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The scattering problem for step-like Jacobi operator

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The direct/inverse problem is solved for the step-like Jacobi operator in the prescribed class of convergence of the operator coefficients to their limits. The characterization of scattering data is given by means of Marchenko approach ([7]).

1. Introduction

Under the step-like Jacobi operator (Jacobi matrix) we mean the operator in $l^2(\mathbb{Z})$

$$(Ly)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \tag{1.1}$$

where $\inf_{n\in\mathbb{Z}} a_n > 0$ and

$$a_n - \frac{1}{2} \to 0, \quad b_n - \operatorname{sign}(n) b \to 0 \qquad (n \to \pm\infty).$$

Here $b \in \mathbb{R} \setminus \{0\}$ is some constant. The discrete operator in $l^2(\mathbb{Z})$, generated by the finite-difference operation

$$(L_0 y)_n = \frac{1}{2} y_{n-1} + \frac{1}{2} y_{n+1} + \operatorname{sign}(n) b y_n$$

is called the Jacobi operator with pure step. This operator arises as an initial data for the Toda lattice in the Toda shock and rarefaction problems as $|b| > 1$ ([12, 9]). The direct/inverse scattering problem for operator L was studied in [12] under the following two assumptions: (1) the coefficients a_n and b_n tend to their asymptotes faster then any polynomial; (2) the step is "big" ($|b| > 1$). We

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study the scattering problem for the small step ($|b| < 1$) and when the rate of approximation is following

$$\sum_{n\in Z} |n| \left\{ |a_n - \frac{1}{2}| + |b_n - \operatorname{sign}(n) b| \right\} < \infty. \tag{1.2}$$

Such operator is considered as an initial data for the Toda lattice in [3], where the asymptotic behavior of solution is studied under an assumption that this solution exists. At the present study we solve the scattering problem for associated L-operator .

Since there is no difference between the cases $b > 0$ and $b < 0$ we restrict ourselves with the case $-1 < b < 0$. Then the continuous spectrum $\sigma_c(L) = [-1+b, 1-b]$ of the step-like operator consists of two parts : of multiplicity one $[-1+b, -1-b] \cup [1+b, 1-b]$ and of multiplicity two $[-1-b, 1+b]$. The discrete spectrum of this operator is finite (see Section 2) .

Let $\lambda \in \mathbb{C}$ be the spectral parameter of problem:

$$Ly = \lambda y. \tag{1.3}$$

Introduce two more spectral parameters z_+ and z_-, related with λ by the Jukovski transformation

$$\lambda \mp b = \frac{1}{2}(z_\pm + z_\pm^{-1}). \tag{1.4}$$

The functions $Z_\pm : \lambda \to z_\pm$ map the upper (resp. lower) half-plain onto the lower (resp. upper) unit half-disks. Denote

$$\mathbb{T}^\pm = \{z_\pm : |z_\pm| = 1\}, \qquad \mathbb{D}^\pm = \{z_\pm : |z_\pm| < 1\}. \tag{1.5}$$

and put

$$\mathbb{P}^+ = \mathbb{D}^+ \setminus Z_+((1+b, 1-b]), \quad \mathbb{P}^- = \mathbb{D}^- \setminus Z_-([-1+b, -1-b)). \tag{1.6}$$

The maps $Z_\pm : \mathbb{C} \setminus [-1+b,\ 1-b] \to \mathbb{P}^\pm$ are single-valued. Introduce also notations

$$\Delta_0^\pm = Z_\pm([\pm 1 + b,\ \pm 1 - b]), \quad \Delta_1^\pm = Z_\pm([-1-b,\ 1+b]),$$

$$\Delta_2^\pm = \mathbb{T}^\pm \setminus \Delta_1^\pm. \tag{1.7}$$

Thus $\Delta_0^\pm$ and $\Delta_2^\pm$ represent the images under the Jukovski transformations $Z_\pm$ of spectrum of multiplicity one, the set $\Delta_0^\pm$ corresponds to the real part of spectrum on $z_\pm$ -plain . $\Delta_1^\pm$ are the images of spectrum of multiplicity two.

Let $f_n^{\pm}(z_\pm) \sim {z_\pm}^{\pm n}$ as $n \to \pm\infty$ be the Jost solutions of the problem (1.1)–(1.3). In the domains $\mathbb{P}^{\pm}$ they are single-valued functions of λ. In the domains $|z_\pm| \le 1$ they admit the representation through the transformation operators ([8, 11])

$$f_n^{\pm}(z_\pm) = \sum_{m=n}^{\pm\infty} K_\pm(n,m){z_\pm}^{\pm m}. \tag{1.8}$$

Consider the scattering relations

$$T^{\pm} f^{\mp} = R^{\pm} f^{\pm} + \overline{f^{\pm}}, \qquad z_\pm \in \mathbb{T}^{\pm}. \tag{1.9}$$

They imply the following properties **A**–**E** of the transmission $T^{\pm}$ and reflection $R^{\pm}$ coefficients (for proofs see Section 2).

A. *The functions $R^{\pm}(z_\pm)$ and $T^{\pm}(z_\pm)$ are continuous functions on the sets $\mathbb{T}^{\pm} \setminus \{\mp 1\}$. The relations are valid*

$$R^{\pm}(z_\pm^{-1}) = \overline{R^{\pm}(z_\pm)}, \quad T^{\pm}(z_\pm^{-1}) = \overline{T^{\pm}(z_\pm)}, \qquad z_\pm \in \mathbb{T}^{\pm}\,, \tag{1.10}$$

$$R^{\pm}(z_\pm) = \frac{T^{\pm}(z_\pm)}{\overline{T^{\pm}(z_\pm)}}, \qquad z_\pm \in \Delta_2^{\pm}\,, \tag{1.11}$$

$$|T^{\pm}|^2 \frac{z_\mp - z_\mp^{-1}}{z_\pm - {z_\pm}^{-1}} = 1 - |R^{\pm}|^2, \qquad z_\pm \in \Delta_1^{\pm}, \tag{1.12}$$

$$\frac{\overline{R^+}}{\overline{T^+}}(z_+ - {z_+}^{-1}) = -\frac{R^-}{T^-}(z_- - {z_-}^{-1}), \qquad z_\pm \in \Delta_1^{\pm}. \tag{1.13}$$

In equalities (1.12) and (1.13) the points z_+ and z_- correspond to one point λ, that is $z_\pm = Z_\pm(Z_\mp^{-1}(z_\mp))$.

