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Gauss type complex quadrature formulae, power moment problem and elliptic curves

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A complex-valued Borel measure ω on $\mathbb{C}$ is called n-reducible if there is a quadrature formula with n complex nodes which is exact for all polynomials of degree $\leq 2n-1$. A criterion of n-reducibility is given on the base of a solvability criterion for a complex power moment problem. The latter is an analytic version of a Sylvester theorem from the theory of binary form invariants. The 2-reducibility of measures ω with $|\mathrm{supp}\omega| = 3$ is closely related to the modular invariants of elliptic curves.*

Dedicated to the 100th anniversary of the birth of Naum Il'ich Akhiezer

Let ω be a Borel complex-valued nonzero measure on $\mathbb{C}$ and let n be a positive integer. Suppose that

$$\int |z^{2n-1}| \cdot |\mathrm{d}\omega| < \infty \tag{1}$$

and denote by $\mathcal{P}_{2n-1}$ the linear space of all polynomials of degrees $\leq 2n-1$. We define a *Gauss type quadrature formula of degree* $2n-1$ for the measure ω as a relation

$$\int f \, \mathrm{d}\omega = \sum_{k=1}^{n} r_k f(z_k), \quad f \in \mathcal{P}_{2n-1}, \tag{2}$$

with some pairwise distinct *nodes* $z_1, \ldots, z_n$ in $\mathbb{C}$ and some nonzero complex coefficients $r_1, \ldots, r_n$. The classical *Gauss quadrature formula* is

$$\frac{1}{2}\int_{-1}^{1} f \, \mathrm{d}x = \sum_{k=1}^{n} \rho_k f(x_k), \quad f \in \mathcal{P}_{2n-1}, \tag{3}$$

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where x_k are the roots of the n-th Legendre polynomial and ρ_k are some positive (uniquely determined) coefficients. In fact, a similar formula exists for any ω such that

$$\omega \geq 0,\ \operatorname{supp}\omega \subset \mathbb{R},\ |\operatorname{supp}\omega| \geq n. \tag{4}$$

In particular, this exists for all n if the support of ω is infinite. The point is that the quadrature formula (2) is equivalent to the system of equations

$$\sum_{k=1}^{n} r_k z_k^j = \int z^j \,\mathrm{d}\omega,\ \ 0 \leq j \leq 2n-1. \tag{5}$$

But for given real a_j, $0 \leq j \leq 2n-1$, the system

$$\sum_{k=1}^{n} r_k z_k^j = a_j,\ \ ,0 \leq j \leq 2n-1, \tag{6}$$

has a solution $\{(r_k, z_k)\}_1^n$ with pairwise distinct $z_k \in \mathbb{R}$ and $r_k > 0$ if and only if the Hankel form

$$H(\xi_1, \dots, \xi_n) = \sum_{j,l=0}^{n-1} a_{j+l}\xi_j\xi_l \tag{7}$$

is definite positive. Indeed, (6) is a power moment problem with respect to the measure μ with $\operatorname{supp}\mu = \{z_k\}_1^n$ and $\mu\{z_k\} = r_k$, $1 \leq k \leq n$. The above mentioned criterion is a truncated version of the classical one, see [1, Ch. 2, §1]. It remains to note that for

$$a_j = \int z^j \,\mathrm{d}\omega,\ \ 0 \leq j \leq 2n-1, \tag{8}$$

the form (7) turns into

$$H(\xi_1, \dots, \xi_n) = \int \left(\sum_{j=0}^{n-1} \xi_j z^j \right)^2 \mathrm{d}\omega \tag{9}$$

which is positive definite under conditions (4). The complex version of the system (6) was introduced by Sylvester [7] in the context of the binary form theory. Later Ramanujan independently solved this system in the short note [4]. However, these remarkable papers do not contain an explicitly formulated solvability criterion. Nevertheless, it turns out that a criterion can be extracted from [7]. It was done in [3, Section 5] written in the language of the modern invariant theory. Below we formulate and prove the "Sylvester theorem" in our analytic context.

To this end it is convenient to reformulate (6) as a power moment problem

$$\int z^j \,\mathrm{d}\mu = a_j, \;\; 0 \le j \le 2n-1, \tag{10}$$

with respect to a complex-valued measure μ, $\operatorname{supp}\mu \subset \mathbb{C}$, such that

$$|\operatorname{supp}\mu| = n. \tag{11}$$

Theorem 1. *Given the complex numbers a_j, the power moment problem (10) has a solution satisfying (11) if and only if the polynomial*

$$G_n(z) = \begin{vmatrix} a_0 & a_1 & \dots & a_{n-1} & a_n \\ a_1 & a_2 & \dots & a_n & a_{n+1} \\ . & . & \dots & . & . \\ a_{n-1} & a_n & \dots & a_{2n-2} & a_{2n-1} \\ 1 & z & \dots & z^{n-1} & z^n \end{vmatrix} \tag{12}$$

has exactly n pairwise distinct roots.

In particular, this means that $G_n \ne 0$ and, moreover, $\deg G_n = n$, i.e., the Hankel determinant

$$\Delta_n = \begin{vmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \\ . & . & \dots & . \\ a_{n-1} & a_n & \dots & a_{2n-2} \end{vmatrix} \tag{13}$$

is different from zero. Indeed, Δ_n is the formal leading coefficient of $G_n(z)$.

