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Functional models for almost periodic Jacobi matrices and the Toda hierarchy

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We construct and solve a class of evolution equations for (almost periodic) Jacobi matrices which contains as a special case the well known Toda hierarchy. Our construction arises naturally within the framework of functional models on a Riemann surface for almost periodic Jacobi matrices.

1. Introduction

We first recall the Toda hierarchy (see, e.g., [7]). We fix an arbitrary polynomial $p(z)$ with real coefficients (it is usually normalized to have leading coefficient 1 and free coefficient 0). For any bounded doubly infinite self adjoint Jacobi matrix J (a tridiagonal matrix, where we assume as usual that the elements on the diagonals above and below the main one are real and positive) we set $M_p(J) = p(J)$ and decompose the matrix $M_p(J)$ (which has a finite number of non zero diagonals) into a sum of an upper triangular matrix $M_p^+(J)$ and of a lower triangular matrix $M_p^-(J)$ with $M_p^+(J) = \left(M_p^-(J)\right)^*$. We then define

$$\widetilde{M_p}(J) = M_p^+(J) - M_p^-(J)$$

(which is obviously a bounded skew self adjoint operator on $l^2(\mathbb{Z})$), and the equation corresponding to the polynomial $p(z)$ in the Toda hierarchy is given by

$$\dot{J} = [\widetilde{M_p}(J), J]. \tag{1}$$

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Of course, the equation (1) can be rewritten as a system of difference-differential evolution equations for the entries of the Jacobi matrix J. (The usual Toda chain corresponds to $p(z) = z$.)

Theorem 1.1. *[7]. Suppose J_0 is a bounded doubly infinite self adjoint Jacobi matrix. Then there exists a unique integral curve $t \mapsto J(t)$ of the Toda equations, that is,* (1) *holds and $J(0) = J_0$.*

In the case when J_0 is almost periodic with the spectrum E consisting of a finite union of intervals (the finite band case, see, e.g., [2, 7]), it is well known that the evolution equations of the Toda hierarchy are (uniquely) solvable on all of $\mathbb{R}$, with the solution $J(t)$ being again almost periodic for all t. Furthermore, almost periodic Jacobi matrices with spectrum E are naturally parametrized by a finite dimensional torus (the real part of the Jacobian variety of the associated compact hyperelliptic Riemann surface), and the evolution equation (1) corresponds to a linear evolution on the torus in the direction determined by the polynomial $p(z)$.

We shall establish similar results for the evolution equation (1) for an arbitrary almost periodic Jacobi matrix J_0 whose spectrum E (containing, in general, an infinitely many gaps) satisfies the so called homogeneity condition (see Section 2 below for a precise definition). We first recall from [6] the parametrization of almost periodic Jacobi matrices with a given homogeneous spectrum.

Theorem 1.2. *There is a continuous one-to-one correspondence between almost periodic Jacobi matrices with homogeneous spectrum E and characters on the group $\Gamma = \pi_1(\mathbb{C}\backslash E)$.*

Here, the group of characters Γ^* substitutes the real part of the Jacobian variety of the hyperelliptic Riemann surface of finite genus when E is a system of intervals (and the corresponding Riemann surface is the double of $\overline{\mathbb{C}}\backslash E$). The correspondence of Theorem 1.2 is given by associating to each character the operator of multiplication by the planar variable z in the character automorphic L^2 space with this character — see Section 2 for details.

In fact, in this general setting it is natural to extend Toda hierarchy (the question arose in a discussion with Fritz Gesztezy). Let J_0 be a bounded self adjoint Jacobi matrix with absolutely continuous spectrum, and denote by E the spectrum of J_0. We fix an arbitrary function $a \in L^\infty(E, \mathbb{R})$ and for any bounded self adjoint Jacobi matrix J with absolutely continuous spectrum whose spectrum is contained in E we set $M_a(J) = a(J)$ (in the sense of the standard functional calculus for self adjoint operators). We decompose the matrix $M_a(J)$ into a sum of an upper triangular matrix $M_a^+(J)$ and of a lower triangular matrix $M_a^-(J)$ with $M_a^+(J) = (M_a^-(J))^*$ (here * denotes simply the formal adjoint of a doubly infinite matrix). Notice that these two matrices do not define in general bounded operators on $l^2(\mathbb{Z})$, but at any rate they do define unbounded operators with domains containing finitely supported sequences (since their action on the standard basis vectors yields simply the columns of $M_a(J)$ truncated to the positive or the negative indices respectively). We now define