Let $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$ be the eigenvalues of operator L , let $z_{\pm,k} = Z_\pm(\lambda_k)$ be their images and $(\alpha_k^{\pm}) = (\|f^{\pm}(z_{\pm,k})\|)^{-1}$ be the associated $(\pm)$–norming constants for eigenfunctions. It is evident that

B. $\{\lambda_1, \dots, \lambda_p\} \in \mathbf{R} \setminus [-1+b,\, 1-b], \quad \alpha_k^+ > 0,\, k = 1, \dots, p\,.$

The connection between corresponding $+$ and $-$ norming constants is given by equality

$$(\frac{dW}{d\lambda}(\lambda_k))^{-2} = (\alpha_k^+ \alpha_k^-)^2, \tag{1.14}$$

where

$$W(\lambda) = \langle f^+, f^- \rangle = a_{n-1}(f_{n-1}^+ f_n^- - f_n^+ f_{n-1}^-) \tag{1.15}$$

is the Wronskian of the Jost solutions (1.8).

C. *Functions* $T^\pm$ *can be continued as meromorphic functions in the domains* $\mathbb{P}^\pm$ *with the only simple poles at the points* $z_{\pm,k}$. *For each* $\lambda \in \mathbb{C} \setminus [-1+b,\, 1-b]$ and $z_\pm = Z_\pm(\lambda)$ *the equality holds*

$$\frac{z_+ - {z_+}^{-1}}{2T^+(z_+)} = \frac{z_- - {z_-}^{-1}}{2T^-(z_-)} = \tilde{W}(\lambda). \tag{1.16}$$

Function $\tilde{W}(\lambda)$ *is holomorphic in the domain* $\mathbb{C} \setminus \{-1+b,\, 1-b\}$ *and continuous till boundary. Besides,*

$$\lim_{\varepsilon \downarrow 0} \tilde{W}(\lambda \pm i\varepsilon) \neq 0 \,\text{as}\, \lambda \in (-1+b,\, 1-b). \tag{1.17}$$

If $\tilde{W}(-1+b) = 0$ *(resp.* $\tilde{W}(1-b) = 0$ *) then* $\frac{d\tilde{W}}{dz_+} \neq 0$ *(resp.*$\frac{d\tilde{W}}{dz_-} \neq 0$*).*

Note, that in fact

$$\tilde{W}(\lambda) = W(\lambda), \tag{1.18}$$

where $W(\lambda)$ is the Wronskian of the Jost solutions.

D. *The transmission coefficients are bounded as* $z_\pm \to 0$ and

$$T^+(0)T^-(0) = (K_+(n,n)K_-(n,n))^{-2}, \tag{1.19}$$

where the product $K_+(n,n)K_-(n,n)$ *does not depend on* n.

The Marchenko equation of problem (1.1)–(1.3) is derived in [2, 3]. It has a standard form (see, for example, [11]):

$$\frac{\delta(n,m)}{K_\pm(n,n)} = K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l)F_\pm(l+m), \qquad \pm m \geq \pm n, \tag{1.20}$$

where $\delta(n,m)$ is the Kronecker symbol. The kernel $F_\pm$ of this equation is represented as

$$F_\pm(m) = \sum_{k=1}^{p} (\alpha_k^\pm)^2 (z_{\pm,k})^{\pm m} \pm \frac{1}{\pi} \int_{\hat{z}_\pm}^{\pm 1} h_\pm(z) z^{\pm m - 1} dz + \frac{1}{2\pi} \int_0^{2\pi} R^\pm(e^{i\theta}) e^{\pm im\theta} d\theta, \tag{1.21}$$

where

$$h^{\pm}(z_{\pm}) = \frac{1}{2i}\frac{z_{\mp} - z_{\mp}^{-1}}{z_{\pm}^{-1} - z_{\pm}}|T^{\pm}|^2(z_{\pm} + i0) \tag{1.22}$$

and

$$\hat{z}_{\pm} = Z_{\pm}(\pm 1 \mp b). \tag{1.23}$$

The points $z_{\pm}$ in formula (1.22) correspond to one point λ. The following condition **E** establish the connection between the smoothness of function $F_{\pm}(m)$ and the rate of approximation (1.2). Denote by

$$\tilde{R}_{\pm}(n) = \frac{1}{2\pi}\int_0^{2\pi} R^{\pm}(e^{i\theta})\, e^{\pm in\theta}\, d\theta, \tag{1.24}$$

the Fourier coefficients of the reflection coefficients. Then

E. *The following estimates are valid under condition (1.2)*

$$\sum_{n=l}^{\pm\infty} |n\,(\tilde{R}_{\pm}(n+2) - \tilde{R}_{\pm}(n))| < \infty. \tag{1.25}$$

In Section 2 we prove that the part of the Marchenko equation kernel

$$\tilde{H}_{\pm}(n) = \pm\frac{1}{\pi}\int_{\hat{z}_{\pm}}^{\pm 1} h_{\pm}(z) z^{\pm n-1} dz \tag{1.26}$$

and the part, corresponding to the discrete spectrum, also satisfy estimate (1.25).