P r o o f. "Only if". It follows from (10) and (12) that

$$\int z^j G_n \,\mathrm{d}\mu = 0, \;\; ,0 \le j \le n-1, \tag{14}$$

since the integral (14) is equal to a determinant where some two rows coincide. Hence,

$$\int f G_n \,\mathrm{d}\mu = 0, \;\; f \in \mathcal{P}_{n-1}. \tag{15}$$

Note that each function on $\mathcal{M} = \mathrm{supp}\mu$ is the restriction to $\mathcal{M}$ of its Lagrange interpolation polynomial. Therefore, (15) implies

$$G_n|\mathrm{supp}\mu = 0. \tag{16}$$

Since $\deg G_n \leq n$ and $|\mathrm{supp}\mu| = n$, we conclude that either $G_n = 0$ or $\mathrm{supp}\mu$ is the set of roots of G_n. The latter is just what we need. The former is impossible as we prove below.

Suppose to the contrary that $G_n = 0$. Then $\Delta_n = 0$ and, therefore, the linear system

$$\sum_{i=0}^{n-1} \gamma_i a_{i+j} = 0, \ \ 0 \leq j \leq n-1, \tag{17}$$

has a nontrivial solution $(\gamma_i)_0^{n-1}$. By (10) the system (17) can be rewritten as

$$\int z^j g(z)\, \mathrm{d}\mu = 0, \ \ 0 \leq j \leq n-1,$$

where

$$g(z) = \sum_{i=0}^{n-1} \gamma_i z^i \not\equiv 0.$$

Hence, $g|\mathrm{supp}\mu = 0$ as before but now $\deg g < n$, the contradiction.

"If". Let $\mathcal{N}$ be the set of roots of $G_n(z)$. By assumption, $|\mathcal{N}| = n$. Consider the n-dimensional linear space $X_{\mathcal{N}}$ of measures μ such that $\mathrm{supp}\mu \subset \mathcal{N}$, i.e., $G_n|\mathrm{supp}\mu = 0$. The formula

$$T\mu = \left(\int z^j \, \mathrm{d}\mu \right)_0^{n-1}$$

defines a linear mapping $T : X_{\mathcal{N}} \to \mathbb{C}^n$. Since $\mathrm{Ker}T = 0$, we have $\mathrm{Im}T = \mathbb{C}^n$, in particular, there exists a measure μ such that $\mathrm{supp}\mu \subset \mathcal{N}$ (so that $|\mathrm{supp}\mu| \leq n$) and

$$\int z^j \, \mathrm{d}\mu = a_j, \ \ 0 \leq j \leq n-1. \tag{18}$$

Setting

$$c_j = \int z^j \, \mathrm{d}\mu, \ \ 0 \leq j \leq 2n-1,$$

we already have $c_j = a_j$, $0 \leq j \leq n-1$. Further,

$$\sum_{i=0}^{n} g_i c_{i+j} = 0, \ \ 0 \leq j \leq n-1,$$

where g_i are the coefficients of $G_n(z)$, in particular, $g_n = \Delta_n \neq 0$. On the other hand,

$$\sum_{i=0}^{n} g_i a_{i+j} = 0, \ \ 0 \le j \le n-1,$$

by standard properties of minors applying to those which come from the development of (12) along the last row. Thus, both sequences $(c_j)_0^{2n-1}$ and $(a_j)_0^{2n-1}$ satisfy the same linear difference equation of order n and the same initial conditions. Hence, they coincide, i.e, (18) extends to (10). It remains to prove that $|\mathrm{supp}\mu| = n$.

To this end we introduce the polynomial

$$h(z) = \prod_{\zeta \in \mathrm{supp}\mu} (z - \zeta) \equiv \sum_{j=0}^{p} \lambda_j z^j, \ \ p = |\mathrm{supp}\mu| \le n,$$

and consider the subspace $M \subset \mathcal{P}_n$ of those polynomials of degrees $\le n$ which vanish on $\mathrm{supp}\mu$. Obviously, $M = h\mathcal{P}_{n-p}$ so, $\dim M = n - p + 1 \ge 1$. On the other hand, if a polynomial

$$a(z) = \sum_{i=0}^{n} \alpha_i z^i$$

belongs to M then

$$\int z^j a(z)\, \mathrm{d}\mu = 0, \ \ j \ge 0,$$

since $a|\mathrm{supp}\mu = 0$. A fortiori,

$$\sum_{i=0}^{n} \alpha_i a_{i+j} = 0, \ \ 0 \le j \le n.$$

This is a homogeneous system of linear equations for $(\alpha_i)_0^n$ of rank $\ge n$ since $\Delta_n \neq 0$. Hence, $\dim M \le 1$ so, $\dim M = 1$. Thus, $n - p + 1 = 1$, i.e., $p = n$. ∎

In the course of the above proof we have found that $\mathrm{supp}\mu$ must coincide with the set of roots of $G_n(z)$. Let the latter be $\{z_k\}_1^n$. Then the Vandermond system

$$\sum_{k=1}^{n} r_k z_k^j = a_j, \ \ 0 \le j \le n-1, \tag{19}$$

uniquely determines the values $r_k = \mu\{z_k\}$. As a result, we obtain

Theorem 2. *If the moment problem (10) with condition (11) is solvable, then the solution is unique.*

In addition, let us remark that $G_n(z)$ *is the* n*-th* μ*-orthogonal polynomial* as (15) shows. However, in order to get the intermediate μ-orthogonal polynomials $\{G_k(z)\}_1^{n-1}$ of degrees $1, 2, \dots, n-1$ we have to impose the extra assumptions $\Delta_k \neq 0$, $1 \leq k \leq n-1$, on the principal minors of Δ_n. Let us give a simple illustrating example.

