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Dynamical entropy for a class of algebraic origin automorphisms

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We consider an automorphism of C^*-algebra which arises in connection with Gibb's states in the Ruelle theory of dynamical systems, and estimate the Voiculesku's topological entropy for such automorphisms using Choda's approach to the entropy.

Initially a non-commutative version of the Kolmogorov–Sinai entropy was introduced by Connes and Störmer [CS] in 1976 for trace preserving automorphisms of finite von Neumann algebras. This notion was extended later by Connes, Narnhofer and Thirring [CNT] in 1987 onto the dynamical entropy $h_\phi(\alpha)$ for an automorphism α of a C^*-algebra A which preserves a given state ϕ. We also consider the topological entropy $ht(\alpha)$ for automorphisms of C^*-algebras that was introduced by Voiculescu [Vo] for nuclear C^*-algebras and which was extended by Brown onto automorphisms of exact C^*-algebras. In the general case $ht(\alpha) \geq h_\phi(\alpha)$ ([Vo, Pr. 4.6]).

In this paper we estimate the dynamical topological entropy for some class of II_1 factor automorphisms which arise in the context of Ruelle theory of Gibb's states for automorphisms of a complete metric space (see [Ca]). Though we consider a particular case, this result is apparently extensible to all the expansive automorphisms of torus.

We will use the definition of the entropy for an automorphism of an amenable discrete countable group $h_{Ch}(\alpha)$ that was introduced by Choda [Ch] and the notion of the topological entropy for an automorphism of compact topological space $h_{top}(\alpha)$ (see [Wa]).

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Theorem 1. *Let G be a discrete countable Abelian group and let α be an automorphism of this group. Let Γ be the (compact) dual group of G and let $\hat{\alpha}$ be the adjoint automorphism. Then*

$$h_{Ch}(\alpha) = h_{top}(\hat{\alpha}). \tag{1}$$

T h e p r o o f follows from the works of Peters [Pe] and Choda [Ch]. In the paper [Ch, Col. 3.6] it is shown that $h_{top}(\hat{\alpha}) \leq h_{Ch}(\alpha)$. From the paper [Pe, Th. 6] we conclude that the entropy for an automorphism of a discrete countable Abelian group $h_{Pet}(\alpha)$ proposed by Peters satisfies the following equality $h_{top}(\hat{\alpha}) = h_{Pet}(\alpha)$. Since $h_{Ch}(\alpha) \leq h_{Pet}(\alpha)$ we obtain the required equality: $h_{Ch}(\alpha) = h_{Pet}(\alpha) = h_{top}(\hat{\alpha})$. ■

E x a m p l e 1. Consider the case of $G = \mathbb{Z}^2$. Let γ be an automorphism given by some integer matrix T with $\det T = 1$. The dual group for $\mathbb{Z}^2$ is $\mathbb{T}^2$ and the adjoint automorphism $\hat{\gamma}$ is given by the matrix $\hat{T} = (T^{-1})^*$. Since $\det T = 1$, $\hat{T}$ has the same eigenvalues λ_1 and λ_2 as T. Therefore we have $h_{Ch}(\gamma) = h_{top}(\hat{\gamma}) = \log \max\{|\lambda_1|, |\lambda_2|\}$ according to [Li].

E x a m p l e 2. Let γ be an automorphism of $\mathbb{T}^2$ determined by the matrix $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, then $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2}$ are the eigenvalues of this matrix, and let E_{λ_1} and E_{λ_2} be their corresponding eigenspaces.

Define the subgroup $A \subset \mathbb{T}^2$ in the following way $A = \{(a_1, a_2) : \exists\, k, l, m, n \in \mathbb{Z} :\ (a_1 + k, a_2 + l) \in E_{\lambda_1},\ (a_1 + m, a_2 + n) \in E_{\lambda_1}\}$. The automorphism γ_A of $\mathbb{T}^2$ whose matrix is $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ induces an automorphism of A in a natural way.

According to [Ca, Ex. 2.2] the group A is a dense subgroup of the torus invariant with respect to γ. It has the following property: two elements $x, y \in \mathbb{T}^2$ are conjugate if and only if $x - y \in A$. That is, $d(\gamma^n x, \gamma^n y) \xrightarrow[n\to\infty]{} 0$ if and only if $x - y \in A$, where d is the ordinary distance on the torus. Hence we obtain in particular $d(\gamma^n(a + t), \gamma^n t) \xrightarrow[n\to\infty]{} 0$, for any $a \in A,\ t \in \mathbb{T}^2$.