The set

$$(R^{+}(z_{+}),\, T^{+}(z_{+}),\, R^{-}(z_{-}),\, T^{-}(z_{-}),\, \lambda_1, \dots, \lambda_p,\, \alpha_1^{+}, \dots, \alpha_p^{+}) \tag{1.27}$$

forms the scattering data of problem (1.1)–(1.3). The full characterization of scattering data is given by the properties **A**–**E**. Namely, the main result is the following

Theorem 1.1. *Let $-1 < b < 0$. The necessary and sufficient conditions that a set (1.27) be the scattering data of unique operator (1.1), satisfying the condition (1.2), are the conditions* **A**–**E**.

The proof of sufficiency (the solution of inverse problem) is given in Section 3.

2. The direct scattering problem

Here we give the proof of conditions **A**–**E**. First of all, we discuss the analytical properties of Jost solutions.

Let $G_\pm(n,m,z_\pm)$ be the Green function

$$G_\pm(n,m,z_\pm) = \pm 2 \begin{cases} \frac{{z_\pm}^{m-n} - {z_\pm}^{n-m}}{z_\pm - {z_\pm}^{-1}}, & \pm m > \pm n, \\ 0, & \pm m \le \pm n, \end{cases} \tag{2.1}$$

and let

$$\phi_n^+(z_+) = \prod_{j=n}^{+\infty} (2a_j)\, f_n^+(z_+), \qquad \phi_n^-(z_-) = \prod_{j=-\infty}^{n-1} (2a_j)\, f_n^-(z_-). \tag{2.2}$$

From the asymptotic behavior of the Jost solutions it follows that the functions $\phi_n^\pm$ satisfy the discrete integral equations

$$\phi_n^\pm(z_\pm) = {z_\pm}^{\pm n} + \sum_{m=n\pm 1}^{\pm\infty} J_\pm(n,m,z_\pm)\phi_m^\pm(z_\pm), \tag{2.3}$$

where

$$J_+(n,m,z_+) = (b - b_m)G_+(n,m) + (\frac{1}{2} - 2a_{m-1}^2)G_+(n,m-1), \tag{2.4}$$

$$J_-(n,m,z_-) = -(b + b_m)G_-(n,m) + (\frac{1}{2} - 2a_m^2)G_-(n,m+1). \tag{2.5}$$

Applying to equations (2.3)–(2.5) the method of successive approximations (see, for example, [6]) we obtain, that the functions $f_n^\pm(z_\pm)$ as the functions of variables $z_\pm$ are holomorphic in the domains $\mathbb{D}^\pm \setminus \{0\}$, continuous till the boundaries $\mathbb{T}^\pm$ and

Lemma 2.1 ([6, 10]). *The estimates hold*

$$|z_+^{-n} f_n^+(z_+) - \prod_{j=n}^{+\infty} (2a_j)^{-1}| \le C|z_+| \sum_{m=n}^{+\infty} |m|\{|a_m - 1/2| + |b_m - b|\}, \ |z_+| \le 1, \tag{2.6}$$

$$|z_-^{n} f_n^-(z_-) - \prod_{j=-\infty}^{n-1} (2a_j)^{-1}| \le C|z_-| \sum_{-\infty}^{m=n} |m|\{|a_m - 1/2| + |b_m + b|\}, \ |z_-| \le 1, \tag{2.7}$$

where C is some constant, depending on $\inf_n a_n$, b, $\|L - L_0\|$ and the sum in right-hand side of inequality (1.2).

From estimates (2.6), (2.7) it follows that the functions $f^{\pm}$ as functions of spectral parameter λ are analytic in the domain $\mathbb{C} \setminus [-1+b, 1-b]$ and their Wronskian $W(\lambda)$ is also analytic function at the same domain, continuous till boundary. Its behavior as $\lambda \to \infty$ is studied in Lemma 2.3 below. Estimates (2.6), (2.7) imply also representation (1.8), where

$$K_{\pm}(n,m) = \frac{1}{2\pi} \int_0^{2\pi} f_n^{\pm}(e^{i\theta})\, e^{\mp i m\theta} d\theta \tag{2.8}$$

are the coefficients of the transformation operators. From (2.8) it follows that $L\, K_{\pm} = K_{\pm}\, L_{0,\pm}$, where

$$(L_{0,\pm} y)_n = \frac{1}{2} y_{n-1} + \frac{1}{2} y_{n+1} \pm b y_n. \tag{2.9}$$

The comparison of the main diagonal entries in these equalities imply the representation for the coefficients of operator L through the transformation operators elements:

$$2a_n = \frac{K_+(n+1,n+1)}{K_+(n,n)} = \frac{K_-(n,n)}{K_-(n+1,n+1)}, \tag{2.10}$$

$$b_n \mp b = \frac{1}{2}(\kappa_{\pm}(n, n \pm 1) - \kappa_{\pm}(n \mp 1, n)), \tag{2.11}$$

where we denote

$$\kappa_{\pm}(n,m) = \frac{K_{\pm}(n,m)}{K_{\pm}(n,n)}. \tag{2.12}$$

Note, that from formulas (2.10) and from asymptotic behavior of coefficients a_j it follows that

$$K_+(n,n) = \prod_n^{+\infty} (2a_j)^{-1}, \quad K_-(n,n) = \prod_{-\infty}^{n-1} (2a_j)^{-1}, \tag{2.13}$$

and, therefore, the product

$$K_+(n,n)\, K_-(n,n) = \prod_{-\infty}^{+\infty} (2a_j)^{-1} \tag{2.14}$$

does not depend on n (see property **D**).