Example 3. For $n = 2$ and $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 0$ we have $\Delta_2 = -1$ and $G_2(z) = -z^2 + 2z - 4$. We are in conditions of Theorem 1. The corresponding measure μ is supported on the set of roots $\{1 + i\sqrt{3}, 1 - i\sqrt{3}\}$ of $G_2(z)$. However, $\Delta_1 = 0$, so that $\deg G_1 = 0$. ■

Now we pass to a construction of the quadrature formula (2). The latter is equivalent to the power moment problem

$$\int f \, \mathrm{d}\omega = \int f \, \mathrm{d}\mu, \ \ f \in \mathcal{P}_{2n-1}, \tag{20}$$

for a complex–valued measure μ on $\mathbb{C}$ under condition (11).

Definition 4. *A measure* ω *is called* n-reducible *if a quadrature formula (20) with* $|\mathrm{supp}\mu| = n$ *does exist.*

By Theorem 2 if ω is n-reducible then the corresponding quadrature formula is unique.

It is easy to see that *a measure* ω *is* 1*-reducible if and only if*

$$\int \mathrm{d}\omega \neq 0. \tag{21}$$

In order to formulate a general criterion of n-reducibility we introduce the ω-inner product

$$(f_1, f_2)_\omega = \int f_1 f_2 \, \mathrm{d}\omega \tag{22}$$

of $f_1 \in \mathcal{P}_i$, $f_2 \in \mathcal{P}_k$ ($i + k \leq 2n-1$). In particular, (22) is defined on the space $\mathcal{P}_{n-1}$.

This complex-valued inner product may not be real even for $f_1 = f_2$. Also, it may degenerate, which means that $(f_1, f_2)_\omega = 0$ for some $f_1 \neq 0$ and all f_2. For instance, if the first $2n-2$ power moments of ω are equal to zero, then the ω-inner product vanishes on $\mathcal{P}_{n-1}$.

Lemma 5. *If the* ω*-inner product on* $\mathcal{P}_{n-1}$ *does not degenerate then there exists a polynomial* $G(z)$ *of degree* n *which is orthogonal to* $\mathcal{P}_{n-1}$*. This polynomial is unique up to proportionality.*

P r o o f. The nondegeneracy means that if $f \in \mathcal{P}_{n-1}$ and f is ω-orthogonal to $\mathcal{P}_{n-1}$ then $f = 0$. Equivalently, this means that the homogeneous system of linear equations

$$\left(f, z^j\right)_\omega = 0, \ \ 0 \le j \le n-1, \tag{23}$$

has the only solution $f = 0$ in $\mathcal{P}_{n-1}$. Hence, the corresponding nonhomogeneous system

$$\left(g, z^j\right)_\omega = c_j, \ \ 0 \le j \le n-1, \tag{24}$$

is uniquely solvable for any sequence $(c_j)_0^{n-1}$.

Now we consider $G(z) = z^n + g(z)$ where $g \in \mathcal{P}_{n-1}$. Then G is orthogonal to $\mathcal{P}_{n-1}$ if and only if G satisfies (24) with $c_j = -\left(z^n, z^j\right)_\omega$, $0 \le j \le n-1$. Hence, a unique $G \perp \mathcal{P}_{n-1}$ does exist. Later on we denote it by $G_{n,\omega}$, so that $G_{n,\omega} \perp \mathcal{P}_{n-1}$ and the leading coefficient of $G_{n,\omega}$ is equal to 1 ($G_{n,\omega}$ is a *monic* polynomial). ■

Theorem 6. *A measure ω is n-reducible if and only if the ω-inner product on $\mathcal{P}_{n-1}$ does not degenerate and the polynomial $G_{n,\omega}$ has no multiple roots. The corresponding Gauss type quadrature formula is unique.*

P r o o f. "Only if". By definition of n-reducibility, the moment problem (10) with

$$a_j = \int z^j \, \mathrm{d}\omega, \ \ 0 \le j \le 2n-1, \tag{25}$$

has a solution μ, $|\mathrm{supp}\mu| = n$. Accordingly,

$$(f_1, f_2)_\omega = (f_1, f_2)_\mu \tag{26}$$

for any polynomials f_1, f_2 with $\deg f_1 + \deg f_2 \le 2n-1$. In particular, for $f \in \mathcal{P}_{n-1}$ the system (23) can be rewritten as

$$\int f z^j \, \mathrm{d}\mu = 0, \ \ 0 \le j \le n-1,$$

which yields $f = 0$ as we already know. Hence, the ω-inner product on $\mathcal{P}_{n-1}$ does not degenerate.

By Lemma 5 the n-th μ-orthogonal polynomial $G_n(z)$ associated with μ is proportional to $G_{n,\omega}(z)$. The leading coefficient Δ_n of $G_n(z)$ is not zero by Theorem 1. Hence,

$$G_n(z) = \Delta_n G_{n,\omega}(z) \tag{27}$$

and, applying Theorem 1 again, we conclude that the polynomial $G_{n,\omega}(z)$ has no multiple roots.