Besides it turns out that we can construct the isomorphism $\theta : A \to \mathbb{Z}^2$ and the automorphism γ induces the automorphism $\gamma_{\mathbb{Z}^2}$ on $\mathbb{Z}^2$ given by the matrix $\widetilde{T} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ with the same eigenvalues λ_1 and λ_2. This together with Example 1 implies $h_{Ch}(\gamma_A) = h_{Ch}(\gamma_{\mathbb{Z}^2}) = \log \frac{3+\sqrt{5}}{2}$.

Denote by $\mathcal{M} = C(\mathbb{T}^2)$ the C^*-algebra of continuous complex function on the torus.

The automorphism γ induces on $\mathcal{M}$ the automorphism $\gamma_{\mathcal{M}}$

$$(\gamma_{\mathcal{M}} f)(t) = f(\gamma t), \text{ where } f \in \mathcal{M},\ t \in \mathbb{T}^2.$$

According to [Vo, Pr. 4.8] $ht(\gamma_{\mathcal{M}}) = h_{top}(\gamma) = \log|\lambda_1| = \log\frac{3+\sqrt{5}}{2}$.

Consider the action σ of A on $\mathcal{M}$ by left shifts, i.e., $\sigma_a(f)(t) = f(a+t)$, where $a \in A,\ f \in \mathcal{M},\ t \in \mathbb{T}^2$. Let $\mathcal{N} = \mathcal{M} \rtimes_\sigma A$ be a C^*-cross product, i.e., it is the C^*-algebra generated by the algebra $\pi(\mathcal{M})$ and the unitary group $U_A = \{U_a : a \in A\}$, where π is a representation of $\mathcal{M}$ on $l^2(A, \mathcal{H})$ ($\mathcal{H} = L^2(\mathbb{T}^2)$)

$$(\pi(f)\xi)_a = \sigma_a^{-1}(f)\xi_a,\ f \in \mathcal{M},\ \xi \in l^2(A, \mathcal{H}),\ a \in A;$$

U is a representation of A on $l^2(A, \mathcal{H})$

$$(U_a\xi)_b = \xi_{b-a},\ a, b \in A,\ \xi \in l^2(A, \mathcal{H}).$$

We can induce on $\mathcal{N}$ the automorphism $\gamma_{\mathcal{N}}$ in a natural way, that is,

$$\gamma_{\mathcal{N}}(\pi(f)) = \pi(\gamma_{\mathcal{M}}(f)),\ f \in \mathcal{M}, \quad \gamma_{\mathcal{N}}(U_a) = U_{\gamma_A a},\ a \in A. \tag{2}$$

Let us remind that an algebra $\mathcal{L}$ is asymptotically commutative with respect to an automorphism α iff

$$\forall\, a, b \in \mathcal{L} \quad \|a\alpha^n(b) - \alpha^n(b)a\| \xrightarrow[n\to\infty]{} 0.$$

Theorem 2. *The algebra $\mathcal{N} = \mathcal{M} \rtimes_\sigma A$ is asymptotically commutative with respect to $\gamma_{\mathcal{N}}$.*

P r o o f. It suffices to prove the asymptotic commutativity of elements of the form $\pi(f)U_a$. That is, we verify the convergence to zero of the following expression

$$I = \|\pi(f)U_a\gamma_{\mathcal{N}}^n(\pi(g)U_b) - \gamma_{\mathcal{N}}^n(\pi(g)U_b)\pi(f)U_a\| \xrightarrow[n\to\infty]{} 0.$$

Below in the proof we write γ instead of $\gamma_{\mathcal{N}}$.

Since $\pi(f)\pi(\gamma^n g) = \pi(\gamma^n g)\pi(f)$ and $U_aU_{\gamma^n b} = U_{\gamma^n b}U_a$, then

$$I = \|\pi(f)(U_a\pi(\gamma^n g) - \pi(\gamma^n g)U_a)U_{\gamma^n b} + \pi(\gamma^n g)(\pi(f)U_{\gamma^n b} - U_{\gamma^n b}\pi(f))U_a\|$$

$$\leq C(\|U_a\pi(\gamma^n g) - \pi(\gamma^n g)U_a\| + \|\pi(f)U_{\gamma^n b} - U_{\gamma^n b}\pi(f)\|), \tag{3}$$

where C $= \max\{\|\pi(f)\|, \|\pi(g)\|\}$. Consider

$$I_1 = \|U_a\pi(\gamma^n g) - \pi(\gamma^n g)U_a\| = \|U_a\pi(\gamma^n g)U_a^{-1} - \pi(\gamma^n g)\|.$$

