantum Shilov boundary from the quantum matrix space. This is certainly equivalent to describing the $*$-algebra $\mathrm{Pol}(S(\mathbb{U}))_q$ in terms of generators and relations.

Proposition 6.1. *In* $\mathrm{Pol}(S(\mathbb{U}))_q$ *the following relations are valid:*

$$\sum_{j=1}^n q^{2n-\alpha-\beta}z_j^\alpha(z_j^\beta)^*-\delta^{\alpha\beta}=0,\qquad \alpha,\beta=1,2,\ldots,n. \tag{6.1}$$

The left hand sides of these equations generate the kernel of the canonical homomorphism $\psi:\mathrm{Pol}(\mathrm{Mat}_n)_q\to\mathrm{Pol}(S(\mathbb{U}))_q$.

P r o o f. The first statement is due to the fact that in $\mathbb{C}[U_n]_q$

$$\sum_{j=1}^n z_j^\alpha(z_j^\beta)^\star-\delta^{\alpha\beta}=0,\qquad \alpha,\beta=1,2,\ldots,n.$$

Let $\mathcal{I}$ be the two-sided ideal of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ generated by the left hand sides of (6.1) and $A=\mathrm{Pol}(\mathrm{Mat}_n)_q/\mathcal{I}$. One has to prove that the canonical onto map $j:A\to\mathrm{Pol}(S(\mathbb{U}))_q$ is in fact an isomorphism.

First prove that in A

$$\det_q \boldsymbol{z} \cdot (\det_q \boldsymbol{z})^* = q^{-n(n-1)}. \tag{6.2}$$

Equip A with a $\mathbb{Z}$-grading as follows:

$$\deg(z_a^\alpha) = 1, \qquad \deg((z_a^\alpha)^*) = -1, \qquad a, \alpha = 1, 2, \dots, n.$$

It was demonstrated in Section 2 that the relation (6.2) is valid in $\mathrm{Pol}(S(\mathbb{U}))_q$. What remains is to use the fact that the algebra A is a $U_q\mathfrak{sl}_n$-module algebra, together with the following statement:

Lemma 6.2.
i) $\det_q \boldsymbol{z} \cdot (\det_q \boldsymbol{z})^* \in A$ *is a* $U_q\mathfrak{sl}_n$*-invariant of degree zero.*
ii) The subalgebra of $U_q\mathfrak{sl}_n$*-invariants of degree zero in* A *is one-dimensional.*

P r o o f of Lemma 6.2. The first statement is obvious. Turn to the second statement. For any $U_q\mathfrak{sl}_n$-invariant of degree zero $f \in A$ there exists such $U_q\mathfrak{sl}_n$-invariant of degree zero $\widehat{f} \in \mathrm{Pol}(\mathrm{Mat}_n)_q$ that $f = j \cdot \widehat{f}$.[1] On the other hand, $\mathrm{Pol}(\mathrm{Mat}_n)_q$ is a free $\mathbb{C}[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]$-module generated by $1 \in \mathrm{Pol}(\mathrm{Mat}_n)_q$:

$$f_1 \otimes f_2 : g_1 \otimes g_2 \mapsto g_1 f_1 \otimes f_2 g_2.$$

Consider $\widetilde{f} \in \mathbb{C}[\mathrm{Mat}_n \times \overline{\mathrm{Mat}}_n]$ such that $\widehat{f} = \widetilde{f} \circ 1$. It suffices to prove that $\widetilde{f}$ is in the subalgebra generated by $\sum\limits_{j=1}^{n} q^{2n-\alpha-\beta} z_j^\alpha \otimes (z_j^\beta)^*$, $\alpha, \beta = 1, 2, \dots, n$. This is a consequence of the fact that $\widetilde{f}$ is a $U_q\mathfrak{sl}_n$-invariant of degree zero, and a q-analogue of the first main theorem of the theory of invariants [3]. ■

It follows from (6.2) that $\det_q \boldsymbol{z} \in A$ is invertible. Hence one has a well defined homomorphism of algebras $j' : \mathrm{Pol}(S(\mathbb{U}))_q \to A$, $j' : z_a^\alpha \mapsto z_a^\alpha$, $a, \alpha = 1, 2, \dots, n$. Obviously, $jj' = \mathrm{id}$. What remains is to prove that j' is onto, that is the elements $(\det_q \boldsymbol{z})^{-1}$, z_a^α, $a, \alpha = 1, 2, \dots, n$, generate the algebra A. It follows from (6.1) and well known properties of quantum determinants that in A one has

$$(z_a^\alpha)^* = (-q)^{a+\alpha-2n} (\det_q \boldsymbol{z})^{-1} \det_q \boldsymbol{z}_a^\alpha, \qquad a, \alpha = 1, 2, \dots, n.$$

So, j is invertible. The proof of Proposition 6.1 is complete. ■

Thus we get another description of the $*$-algebra $\mathrm{Pol}(S(\mathbb{U}))_q$. It is to be used for producing a q-analogue for the Shilov boundary of a unit ball in the space of rectangular matrices $\mathrm{Mat}_{m,n}$, $m < n$.