"If". Consider the difference

$$R(z) = G_n(z) - \Delta_n G_{n,\omega}(z)$$

where $G_n(z)$ and Δ_n are defined by (12) and (13) respectively with a_j coming from (25). Then $R \in \mathcal{P}_{n-1}$ and

$$\int z^j R(z)\, \mathrm{d}\mu = 0, \;\; 0 \le j \le n-1,$$

by (14) and by definition of $G_{n,\omega}$. Hence, $R = 0$ since the ω-inner product does not degenerate. Thus, we obtain (27) again. Moreover, $\Delta_n \neq 0$ since

$$a_{i+j} = \left(z^i, z^j\right)_\omega, \;\; 0 \le i, j \le n-1, \tag{28}$$

so that Δ_n is just the Gram determinant of the nondegenerating ω-inner product. As a result, $G_n(z)$ is a polynomial of degree n without multiple roots. It remains to refer to Theorem 1 in the inverse direction.

The uniqueness follows from Theorem 2. ■

In fact, if ω is n-reducible then the nodes $z_, \dots, z_n$ of the corresponding quadrature formula are the roots of the polynomial $G_{n,\omega}(z)$, i.e., of the n-th ω-orthogonal monic polynomial. The coefficients $r_1, \dots, r_n$ are uniquely determined by the formula

$$r_k = \frac{1}{\left(G'_{n,\omega}(z_k)\right)^2} \int \left(\frac{G_{n,\omega}(z)}{z - z_k}\right)^2 \mathrm{d}\omega, \;\; 1 \le k \le n. \tag{29}$$

In the classical real-positive case (29) is the well-known Cristoffel formula. Let us repeate the classical proof in our context. The point is that

$$L_k(z) = \frac{G_{n,\omega}(z)}{G'_{n,\omega}(z_k)(z - z_k)} \tag{30}$$

is a polynomial of degree $n-1$ such that $L_k(z_j) = \delta_{jk}$ (a basis Lagrangian polynomial). Applying (20) to $f = L_k^2$ we get (29) since

$$r_k = \mu\left\{z_k\right\} = \int L_k^2\, \mathrm{d}\mu = \int L_k^2\, \mathrm{d}\omega.$$

Certainly, (29) has to be called the *Cristoffel formula* as well.

Corollary 7. *If a measure ω is n-reducible then*

$$|\text{supp } \omega| \geq n. \tag{31}$$

P r o o f. If $|\text{supp}\omega| = m < n$ then the minimal vanishing on $\text{supp}\omega$ polynomial belongs to $\mathcal{P}_{n-1}$. On the other hand, it is ω-orthogonal to $\mathcal{P}_{n-1}$. This is impossible since the ω-inner product does not degenerate. ∎

A measure ω on $\mathbb{C}$ is called *totally reducible* if it is n-reducible for all n under restriction (31). For instance, so is any nonnegative measure on $\mathbb{R}$. More generally, let Γ be a simple smooth arc in $\mathbb{C}$ and let $\phi(z)$ be a nonzero regular analytic function in a simply connected domain $\mathcal{D} \supset \Gamma$. Consider the segment $I = [w_1, w_2]$ where w_1 and w_2 are the endpoints of Γ, $w_1 \neq w_2$. Assume that $I \subset \mathcal{D}$ and $\phi(z) \geq 0$ for $z \in I$. Then the measure $\mathrm{d}\omega = \phi(z)\mathrm{d}z|\Gamma$ is totally reducible. (In particular, so is $\mathrm{d}z$ on Γ.) Indeed,

$$\int_{\Gamma} f(z)\phi(z)\,\mathrm{d}z = \int_{I} f(z)\phi(z)\,\mathrm{d}z \tag{32}$$

for all polynomials f, in particular, for $f \in \mathcal{P}_{2n-1}$. It remains to pass from I to $[0,1]$ by parametrization $z = (w_2 - w_1)t + w_1$, $0 \leq t \leq 1$.

In contrast, if Γ is a simple smooth closed contour, $\phi(z)$ is a regular analytic function inside of Γ and $\phi(z)$ is continuous up to Γ then $\mathrm{d}\omega = \phi(z)\mathrm{d}z$ is not n-reducible for each n. Indeed, now the ω-inner product vanishes in $\mathcal{P}_{n-1}$.

The situation becomes much more interesting if $\phi(z)$ has a finite set of single poles (and no other singularities) inside of Γ. Let the poles be $z_1, \ldots, z_n$. Then

$$\int_{\Gamma} f(z)\phi(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} f(z_k)\mathrm{Res}[\phi(z)]_{z=z_k} \tag{33}$$

for all polynomials f again. The measure $\phi(z)\mathrm{d}z|\Gamma$ turns out to be n-reducible. However, it is not m-reducible for $m > n$. This is a particular case of the following situation.

A quadrature formula for a measure ω is called *universal* if it is valid for all polynomials. For example, so are (32) and (33) under the conditions we have imposed there.