Lemma 2.2 ([10]). *Let $\pm m > \pm n$. Then the estimate is valid*

$$|K_\pm(n,m)| \le C \sum_{[\frac{m+n\pm 1}{2}]}^{\pm\infty} \left(|a_n - \frac{1}{2}| + |b_n \mp b| \right), \tag{2.15}$$

where the constant C is of the same nature as the constant from Lemma 2.1.

The following lemma studies the asymptotic behavior of the Wronskian $W(\lambda)$ as $\lambda \to \infty$ (i.e., $z_\pm \to 0$).

Lemma 2.3. *The asymptotic behavior holds as $z_\pm \to 0$*

$$2\langle f^+,\, f^-\rangle = \prod_{-\infty}^{+\infty}(2a_j)^{-1}\{z_\pm^{-1} - 2(\sum_{-\infty}^{-1}(b_n+b) + \sum_{0}^{+\infty}(b_n - b))\} + O(z_\pm). \tag{2.16}$$

P r o o f. Since $\frac{dz_-}{dz_+} = 1 + O(z_+)$ as $z_+ \to 0$ then due to (2.12)–(2.14) we have

$$2\langle f^+,\, f^-\rangle_{-1} = 2a_{-1}\,\{\sum_{m=-1}^{+\infty} K_+(-1,m)z_+^m \sum_{l=-\infty}^{0} K_-(0,l)z_-^{-l}$$

$$- \sum_{m=0}^{+\infty} K_+(0,m)z_+^m \sum_{l=-\infty}^{-1} K_-(-1,l)z_-^{-l}\} = 2a_{-1}\, K_+(-1,-1)\, K_-(0,0)\{z_+^{-1}$$

$$+\kappa_+(-1,0) + \kappa_-(0,-1)\} + O(z_+).$$

By (2.11)

$$\frac{1}{2}\kappa_+(-1,0) = \sum_{0}^{+\infty}(b - b_n), \qquad \frac{1}{2}\kappa_-(0,-1) = -\sum_{-\infty}^{-1}(b + b_n),$$

that implies formula (2.16).

The proof of properties **A**, **B** and **C** can be done now by standard approach. From the scattering relations (1.9) it follows that

$$T^\pm = \pm\frac{\langle f^\pm,\, \overline{f^\pm}\rangle}{\langle f^+,\, f^-\rangle}, \qquad R^\pm = \pm\frac{\langle f^\mp,\, \overline{f^\pm}\rangle}{\langle f^+,\, f^-\rangle}, \tag{2.17}$$

and since $\overline{f^\pm(z_\pm)} = f^\pm(z_\pm^{-1})$, as $z_\pm \in \mathbb{T}^\pm$ and

$$\langle f^\pm,\, \overline{f^\pm}\rangle = \pm\frac{1}{2}(z_\pm^{-1} - z_\pm), \tag{2.18}$$

the relations (1.9)–(1.16) follow immediately. The analytical properties of the Jost solutions imply relevant properties of their Wronskian $W(\lambda)$. Since the discrete spectrum of operator coincides with the set of zeros of this Wronskian, it can have the only accumulation points at the edges of continuous spectrum. But from Lemma 2.1, using the same arguments as in [7, p. 266–269], we obtain, that the discrete spectrum is finite.

Property **D** follows from Lemma 2.3 and (1.18).

Consider now the behavior of the function $W(\lambda)$ at the edges of continuous spectrum. Note, that $W(\lambda) \neq 0$ as $\lambda \in (-1+b, 1-b)$. It follows from the independence of solutions $f_n^{\pm}$, $\overline{f_n^{\pm}}$ as $\lambda \in (-1 \mp b, 1 \mp b)$. At the points $\lambda = -1+b$ and $\lambda = 1-b$ the Wronskian $W(\lambda)$ can have zeros (so called resonance cases), but of the order not more then $1/2$.

Lemma 2.4. *In the resonance case, when $W(\lambda) = 0$ at $\lambda = -1+b$ (or, resp., $\lambda = 1-b$) then $\frac{dW}{dz_+}\big|_{z_+=-1} \neq 0$ (resp., $\frac{dW}{dz_-}\big|_{z_-=1} \neq 0$).*

P r o o f. Let, for example, $W(-1+b) = 0$. Put $\hat{z}_- = Z_-(-1+b)$. Then $f_-(\hat{z}_-) \in l^2(\mathbb{Z}^-)$. Since $Z_+(-1+b) = -1$ and $\frac{d\lambda}{dz_+}\big|_{z_+=-1} = 0$ then the vectors $f^+(-1)$ and $\dot{f}^+(-1) = \frac{d}{dz_+} f^+(-1)$ are independent solutions of equation (1.2) and

$$\langle f^+, \dot{f}^+ \rangle = \frac{1}{2} \lim_{n\to\infty} \lim_{z_+\to -1} (z_+^n (n+1) z_+^n - n z_+^{2n}) = \frac{1}{2}.$$

If $W(-1+b) = 0$ then $f^+(-1) = C f^-(\hat{z}_-)$, where $C \neq 0$. Let N be a big natural number. Then

$$\frac{dW}{dz_+}\Big|_{z_+=-1} = \langle \dot{f}^+, f^- \rangle_{-N} + \langle f^+, \dot{f}^- \rangle_{-N} = C^{-1} \langle \dot{f}^+, f^+ \rangle$$
$$+ \frac{1}{2} C \frac{dz_-}{dz_+}\Big|_{z_+=-1} (\hat{z}_-^{-2N} + O(z_-^{-2N-1})) = C^{-1},$$

since $\frac{dz_-}{dz_+}\big|_{z_+=-1} = 0$.