Let $\xi \in l^2(A, \mathcal{H})$, $s \in \mathcal{M}$, $b, c \in A$. Then

$$((U_c \pi(s) U_c^{-1})\xi)_b = ((\pi(s) U_c^{-1})\xi)_{b-c} = \sigma_{b-c}^{-1}(s)(U_c^{-1}\xi)_{b-c}$$

$$= \sigma_{b-c}^{-1}(s)\xi_b = \sigma_b^{-1}(\sigma_c(s))\xi_b = (\pi(\sigma_a(s))\xi)_b.$$

Hence

$$U_c \pi(s) U_c^{-1} = \pi(s_c), \text{ where } s_c(t) = s(c+t), \ t \in \mathbb{T}^2. \tag{4}$$

According to (4) we obtain

$$I_1 = \|\pi((\gamma^n g)_a) - \pi(\gamma^n g)\| = \|\pi((\gamma^n g)_a - \gamma^n g)\| = \|(\gamma^n g)_a - \gamma^n g\|.$$

Consider an arbitrary $t \in \mathbb{T}^2$. Then

$$|(\gamma^n g)_a(t) - \gamma^n g(t)| = |g(\gamma^n(a+t)) - g(\gamma^n t)| \xrightarrow[n\to\infty]{} 0$$

since the function g is continuous and by virtue of the properties of A (see Example 2) $d(\gamma^n(a+t), \gamma^n t) \xrightarrow[n\to\infty]{} 0$. Since $\mathbb{T}^2$ is a compact metric space then $I_1 = \|(\gamma^n g)_a - \gamma^n g\| \xrightarrow[n\to\infty]{} 0$ too.

Similarly we obtain

$$I_2 = \|\pi(f) U_{\gamma^n b} - U_{\gamma^n b}\pi(f)\| = \|U_{\gamma^n b}^{-1}\pi(f) U_{\gamma^n b} - \pi(f)\|$$

$$= \|\pi(f_{\gamma^n b}) - \pi(f)\| = \|f_{\gamma^n b} - f\| = \sup_{t\in\mathbb{T}^2} |f(\gamma^n b + t) - f(t)|$$

$$= \sup_{t\in\mathbb{T}^2} |f(\gamma^n b + \gamma^n t) - f(\gamma^n t)| = \|(\gamma^n f)_b - \gamma^n f\| \xrightarrow[n\to\infty]{} 0.$$

Applying the inequality (3) we conclude $I \xrightarrow[n\to\infty]{} 0$, and the proof of the theorem is complete. ∎

Theorem 3. *For the automorphism $\gamma_{\mathcal{N}}$ of $\mathcal{N} = \mathcal{M} \rtimes_\sigma A$ (see (2)) the following estimate on entropy holds:*

$$ht(\gamma_{\mathcal{N}}) \le 2 \log \frac{3+\sqrt{5}}{2}. \tag{5}$$

P r o o f. According to [Ch, Pr. 3.3]

$$ht(id_{\mathcal{N}}, \gamma_{\mathcal{N}}, \omega_K, \delta) \le h_{Ch}(\gamma_A) + \limsup_{n\to\infty} \frac{1}{n} \log rcp(id_{\mathcal{M}}, \bigcup_{b\in F} \sigma_b^{-1}(\bigcup_{i=0}^{n-1} \gamma_{\mathcal{M}}^i(\omega)), \frac{\delta}{2}),$$

where K is a finite subset of A, ω is a finite subset of the unit ball of M, $\omega_K = \{\pi(f)U_a : \ f \in \omega, \ a \in K \}$, F is a Fölner set for $(\bigcup\limits_{i=0}^{n-1} \gamma_A^i(K), \frac{\delta}{2})$ with $|F| = c(\bigcup\limits_{i=0}^{n-1} \gamma_A^i(K), \frac{\delta}{2})$.

In what follows we consider sets ω consisting of a finite number of functions of the form

$$f_{m,n}(t) = e^{i2\pi m t_1} e^{i2\pi n t_2}, \quad t = (e^{i2\pi t_1}, e^{i2\pi t_2}) \in \mathbb{T}^2. \tag{6}$$

Let $b = (e^{i2\pi b_1}, e^{i2\pi b_2}) \in A$. Then

$$\sigma_b^{-1}(f_{m,n}(t)) = f_{m,n}(t-b) = e^{i2\pi m(t_1-b_1)} e^{i2\pi n(t_2-b_2)}$$
$$= e^{-i2\pi(mb_1+nb_2)} f_{m,n}(t) = p_{m,n,b} f_{m,n}(t).$$