Proposition 8. *If a measure ω admits an universal quadrature formula with n nodes then it is not m-reducible for $m > n$.*

P r o o f. Let

$$\int f\,\mathrm{d}\omega = \int f\,\mathrm{d}\mu \tag{34}$$

for all polynomials, $|\mathrm{supp}\mu| = n$. Suppose to the contrary that

$$\int f \, \mathrm{d}\omega = \int f \, \mathrm{d}\rho, \;\; f \in \mathcal{P}_{2m-1}, \tag{35}$$

where $m = |\mathrm{supp}\rho| > |\mathrm{supp}\mu|$. Comparing (35) to (34) we see that μ is m-reducible which contradicts Corollary 7. ■

Corollary 9. *The number n of poles of $\phi(z)$ in (33) is the maximal m such that the measure $\phi(z)\mathrm{d}z|\Gamma$ is m-reducible.*

The problem arises: *is (33) m-reducible for some $m < n$?*

In the case of negative answer the residue formula (33) is the only Gauss type quadrature formula of degree $2n-1$ for the contour integral. Since the poles and the residues of $\phi(z)$ can be taken arbitrarily, our problem can be posed in the following more abstract form.

Problem A. *Given a measure ω, $|\mathrm{supp}\omega| = n+1$ and let $m \le n$. Is ω m-reducible? In particular, is ω totally reducible?*

Of course, a formal answer is contained in Theorem 6 with m in role of n. However, sometimes it can be given in a more concrete form.

Let us study the problem of 2-reducibility for $|\mathrm{supp}\omega| = 3$, say, $\mathrm{supp}\omega = \{z_1, z_2, z_3\}$. To do the situation purely geometric we set all $\omega\{z_k\} = 1$, i.e., we will consider only *uniform* measures. We call a triangle $\{z_1, z_2, z_3\}$ *reducible* if the corresponding uniform measure is 2-reducible. Otherwise, the triangle is called *irreducible.*

Since our problem is complex affine invariant, we can normalize it to $z_1 = 0, z_2 = 1, z_3 = \zeta$. Thus, the 2-reducibility depends only on the parameter ζ running over $\mathbb{C} \setminus \{0, 1\}$. In terms of initial nodes we have

$$\zeta = \frac{z_3 - z_1}{z_2 - z_1}. \tag{36}$$

Under all possible permutations of z_1, z_2, z_3 the ratio ζ takes six values

$$\zeta, \;\; \frac{1}{\zeta}, \;\; 1-\zeta, \;\; \frac{1}{1-\zeta}, \;\; \frac{\zeta}{\zeta-1}, \;\; \frac{\zeta-1}{\zeta} \tag{37}$$

which we call *adjoint* to ζ. Actually, all of them are adjoint to each of them. (This is the so-called *anharmonic* 6-*tuple*.) The reducibility of the triangle $\{0, 1, \zeta\}$ is a common property of the values (37). Moreover, if some triangles $\{z_1, z_2, z_3\}$ and $\{z'_1, z'_2, z'_3\}$ are similar then the ratio

$$\zeta' = \frac{z'_3 - z'_1}{z'_2 - z'_1}$$

is equal to one of values adjoint to ζ, and *vice versa*. Therefore, any provisional answer to our question depends only on the similarity class of the triangle $\{z_1, z_2, z_3\}$.

R e m a r k. Two different expressions (37) take the same value if and only if

$$\zeta = -1,\ \frac{1}{2},\ 2,\ e^{\frac{\pi i}{3}},\ e^{-\frac{\pi i}{3}}. \tag{38}$$

Actually, $\frac{1}{2}$ and 2 are adjoint to -1 and $e^{-\frac{\pi i}{3}}$ is adjoint to $e^{\frac{\pi i}{3}}$. We call the values (38) *exceptional.*

By Theorem 6 a measure ω with $|\mathrm{supp}\omega| = 3$ is not 2-reducible if and only if its power moments

$$a_j = \int z^j \,\mathrm{d}\omega,\ \ 0 \le j \le 3,$$

are such that

$$\Delta_2 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} = 0 \tag{39}$$

or the quadratic polynomial

$$G_2(z) = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ 1 & z & z^2 \end{vmatrix} \tag{40}$$

has a double root. We have

$$a_0 = 3, a_j = 1 + \zeta^j,\ \ 1 \le j \le 3,$$

since ω is uniform. Therefore, (39) takes the form

$$\zeta^2 - \zeta + 1 = 0, \tag{41}$$

so that

$$\zeta = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm\frac{\pi i}{3}} \tag{42}$$

which, in turn, means that *the triangle* $\{0, 1, \zeta\}$ *is equilateral.*

Now we have to equate the discriminant of $G_2(z)$ to zero, i.e.

$$\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}^2 - 4 \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = 0.$$

According to (40) we obtain the equation

$$D(\zeta) \equiv 4\zeta^6 - 12\zeta^5 + 21\zeta^4 - 22\zeta^3 + 21\zeta^2 - 12\zeta + 4 = 0. \tag{43}$$

Note that the exceptional values (38) do not satisfy (43). Hence, if ζ is a root then the set of roots consists of those six different numbers which are adjoint to ζ (including ζ itself). Since (43) is a recurrent equation, one can transform it in a cubic equation by division on ζ^3 and compatible substitutions

$$\zeta + \frac{1}{\zeta} = \eta, \ \zeta^2 + \frac{1}{\zeta^2} = \eta^2 - 2, \ \zeta^3 + \frac{1}{\zeta^3} = \eta^3 - 3\eta. \tag{44}$$