Lemmas 2.1, 2.4, formulas (2.17), (1.18) imply property **C** and the following

Corollary 1. *Let*

$$h^{\pm}(z_{\pm}) = \frac{1}{2i} \frac{z_{\mp} - z_{\mp}^{-1}}{z_{\pm}^{-1} - z_{\pm}} \left| T^{\pm}(z_{\pm} + i0) \right|^2, \tag{2.19}$$

where $z_{\mp} = Z_{\mp}(\lambda(z_{\pm} + i0))$. These functions are continuous and positive on the upper side of cut $\Delta_0^{\pm} \setminus \{\hat{z}_{\pm}\}$ and are L^1–integrable at the points $\hat{z}_{\pm} = Z_{\pm}(\pm 1 \mp b)$. Moreover, $h^{\pm}(\pm 1) = 0$.

Consider the connection between + and − norming constants, defined by formula (1.14). By standard approach (see, for example, [6] or [1]) we obtain that $\frac{dW}{d\lambda}(\lambda_k) \neq 0$ and, therefore, the discrete spectrum is simple. The eigenvectors $f^+(z_{+,k})$ and $f^-(z_{-,k})$ are dependent with some constants d_k, $f_n^{\mp}(z_{\mp,k}) = d_k^{\pm 1} f_n^{\pm}(z_{\pm,k})$. Following [11], we obtain then that

$$d_k^{\mp 1} \frac{dW}{dz_\pm}(z_{\pm,k}) = \frac{d\lambda}{dz_\pm}(z_{\pm,k})(\alpha_k^{\pm})^{-2}, \tag{2.20}$$

that implies formula (1.14).

To prove the property **E** represent the Marchenko equation (1.20) in the form

$$\kappa_\pm(n,m) + F_\pm(n+m) + \sum_{l=n\pm 1}^{\pm\infty} \kappa_\pm(n,l) F_\pm(l+m) = 0, \quad \pm m > \pm n, \tag{2.21}$$

$$\frac{1}{K_\pm^2(n,n)} = 1 + F_\pm(2n) + \sum_{l=n\pm 1}^{\pm\infty} \kappa_\pm(n,l) F_\pm(l+n), \tag{2.22}$$

where the functions $\kappa_\pm(n,m)$ and $F_\pm(n)$ are defined by (2.12) and (1.21) respectively. The further considerations are identical for both half-axes and we give them only for positive half-axis, omitting the sign + for simplicity.

Let n, s be some fixed large natural numbers and $s > n$. Summing (2.21), we have

$$\sum_{m=s}^{\infty} |F(n+m)| \leq \sum_{m=s}^{\infty} |\kappa(n,m)| + \sum_{l=n+1}^{\infty} \sum_{m=s+l}^{\infty} |F(m)||\kappa(n,l)|.$$

Changing the sums in the last summand, we obtain

$$\sum_{m=s}^{\infty} |F(m)|(1 - \sum_{l=n+1}^{\infty} |\kappa(n,l)|) \leq \sum_{m=n+1}^{\infty} |\kappa(n,m)|.$$

From Lemma 2.2 and (2.13) it follows that the right hand side of this equality is finite, and, therefore, for some n_0,

$$\sum_{m=n_0}^{\pm\infty} |F_\pm(m)| < \infty. \tag{2.23}$$

The proof of estimate (1.25) we also give for right half-axis. Consider two cases: odd and even arguments of function $F(n)$. Prove, first, the estimate

$$\sum_{j=n_0}^{\infty} |j||F(2j+1) - F(2j-1)| < \infty. \tag{2.24}$$

Subtract from equation (2.21) with $n = j$, $m = j+1$ the same equation with $n = j-1$, $m = j$. Since by (2.11) $\kappa(j,j+1) - \kappa(j-1,j) = 2(b_j - b)$ then

$$|F(2j+1)-F(2j-1)| \le 2|b_j-b|+\sum_{l=j}^{\infty}|\kappa(j-1,l)F(j+l)|+\sum_{l=j+1}^{\infty}|\kappa(j,l-1)F(j+l)|$$

Multiplyingthis inequality by j, summing and using (1.2) and (2.23), we obtain (2.24).

To prove that

$$\sum_{j=n_0}^{\infty}|j||F(2j+2)-F(2j)| < \infty. \tag{2.25}$$

we use the difference between equations (2.22) taken as $n = j+1$ and $n = j$. Then

$$|F(2j+2)-F(2j)| \le |K^{-2}(j,j) - K^{-2}(j+1,j+1)| + \sum_{l=j}^{\infty}|o((n+l)^{-1})F(j+l)|.$$

From (2.10) we see that

$$|K^{-2}(j,j) - K^{-2}(j+1,j+1) \le |4a_j^2 - 1| \operatorname{Const}(\inf_{j\in\mathbb{Z}} a_j)$$

and from (2.23) and (1.2) inequality (2.25) follows. Since $|z_{\pm,k}| < 1$ it is evident, that the part $F_\pm^d$, corresponding to the discrete spectrum in the kernel of the Marchenko equation, satisfies the estimate

$$\sum_{n=n_0}^{\pm\infty}|n||F_\pm^d(n+2) - F_\pm^d(n)| < \infty.$$

The corollary to Lemma 2.4 implies the same estimate for the function $\tilde{H}_\pm(n)$, defined by formula (1.26), that proves (1.25).