Consider $W = \bigcup\limits_{i=0}^{n-1} \gamma_{\mathcal{M}}^i(\omega)$. W consists of a finite number of functions of the form (6) too. In fact,

$$\gamma_{\mathcal{M}} f_{m,n}(t) = f_{m,n}(\gamma t) = e^{i2\pi m(t_1+t_2)} e^{i2\pi n(t_1+2t_2)}$$
$$= e^{i2\pi(m+n)t_1} e^{i2\pi(m+2n)t_2} = f_{m+n,m+2n}(t).$$

We obtain that $\widetilde{W} = \bigcup\limits_{b\in F} \sigma_b^{-1}(\bigcup\limits_{i=0}^{n-1} \gamma_{\mathcal{M}}^i(\omega))$ consists of $pf_{m,n}$, where $p \in \mathbb{C}$, $|p| = 1$, $f_{m,n} \in W$.

Consider a finite dimensional C^*-algebra $\mathcal{B}$ for which there exist completely positive maps $\varphi : \ \mathcal{M} \to \mathcal{B}, \psi : \mathcal{B} \to B(\mathcal{H})$ such that

$$\|\varphi \circ \psi(pf_{m,n}) - pf_{m,n}\| \leq \frac{\delta}{2} \tag{7}$$

for any function $pf_{m,n} \in \widetilde{W}$. Since the maps φ and ψ are linear and $|p| = 1$ holds, then (7) is equivalent the condition

$$\|\varphi \cdot \psi(f_{m,n}) - f_{m,n}\| \leq \frac{\delta}{2}$$

for any function $f_{m,n} \in W$. Therefore

$$rcp(id_{\mathcal{M}}, \widetilde{W}, \frac{\delta}{2}) = rcp(id_{\mathcal{M}}, W, \frac{\delta}{2})$$

Whence it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log rcp(id_{\mathcal{M}}, \widetilde{W}, \frac{\delta}{2}) = \limsup_{n\to\infty} \frac{1}{n} \log rcp(id_{\mathcal{M}}, W, \frac{\delta}{2})$$

$$= ht(id_{\mathcal{M}}, \gamma_{\mathcal{M}}, W, \frac{\delta}{2}) \leq ht(\gamma_{\mathcal{M}}).$$

Thus

$$ht(id_{\mathcal{N}}, \gamma_{\mathcal{N}}, \omega_K, \delta) \leq h_{Ch}(\gamma_A) + ht(\gamma_{\mathcal{M}}) = 2\log\frac{3+\sqrt{5}}{2}.$$

Since the linear span of the functions (6) is a totally dense subset of the algebra M, then

$$ht(\gamma_{\mathcal{N}}) = \sup_{\{\Omega:\ |\Omega|<\infty\}} \sup_{\delta>0} ht(id_{\mathcal{N}}, \gamma_{\mathcal{N}}, \Omega, \delta) = \sup_{\omega_K} \sup_{\delta>0} ht(id_{\mathcal{N}}, \gamma_{\mathcal{N}}, \omega_K, \delta)$$

$$\leq 2\log\frac{3+\sqrt{5}}{2},$$

and this completes the proof of the theorem. ■

R e m a r k s. 1. If we consider the Haar measure μ on the torus and construct the standard state ϕ by it on $\mathcal{N}$ then Theorem 3 and [Vo, Pr. 4.6] imply

$$h_\phi(\gamma_{\mathcal{N}}) \leq 2\log\frac{3+\sqrt{5}}{2}.$$

2. Let $\mathcal{R} = L^\infty(\mathbb{T}^2, \mu) \rtimes_\sigma A$ be the von Neumann algebra which corresponds to the representation of $\mathcal{N}$ associated to the state ϕ. Let $\gamma_{\mathcal{R}}$ be the extension of the automorphism $\gamma_{\mathcal{N}}$ and $\overline{\phi}$ the extension of the state ϕ. Then according to [CNT, Th. VII.2] we obtain

$$h_{\overline{\phi}}(\gamma_{\mathcal{R}}) \leq 2\log\frac{3+\sqrt{5}}{2}.$$

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Динамическая энтропия для класса автоморфизмов алгебраического происхождения

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Рассматривается автоморфизм C^*-алгебры, возникающий в связи с гиббсовскими состояниями в теории Рюэля динамических систем. Оценивается топологическая энтропия Войкулеску этого автоморфизма с использованием подхода Чоды к энтропии.

Динамічна ентропія для класу автоморфізмів алгебраїчного походження

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Розглядається автоморфізм C^*-алгебри, який виникає у зв'язку з гіббсовськими станами в теорії Рюеля динамічних систем. Оцінюється топологічна ентропія Войкулеску цього автоморфізму з використанням підходу Чоди до ентропії.