As a result,

$$4\eta^3 - 12\eta^2 + 9\eta + 2 = 0.$$

Setting $\eta = 1 - \xi/2$ we obtain the equation

$$\xi^2 - 3\xi - 6 = 0.$$

One of its roots is

$$\xi = \sqrt[3]{3+\sqrt{8}} + \sqrt[3]{3-\sqrt{8}} \approx 2.3553. \tag{45}$$

Hence,

$$\eta = 1 - \frac{1}{2}\left(\sqrt[3]{3+\sqrt{8}} + \sqrt[3]{3-\sqrt{8}}\right) \approx -0.17765 \tag{46}$$

and then we find

$$\zeta = \zeta_1 \equiv \frac{\eta}{2} + i\sqrt{1 - \frac{\eta^2}{4}} \approx -0.0888 + 0.996i \tag{47}$$

as one of roots of (43). Other roots are adjoint to ζ_1 as aforesaid. Obviously, $|\zeta_1| = 1$ so, the triangle $\Lambda = \{0, 1, \zeta_1\}$ is isosceles and its angle between the equal sides is

$$\arg \zeta_1 = \frac{\pi}{2} + \arctan \frac{\eta}{\sqrt{4-\eta^2}} \approx 0.5283\pi. \tag{48}$$

Thus, we prove

Theorem 10. *A triangle $\{z_1, z_2, z_3\}$ is irreducible if and only if it is either equilateral or isosceles with the angle (48) between the equal sides.*

Corollary 11. *Let*

$$\theta(z) = (z - z_1)(z - z_2)(z - z_3)$$

and let Γ be a contour surrounding the points z_1, z_2, z_3. The residue formula

$$\int_\Gamma f(z)\frac{\theta'(z)}{\theta(z)}\,\mathrm{d}z = 2\pi i \sum_{k=1}^{3} f(z_k) \tag{49}$$

is the unique Gauss type quadrature formula of degree 3 for the integration in (49) if and only if the triangle $\{z_1, z_2, z_3\}$ is equilateral or similar to Λ.

In particular, it is applicable to

$$\frac{1}{2\pi i}\int\limits_{|z|=R} f(z)\,\frac{z^2\mathrm{d}z}{z^3-1}=\frac{f(1)+f(\varepsilon)+f(\varepsilon^2)}{3} \tag{50}$$

where $\varepsilon = e^{\frac{2\pi i}{3}}$, $R>1$.

The anharmonic 6-tuple (37) is an orbit of $\zeta \in \mathbb{C}\setminus\{0,1\}$ under action of the group $\mathcal{A}$ of linear fractional transformations generated by $\sigma_1(\zeta)=\frac{1}{\zeta}$ and $\sigma_2(\zeta)=1-\zeta$. This group of six order is called the *anharmonic group.* It is isomorphic to the symmetric group $\mathcal{S}_3$.

The equation (43) is $\mathcal{A}$-invariant. Show that $\mathcal{A}$ is its Galois group over the field $\mathbb{Q}$ of rational numbers.

Since all roots of (43) are rational functions of one of them, it is sufficient to show that (43) is irreducible over $\mathbb{Q}$. Suppose to the contrary. The polynomial $D(\zeta)$ must have a quadratic factor with rational coefficients because of absence of real roots. The roots of that factor must be complex conjugate and their common real part must be rational. The only such pair of roots consists of $1/(1-\zeta_1)$ and its conjugate. Indeed, the real parts of ζ_1 and $1-\zeta_1$ are irrational according to (47)&(46). The same is true for $1/\zeta_1=\bar{\zeta}_1$ and $(\zeta_1-1)/\zeta_1=\overline{1-\zeta_1}$ while

$$\begin{aligned}\frac{1}{1-\zeta_1} &= \frac{1-\bar{\zeta}_1}{|1-\zeta_1|^2}=\frac{1}{2}\cdot\frac{1-\bar{\zeta}_1}{1-\mathrm{Re}\zeta_1}\\ &= \frac{1}{2}\left(1+\frac{\mathrm{Im}\zeta_1}{1-\mathrm{Re}\zeta_1}i\right)=\frac{1}{2}\left(1+\sqrt{\frac{2+\eta}{2-\eta}}i\right).\end{aligned} \tag{51}$$

However, the square of modulus of the root (51) is equal to $1/(2-\eta)$ so, it is irrational hence, $1/(1-\zeta_1)$ cannot satisfy a quadratic equation over $\mathbb{Q}$.

It is easy to describe all $\mathcal{A}$-invariant equations of six degree.

Proposition 12. *The general form of $\mathcal{A}$-invariant monic algebraic equations of six degree is*

$$(\zeta^3-\zeta+1)^3-\frac{27}{4}J\zeta^2(\zeta-1)^2=0,\ \ J\in\mathbb{C}. \tag{52}$$

(The normalizing coefficient $\frac{27}{4}$ will be motivated later.)

P r o o f. The $\mathcal{A}$-invariance of (52) is obvious. Conversely, let $f(\zeta)=0$ be an $\mathcal{A}$-invariant monic algebraic equation. If ζ_1 is its root then all its adjoint values are roots as well. The substitution of ζ_1 into (52) determines a value of J. ($\zeta_1\neq 0,1$, otherwise, ζ_1 could not be a member of an anharmonic 6-tuple.) If ζ_1

is not exceptional then the equation (52) with the above defined J has the same six roots as $f(\zeta) = 0$. It remains to consider the exceptional cases.