In fact, the considerations, concerning to proofs of (2.23)–(2.25) are invertible. Namely, from the same reasons as above, the following result is valid

Lemma 2.5. *Let some real-valued functions $F_\pm(n)$ are such that*

$$\sum_{n=n_0}^{\pm\infty}|n||F_\pm(n+2) - F_\pm(n)| < \infty, \tag{2.26}$$

and let corresponding equation (1.20) has the unique solution $K_\pm(n,m)$. *Then the following estimates are valid*

$$0 < \inf_{n\in\mathbb{Z}} K_\pm(n,n) \le \sup_{n\in\mathbb{Z}} K_\pm(n,n) < \infty,$$
$$\sum_{l=n+1}^{\pm\infty} |\kappa_\pm(n,l)| < \infty,$$
$$\sum_{n=n_0}^{\pm\infty} |n||\kappa_\pm(n,n+1) - \kappa_\pm(n-1,n)| < \infty,$$
$$\sum_{n=n_0}^{\pm\infty} |n||K_\pm^2(n+1,n+1) - K_\pm^2(n,n)| < \infty.$$

Corollary 2. *Under the conditions of lemma 2.5 formulas (2.10)–(2.11) imply the condition (1.2).*

3. The inverse scattering problem

Consider an arbitrary collection

$$\{R_+(z_+), T_+(z_+), R_-(z_-), T_-(z_-)\,(z_\pm \in \mathbb{T}^\pm), \lambda_1, \dots, \lambda_p,\, \alpha_1^+, \dots, \alpha_p^+\}, \qquad (3.1)$$

satisfying the conditions **A**–**E**. From this collection we construct due to (1.16), (1.18) and (1.14) the (-)–norming constants and the functions $F_\pm(n,m)$ by formula (1.21). Consider the solutions $K_\pm(n,m)$ of equations (1.20), where n plays the role of a parameter and the equations are considered in the spaces $l^1(\mathbb{Z}_{n,\pm\infty})$ correspondingly. To prove the existence and uniqueness of such solutions,consider corresponding homogeneous equations.

Lemma 3.1. *Let the functions* $F_\pm(m)$ *be defined by formula (1.21) with scattering data, satisfying the conditions* **A** – **C**. *Then the equation*

$$\phi_m + \sum_{l=n\pm1}^{\pm\infty} F_\pm(m+l)\phi_l = 0$$

has no nontrivial solutions in the class $l^1\left(\mathbb{Z}_{[n,\pm\infty)}\right)$ *for any fixed* $n \in \mathbb{Z}$.

P r o o f. Since $\phi \in l^1\left(\mathbf{Z}_{[n,\pm\infty)}\right)$ then $\phi \in l^2\left(\mathbf{Z}_{[n,\pm\infty)}\right)$ and, therefore,

$$\sum_{m=\pm n\pm1}^{\pm\infty} \phi_m\overline{\phi_m} + \sum_{m=n\pm1}^{\pm\infty}\sum_{l=n\pm1}^{\pm\infty} F_\pm(m+l)\phi_l\overline{\phi_m} = 0.$$

Put $\phi_m = 0$ as $\pm m \le \pm n$ and use the Parseval equality . Since the contribution of summands, corresponding to functions $F^d_\pm = F_\pm - F^c_\pm$ and $\tilde{H}_\pm$, defined by (1.21)–(1.16), is nonnegative, then, following [7], we obtain the inequality

$$\frac{1}{\pi}\int\limits_{-\pi}^{\pi} \left|\widetilde{\phi(\theta)}\right|^2 \left(1 - \left|R_\pm\left(e^{i\theta}\right)\right|\right) d\theta \le 0,$$

where

$$\widetilde{\phi_\pm(\theta)} = \sum_{m=n\pm 1}^{\pm\infty} \phi_m e^{\mp im\theta}\,. \tag{3.2}$$

Since $|R_\pm(z_\pm)| < 1$ as $z_\pm \in \Delta_1^\pm$ then $\widetilde{\phi(\theta)} = 0$ on some part of circumference. But due to (3.2) this function can be continued analytically out of unit disk. Therefore, $\phi_m \equiv 0$ as $\pm m \ge \pm n$.

Corollary 3. *The operator $I + F_n^\pm$ defined by the formula*

$$\{(I + F_n^\pm)\phi\}_m = \phi_m + \sum_{l=n\pm 1}^{\pm\infty} F_\pm(l,m)\phi_l \tag{3.3}$$

is invertible. The Marchenko equation (1.20),(1.21) has a unique solution.

Lemma 3.2. *Let $F_\pm(m)$ be the same as in Lemma 3.1 and let $K_\pm(n,m)$ be the solutions of (1.20). Put*

$$\phi_n^\pm(m) = \frac{1}{2}(\kappa_\pm(n \mp 1, m) + (\frac{K_\pm(n\pm 1, n\pm 1)}{K_\pm(n,n)})^2 \kappa_\pm(n\pm 1, m) - \kappa_\pm(n, m\mp 1)$$

$$-\kappa_\pm(n, m\pm 1) + (\kappa_\pm(n, n\pm 1) - \kappa_\pm(n\mp 1, n))\kappa_\pm(n,m)), \tag{3.4}$$

Then

$$\phi_n^\pm(m) + \sum_{l=n\pm 1}^{\pm\infty} \phi_n^\pm(l) F_\pm(l+m) = 0. \tag{3.5}$$

P r o o f of this lemma is a slight modification of proof of Theorem 3.3.1 ([7]), and we omit it.

Note, that by Lemma 3.1 from (3.5) it follows that $\phi_n^\pm(m) = 0$ as $\pm m \ge \pm n$. It is easy to verify that it is equivalent to equality

$$L_\pm K_\pm = K_\pm L_{0,\pm}\,,$$

where the coefficients of Jacobi operators $L_\pm$ are defined as

$$a_n^+ = \frac{1}{2}\frac{K_+(n+1,n+1)}{K_+(n,n)}, \quad a_n^- = \frac{1}{2}\frac{K_-(n,n)}{K_-(n-1,n-1)}\,, \tag{3.6}$$

$$b_n^+ = b + \frac{K_+(n,n+1)}{2K_+(n,n)} - \frac{K_+(n-1,n)}{2K_+(n-1,n-1)}\,, \tag{3.7}$$

$$b_n^- = -b + \frac{K_-(n,n-1)}{2K_-(n,n)} - \frac{K_-(n+1,n)}{2K_-(n+1,n+1)} \tag{3.8}$$

for all $n \in \mathbb{Z}$.