[1] Due to local finiteness of the $U_q\mathfrak{sl}_n$-modules $\mathcal{I}$, $\mathrm{Pol}(\mathrm{Mat}_n)_q$, A.

The classical theory of Cartan domains [4] provides a well known procedure of producing the Shilov boundary $S(\mathbb{U}')$ of the unit ball $\mathbb{U}' \subset \mathrm{Mat}_{m,n}$, together with the associated Cauchy–Szegö kernel for the unit ball $\mathbb{U} \subset \mathrm{Mat}_n$. We are going to apply exactly this method in the quantum case.

Consider the $*$-subalgebra $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q \subset \mathrm{Pol}(\mathrm{Mat}_n)_q$ generated by z_a^α, $\alpha > n-m$, and the $*$-Hopf algebra $U_q\mathfrak{su}_{m,n} \subset U_q\mathfrak{su}_{n,n}$ generated by E_j, F_j, $K_j^{\pm 1}$, $j < m+n$. It is easy to demonstrate that $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$ is a $U_q\mathfrak{su}_{m,n}$-module subalgebra of the $U_q\mathfrak{su}_{m,n}$-module algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$. Up to relabeling the generators, this $U_q\mathfrak{su}_{m,n}$-module algebra coincides with the $*$-algebra of polynomials in the quantum matrix space considered in [7, 8].

Introduce the notation $\mathrm{Pol}(S(\mathbb{U}'))_q$ for the $*$-algebra determined by its generators

$$z_a^\alpha, \qquad a = 1, 2, \ldots, n; \quad \alpha = n-m+1, n-m+2, \ldots, n,$$

the commutation relations loan from $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$ and the additional relations

$$\sum_{j=1}^{n} q^{2n-\alpha-\beta} z_j^\alpha (z_j^\beta)^* = \delta^{\alpha,\beta}, \qquad \alpha, \beta = n-m+1, n-m+2, \ldots, n.$$

Our construction implies a commutative diagram

$$\begin{array}{ccc} \mathrm{Pol}(\mathrm{Mat}_{m,n})_q & \longrightarrow & \mathrm{Pol}(\mathrm{Mat}_n)_q \\ \downarrow & & \downarrow \\ \mathrm{Pol}(S(\mathbb{U}'))_q & \longrightarrow & \mathrm{Pol}(S(\mathbb{U}))_q. \end{array}$$

where the vertical arrows stand for the homomorphisms of restriction of 'polynomials' onto the Shilov boundaries of the corresponding quantum balls. The homomorphism $\mathrm{Pol}(S(\mathbb{U}'))_q \to \mathrm{Pol}(S(\mathbb{U}))_q$ allows one to transfer an invariant integral from $\mathrm{Pol}(S(\mathbb{U}))_q$ onto $\mathrm{Pol}(S(\mathbb{U}'))_q$. The Cauchy–Szegö kernel is defined as in Section 5 except that in the expression for χ_k, the summing up in $J' \subset \{1, 2, \ldots, n\}$ is replaced with summing up in $J' \subset \{n-m+1, n-m+2, \ldots, n\}$. Another result of Section 5, a q-analogue of the integral Cauchy–Szegö representation for the matrix ball $\mathbb{U}'$, implies a similar integral representation for the quantum matrix ball in the space of rectangular matrices.

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Квантовый матричный шар: ядро Коши–Сегё и граница Шилова

Л. Ваксман

Построен q-аналог интегрального представления Коши–Сегё, которое восстанавливает голоморфную функцию в матричном шаре по её значениям на границе Шилова. Кроме того, описывается граница Шилова квантового матричного шара и устанавливается $U_q\mathfrak{su}_{m,n}$-ковариантность $U_q\mathfrak{s}(\mathfrak{u}_m \times \mathfrak{u}_n)$-инвариантного интеграла на этой границе. Последний результат позволяет получить q-аналог основной вырожденной серии унитарных представлений, связанных с границей Шилова матричного шара.

Квантова матрична куля: ядро Коші–Сегьо та межа Шилова

Л. Ваксман

Побудовано q-аналог інтегрального зображення Коші–Сегьо, яке відновлює голоморфну функцію в матричній кулі по її значенням на межі Шилова. Крім того, описано межу Шилова квантової матричної кулі та встановлено $U_q\mathfrak{su}_{m,n}$-коваріантність $U_q\mathfrak{s}(\mathfrak{u}_m \times \mathfrak{u}_n)$-інваріантного інтеграла на цій межі. Останній результат дає можливість отримати q-аналог основної виродженної серії унітарних представлень, які пов'язані з межею Шилова матричної кулі.