1) $\zeta_1 = -1$, other adjoint values are $\frac{1}{2}$ and 2. Hence,

$$f(\zeta) = (\zeta + 1)(\zeta - \frac{1}{2})(\zeta - 2)g(\zeta)$$

where $g(\zeta)$ is a monic polynomial such that $g(\zeta) = 0$ is an $\mathcal{A}$-invariant equation of third degree. In particular, $\zeta^3 g(\frac{1}{\zeta}) = g(\zeta)$ so, $g(-1) = 0$ and then $g(\frac{1}{2}) = 0$, $g(2) = 0$. Finally,

$$g(\zeta) = (\zeta + 1)(\zeta - \frac{1}{2})(\zeta - 2)$$

and

$$f(\zeta) = (\zeta + 1)^2(\zeta - \frac{1}{2})^2(\zeta - 2)^2.$$

The equation $f(\zeta) = 0$ comes from (52) with $J = 1$.

2) $\zeta_1 = e^{\frac{\pi i}{3}}$, other adjoint value is $e^{-\frac{\pi i}{3}}$ so,

$$f(\zeta) = (\zeta^2 - \zeta + 1)h(\zeta)$$

where $\deg h = 4$. The equation $h(\zeta) = 0$ is $\mathcal{A}$-invariant. The roots of the latter must be exceptional again and they can not be $-1, \frac{1}{2}, 2$ since there is no an $\mathcal{A}$-invariant equation of degree 1. Hence, $h(e^{\pm\frac{\pi i}{3}}) = 0$, i.e.

$$f(\zeta) = (\zeta^2 - \zeta + 1)^2 e(\zeta),$$

$\deg e = 2, e(\zeta) = 0$ is $\mathcal{A}$-invariant. Finally, $e(\zeta) = \zeta^2 - \zeta + 1$ and

$$f(\zeta) = (\zeta^2 - \zeta + 1)^3$$

which corresponds to $J = 0$ in (52). ■

In contrast to the ratio ζ the quantity

$$J = \frac{4(\zeta^2 - \zeta + 1)^3}{27\zeta^2(\zeta - 1)^2} \tag{53}$$

is invariant with respect to permutations of the vertices z_1, z_2, z_3 so, J is a function of an unordered triangle. For the equilateral triangle $J = 0$. For the triangle Λ we have $J = \frac{1}{9}$. (Compare (52) to (43).)

The remarkable relation (53) is well known in the classical theory of elliptic functions, see [2, Section 21], where $J = J(\tau)$ and $\lambda = \lambda(\tau)$ are some important modular functions, $\text{Im}\tau > 0$. In particular,

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2} \tag{54}$$

where

$$g_2(\tau) = 60 \sum_{p,q=-\infty}^{\infty} \frac{1}{(p+q\tau)^4}\,, \quad g_3(\tau) = 140 \sum_{p,q=-\infty}^{\infty} \frac{1}{(p+q\tau)^6}\,.$$

Geometrically, the points $p+q\tau \in \mathbb{C}$ $(p, q \in \mathbb{Z})$ are the nodes of the lattice L_τ generated by the vectors 1 and τ. The equation

$$v^2 = 4u^4 - g_2 u - g_3 \tag{55}$$

defines an elliptic curve in $\mathbb{C}^2$ corresponding to the lattice L_τ. The orbit of a fixed τ under the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ yields a class of equivalent lattices which determines a class of isomorphic elliptic curves. The *modular invariant* $J(\tau)$ just distinguishes these classes. (For some reasons it can be expedient to replace J by $j = 2^6 \cdot 3^3 J = 1728J$.)

The modular function λ conformally maps the standard fundamental domain of the modular group onto $\mathbb{C} \setminus (-\infty, 0) \cup (1, \infty)$, see [2, Section 23]. The inverse mapping is just that which Picard used proving his famous theorem on entire functions.

Coming back to our modest problem we can say now that Theorem 10 refers to two (up to isomorphism) elliptic curves with $J = 0$ and $J = \frac{1}{9}$, i.e., $j = 0$ or $j = 2^6 \cdot 3 = 192$. Respectively, we have two elliptic curves in the standard form

$$v^2 = 4u^3 - 1 \ \ (j = 0); \ \ v^2 = 4u^3 + \frac{27}{8}u + \frac{27}{8} \ \ (j = 192). \tag{56}$$

It is curious to find (or investigate) the lattices behind them.

In the case $j = 0$ we have the equation $\lambda(\tau) = e^{\frac{\pi i}{3}}$. A known root of that is $\tau = e^{\frac{\pi i}{3}}$, see [2, Section 23]. In contrast, *all* τ *such that* $j(\tau) = 192$ *are transcendental* as it follows from the arithmetic theory of elliptic curves. Actually, the same is true for all rational nonnegative values of $j(\tau)$ except for

$$2^6 \cdot 3^3, \ \ 2^6 \cdot 5^3, \ \ 0, \ \ 2^3 \cdot 3^3 \cdot 11^3, \ \ 2^4 \cdot 3^3 \cdot 5^3, \ \ 3^3 \cdot 5^3 \cdot 17^3. \tag{57}$$

Indeed, let τ be algebraic and let r be its degree over $\mathbb{Q}$. Since $\mathrm{Im}\tau \neq 0$, we have $r \geq 2$. If $r > 2$ then the number $j(\tau)$ is transcendental according to a Siegel theorem [6]. If $r = 2$ then τ belongs to an imaginary quadratic extension $K \supset \mathbb{Q}$, in other words, the elliptic curve in question admits *complex mutiplication*. For such curves the nonnegative rational values of $j(\tau)$ are listed in (57), see [5].