Lemma 3.3. *Let the conditions* **A**–**E** *be satisfied. Then for any* $\lambda \in \mathbb{C}$ *the functions*

$$f_n^\pm(z_\pm) = \sum_{m=n}^{\pm\infty} K_\pm(n,m) z_\pm^{\pm m} \tag{3.9}$$

satisfy the equations

$$(L_\pm f^\pm)_n = \lambda f_n^\pm\,, \tag{3.10}$$

where $z_\pm$ *is defined by (1.4). The coefficients of operators* $L_\pm$ *are defined by formulas (3.6)–(3.8) and satisfy the conditions*

$$\sum_{n=n_0}^{\pm\infty} |n| \left\{ |a_n^\pm - \frac{1}{2}| + |b_n^\pm \pm b| \right\} < \infty. \tag{3.11}$$

The result of this lemma follows immediately from Lemmas 3.2, 2.5. To establish, that $L_+ = L_-$ we have to prove, that functions (3.9) are connected by scattering relations (1.8) with given scattering data (3.1).

According to the condition **E** the functions $\tilde{R}_\pm(\cdot)$ and

$$\Phi_\pm(n,\cdot) = \sum_{l=n}^{\pm\infty} K_\pm(n,l)\tilde{R}_\pm(l+\cdot)$$

belong to the space $l^2(\mathbb{Z})$ for any fixed n. Moreover,

$$\sum_{m\in\mathbb{Z}} \Phi_\pm(n,m) e^{\mp im\theta} = R_\pm(e^{i\theta}) \sum_{l=n}^{\pm\infty} K_\pm(n,l) e^{\pm il\theta} = R_\pm(e^{i\theta}) f_n^\pm(e^{i\theta}). \tag{3.12}$$

On the other hand, by equation (1.20) for $\pm m > \pm n$ and $z = e^{i\theta}$ we have

$$\sum_{m\in\mathbb{Z}} \Phi_\pm(n,m)z^{\mp m} = \sum_{m=n\mp 1}^{\mp\infty} \Phi_\pm(n,m)z^{\mp m} + \sum_{m=n}^{\pm\infty} z^{\mp m}\left\{\frac{\delta(n,\,m)}{K_\pm(n,n)} - K_\pm(n,m)\right.$$

$$\left. -\sum_{l=n}^{\pm\infty} K_\pm(n,l)\sum_{j=1}^{p}(\alpha_j^\pm)^2 z_{\pm,j}^{\pm(l+m)} \mp \sum_{l=n}^{\pm\infty} K_\pm(n,l)\frac{1}{\pi}\int_{\hat z_\pm}^{\pm 1} h_\pm(\zeta)\zeta^{\pm(l+m)-1}d\zeta\right\}$$

$$= \sum_{m=n\mp 1}^{\mp\infty} \Phi_\pm(n,m)z^{\mp m} + \frac{z^{\mp n}}{K_\pm(n,n)} - \overline{f_n^\pm}(z)$$

$$-\sum_{j=1}^{p}(\alpha_j^\pm)^2 f_n^\pm(z_{\pm,j})\sum_{m=n}^{\pm\infty}(z_{\pm,j})^{\pm m}z^{\mp m} \mp \sum_{m=n}^{\pm\infty} z^{\mp m}\frac{1}{\pi}\int_{\hat z_\pm}^{\pm 1} h_\pm(\zeta)f_n^\pm(\zeta)\zeta^{\pm m-1}d\zeta. \tag{3.13}$$

Comparing (3.12) with (3.13), we obtain the equalities

$$R_\pm(z)f_n^\pm(z) + \overline{f_n^\pm(z)} = T^\pm(z)g_n^\mp(z), \qquad z\in\mathbb{T}^\pm, \tag{3.14}$$

where

$$g_n^\mp(z) = \frac{z^{\mp n}}{T^\pm(z)}\left(\frac{1}{K_\pm(n,n)} + \sum_{m=n\mp 1}^{\mp\infty}\Phi_\pm(n,m)z^{\pm n\mp m}\right.$$

$$\left. -z\sum_{j=1}^{p}(\alpha_j^\pm)^2 f_n^\pm(z_{\pm,j})\frac{(z_{\pm,j})^{\pm n}}{z - z_{\pm,j}} \mp \frac{z}{\pi}\int_{\hat z_\pm}^{\pm 1} h_\pm(\zeta)f_n^\pm(\zeta)\frac{\zeta^{\pm n-1}}{z-\zeta}d\zeta\right). \tag{3.15}$$

The functions $g_n^\mp$ are defined on the sets $\mathbb{T}^\pm$ correspondingly and can be continued to the domains $\mathbb{P}^\pm$ with singularities on the bands of cuts of $\Delta_0^\pm$, in the points of discrete spectrum and in the point 0. Determine the character of singularities of these functions at the point $z_{\pm,j}$, $j = 1,\dots,p$. By (1.3) and (1.18)

$$\begin{aligned}\lim_{z\to z_{\pm,j}} g_n^\pm(z) &= -z_{\pm,j}f_n^\pm(z_{\pm,j})(\alpha_j^\pm)^2 \lim_{z\to z_{\pm,j}}\frac{1}{T^\pm(z)(z-z_{\pm,j})} \\ &= f_n^\pm(z_{\pm,j})(\alpha_j^\pm)^2\frac{dW}{d\lambda}\Big|_{\lambda=\lambda_j}.\end{aligned} \tag{3.16}$$

Therefore, the functions $g_n^\pm$ have removable singularities at the points $z_{\pm,j}$. From the property **E** it follows that $\sup_{\pm n>0}\sum_{m=n\mp 1}^{\mp\infty}|\Phi_\pm(n,m)| < \infty$ and, thus

$$g^\mp \in l^2(\mathbb{Z}^\mp) \quad \text{as} \quad z_\pm\in\mathbb{P}^\pm\setminus\{0\}. \tag{3.17}$$

From equality (3.14) it follows that

$$(1-|R_\pm|^2)f_n^\pm = \overline{T^\pm g_n^\mp} - \overline{R_\pm}T^\pm g_n^\mp,$$

and by (1.12) as $z_\pm \in \Delta_1^\pm$ one has

$$|T^\pm|^2 f_n^\pm \frac{z_\mp - z_\mp^{-1}}{z_\pm - z_\pm^{-1}} = \overline{T^\pm g_n^\mp} - \overline{R_\pm}T^\pm g_n^\mp.$$

Therefore, due to (1.16) and (1.13)

$$T^\mp f_n^\pm = \overline{g_n^\mp} - T^\mp g_n^\mp \frac{\overline{R_\pm}}{\overline{T^\pm}} \frac{z_\pm - z_\pm^{-1}}{z_\mp - z_\mp^{-1}} = \overline{g_n^\mp} + R_\mp g_n^\mp, \quad z_\pm \in \Delta_1^\pm$$

and eliminating the reflection coefficients from (3.14) we obtain the equalities

$$T^\mp(f_n^\pm f_n^\mp - g_n^\pm g_n^\mp) = (\overline{g_n^\mp} f_n^\mp - \overline{f_n^\mp} g_n^\mp), \qquad z_\pm \in \Delta_1^\pm. \tag{3.18}$$

When $z_\pm \in \Delta_0^\pm$ then $z_\mp = Z_\mp(Z_\pm^{-1}(z_\pm)) \in \Delta_2^\mp$, and from (1.10) and (1.11) it follows that

$$g_n^\mp(z_\mp + i0) - g_n^\mp(z_\mp - i0) = \frac{R_\pm}{T^\pm} f_n^\pm + \frac{\overline{f_n^\pm}}{T^\pm} - \overline{\left(\frac{R_\pm}{T^\pm} f_n^\pm\right)} - \frac{\overline{f_n^\pm}}{\overline{T^\pm}} = 0,$$

i.e., the functions $g_n^\mp$ have no jumps along the set $\Delta_0^\pm$.

The further considerations are identical for (+) and (-) equalities (3.18) and we give them for (+)-case. Consider the function

$$s_n(z_+) = T^+(f_n^+ f_n^- - g_n^+ g_n^-)$$

which is equal to

$$s_n = \overline{g_n^+} f_n^+ - \overline{f_n^+} g_n^+ \qquad \text{as}\, z_+ \in \Delta_1^+. \tag{3.19}$$

The function s_n can be continued as meromorphic function inside the domain $\mathbb{P}^+$. Consider possible singularities of this function. Since $(f_n^+ f_n^-)(0) = K_+(n,n)K_-(n,n)$ and, by (1.19), (3.15) $(g_n^+ g_n^-)(0) = (T^+(0)T^-(0)K_+(n,n) \times K_-(n,n))^{-1} = K_+(n,n)K_-(n,n)$ then $s_n(0) = 0$. From (3.16) and (1.14) it follows that this function has removable singularities at the points $z_{+,k}$. Determine, what jump it has on the set Δ_0^+. Since the functions f_n^+ and g_n^+ have no jumps on this set, we consider the functions $T^+ f_n^-$ and $T^+ g_n^-$. According to (1.16), (1.10), (3.14) and(1.11), we have

$$T^+ f_n^-(z_+ + i0) - T^+ f_n^-(z_+ - i0) = \frac{z_+ - z_+^{-1}}{z_- - z_-^{-1}}(T^- f_n^- + \overline{T^- f_n^-})$$

$$= \frac{z_+ - z_+^{-1}}{z_- - z_-^{-1}}(T^- f_n^- + |T^-|^2 g_n^+ - R_- \overline{T^-} f_n^-) = \frac{z_+ - z_+^{-1}}{z_- - z_-^{-1}}|T^-|^2 g_n^+(z_+ + i0).$$

From the other side, the only summand, having the jump in the representation (3.15), is

$$h_n(z) = \frac{z^{1-n}}{\pi} \int\limits_{\hat{z}_+}^{1} \frac{h_+(\xi) f_n^+(\xi) \xi^{n-1}}{z - \xi} dz.$$

From formula (2.19) using the Cauchy theorem, we have

$$h_n(z_+ + i0) - h_n(z_+ - i0) = -\frac{z_+ - z_+^{-1}}{z_- - z_-^{-1}} |T^+|^2 f_n^+(z_+ + i0).$$

Therefore, due to (1.16)

$$s_n(z_+ + i0) - s_n(z_+ - i0) = f_n^+ g_n^+ (T^+ \overline{T^-} + T^- \overline{T^+}) = 0.$$

Thus, the function $s_n(z)$ is holomorphic function on the unit disk $\mathbb{D}^+$, $s_n(0) = 0$. Due to (3.19) this function is odd as $z \in \Delta_1^+ : \ s_n(z^{-1}) = -s_n(z)$. These properties allow us to continue the function s_n in the domain $\mathbb{C} \setminus \mathbb{D}$. Since it is holomorphic in $\mathbb{C}$ and $s_n(z) \to 0$ as $z \to \infty$, then by the Liouville theorem we have equalities $g_n^+ g_n^- = f_n^+ f_n^-$ and $\overline{g_n^+} f_n^+ = \overline{f_n^+} g_n^+$. Applying now the arguments of [7, p. 279], we see that $g_n^\pm = f_n^\pm$ and, therefore, scattering relations (1.9) are fulfilled. Theorem 1.1 is proved.

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