For $n > 2$ the problem of n-reducibility of a measure ω with $|\mathrm{supp}\omega| = n+1$ requires a further investigation. Like the case $n = 2$ we say that a $(n+1)$-gon is *n-reducible* if so is the corresponding uniform measure. Below we present some simple statements based on the necessary condition $\Delta_n \neq 0$.

Lemma 13. *Let*

$$\mathrm{supp}\omega = \{z_k\}_1^{n+1}, \;\; \omega\{z_k\} = 1, \;\; 1 \le k \le n. \tag{58}$$

Then

$$\Delta_n \equiv \det\left(\int z^{i+j}\, \mathrm{d}\omega\right)_{i,j=0}^{n-1} = V^2 \sum_{k=1}^{n+1} \frac{1}{[p'(z_k)]^2} \tag{59}$$

where

$$p(z) = \prod_{k=1}^{n+1}(z - z_k), \;\; V = \prod_{1\le k<l\le n+1}(z_k - z_l). \tag{60}$$

P r o o f. We have

$$\Delta_n = \det\left(\sum_{k=1}^{n+1} z_k^{i+j}\right)_{i,j=0}^{n-1} = \det(AA')$$

where $A = (z_k^i)$, $0 \le i \le n-1$, $1 \le k \le n+1$, A' is transposed to A. In turn,

$$\det(AA') = \sum_{m=1}^{n+1} V_m^2, \;\; V_m = \det(z_k^i)_{k\ne m},$$

by the Binet–Cauchy Theorem. It remains to take into account that

$$V_m = \pm \prod_{k<l, k\ne m, l\ne m}(z_k - z_l) = \pm\frac{\prod\limits_{k<l}(z_k - z_l)}{p'(z_m)}.$$

■

Corollary 14. *A $(n+1)$-gon $\{z_k\}_1^{n+1}$ is not n-reducible if*

$$\sum_{k=1}^{n+1} \frac{1}{[p'(z_k)]^2} = 0. \tag{61}$$

As an application we consider a regular $(n+1)$-gon, for definiteness,

$$z_k = \varepsilon^k, \;\; 0 \le k \le n, \;\; \varepsilon = e^{\frac{2\pi i}{n+1}}. \tag{62}$$

Then $p(z) = z^{n+1} - 1$, so that

$$\sum_{k=1}^{n+1} \frac{1}{[p'(z_k)]^2} = \frac{1}{(n+1)^2}\sum_{k=1}^{n+1} \varepsilon^{-2nk} = 0.$$

By Corollary 14 and similarity of all regular $(n+1)$-gons we have

Proposition 15. *Any regular* $(n+1)$*-gon is not* n*-reducible.*

The case $n = 2$ is contained in Theorem 10.

Corollary 16. *The residue formula*

$$\frac{1}{2\pi i}\int\limits_{|z|=R} f(z)\frac{z^n\mathrm{d}z}{z^{n+1}-1} = \frac{1}{n+1}\sum_{k=0}^{n} f(\varepsilon^k), \quad R > 1, \tag{63}$$

is the unique Gauss type quadrature formula of degree $2n+1$ *for the integration in (63).*

Proposition 15 can be generalized as follows.

Proposition 17. *The union of regular* $(n+1)$*-gons*

$$z^{n+1} = w_l, \quad 1 \le l \le m, \tag{64}$$

is not n*-reducible.*

P r o o f. Consider the polynomial

$$q(w) = \prod_{l=1}^{m}(w - w_l)$$

and then let

$$p(z) = q(z^{n+1}).$$

The union of regular $(n+1)$-gons in question coincides with the set of roots of $p(z)$. Taking a root ζ_l of the l-st equation (64), we obtain all roots of all equations (64) as

$$z_{l,k} = \zeta_l \varepsilon^k, \quad 1 \le l \le m, \quad 0 \le k \le n.$$

Hence,

$$\sum_{k,l}\frac{1}{[p'(z_{l,k})]^2} = \frac{1}{(n+1)^2}\sum_{l=1}^{m}\frac{1}{[q'(w_l)]^2}\sum_{k=0}^{n}\varepsilon^{-2nk} = 0$$

so, criterion (61) works again. ■

In conclusion we introduce the *spectrum of reducibility of a measure* ω as the set of m, $2 \le m < |\mathrm{supp}\omega|$, such that ω is m-reducible. We call a measure ω *irreducible* if its spectrum of reducibility is empty. For $|\mathrm{supp}\omega| = 3$ we come back to the irredicibility in the former sense. Respectively, a $(n+1)$-gon is called *irreducible* if so is the corresponding uniform measure.

Proposition 18. *Any regular $(n+1)$-gon is irreducible.*

P r o o f. We have

$$a_j = \sum_{k=0}^{n} \varepsilon^{kj} = \begin{cases} n+1 & j \equiv 0 \ (\mathrm{mod}(n+1)), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $a_j = 0$ for $1 \le j \le m$ since $m \le n$. We see that $\Delta_m = 0$ since its 2-nd row vanishes. ■

This result includes Proposition 15 and provides that with a simpler argument.

Problem B. *Is the set of irreducible $(n+1)$-gons finite for every n?*

As we know the answer is affirmative for $n \le 2$.

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