$$\widetilde{M_a}(J) = M_a^+(J) - M_a^-(J);$$

this is a generally unbounded skew self adjoint operator on $l^2(\mathbb{Z})$ whose domain contains finitely supported sequences. When the function a is a polynomial we recover the evolution equations of the Toda hierarchy as described above.

The following is our main result.

Theorem 1.3. *Let J_0 be an almost periodic Jacobi matrix with homogeneous spectrum E and let $a \in L^\infty(E, \mathbb{R})$. Then the evolution equation* (1) *with initial condition J_0 is solvable on all of $\mathbb{R}$. For all t, $J(t)$ is an almost periodic Jacobi matrix with spectrum E. If $\alpha(t) \in \Gamma^*$ (where $\Gamma = \pi_1(\mathbb{C}\backslash E)$) is the character corresponding to $J(t)$, and $\alpha_0 \in \Gamma^*$ is the character corresponding to J_0, then*

$$\alpha(t) = \alpha_0 \cdot \exp(-t\xi).$$

Here $\xi : \Gamma \to i\mathbb{R}$ is the homomorphism corresponding to the additive periods of the function $\widetilde{a}$ which is harmonic conjugate to the harmonic continuation of a from E to $\overline{\mathbb{C}}\backslash E$.

The idea behind the proof of the theorem is that the solution of the evolution equation (1) produces a pair of commuting operators $\frac{d}{dt} + \widetilde{M_a}(J(t))$ and $J(t)$ on an appropriate space of $l^2(\mathbb{Z})$-valued functions. These operators admit a functional model in the Γ' (the commutator of the group Γ) automorphic L^2 space given by multiplication by $\widetilde{a}$ and multiplication by z respectively. The decomposition of the Γ' automorphic L^2 space into a direct integral of Γ character automorphic L^2 spaces (see [8]) provides a link with functional models for individual Jacobi matrices mentioned above.

2. Functional model

Let S be the shift in $l^2(\mathbb{Z})$. We remind that a Jacobi matrix J is said to be almost periodic if the set of operators $\{S^{-n}JS^n\}_{n\in\mathbb{Z}}$ is a precompact in the operator topology [1, 5].

Let G be a compact Abelian group, $p(\alpha)$ and $q(\alpha)$ be continuous real–valued functions on G and $p(\alpha) > 0$. For a fixed $\tau \in G$, define $T\alpha = \tau\alpha$. Evidently, the Jacobi matrix $J(\alpha)$ of the form

$$(J(\alpha)x)_n = p_n(\alpha)x_{n-1} + q_n(\alpha)x_n + p_{n+1}(\alpha)x_{n+1}, \quad x = \{x_n\}_{n=-\infty}^{\infty} \in l^2(\mathbb{Z}), \tag{2}$$

where $p_n(\alpha) = p(T^n\alpha)$ and $q_n(\alpha) = q(T^n\alpha)$, is almost periodic. Moreover, an arbitrary almost periodic matrix can be obtained by this construction with a suitable choice of G, τ, $p(\alpha)$, $q(\alpha)$ and α.

Let us note that the structure of $J(\alpha)$ is described by the following identity

$$J(\alpha)S = SJ(T\alpha). \tag{3}$$

The last relation indicate strongly that it would be only natural to associate with the family of matrices $\{J(\alpha)\}$ a pair of commuting operators.

Let $L^2_{d\chi}(l^2(\mathbb{Z}))$ be the space of $l^2(\mathbb{Z})$–valued vector functions, $x(\alpha) \in l^2(\mathbb{Z})$, with the norm

$$||x||^2 = \int_G ||x(\alpha)||^2 \, d\chi,$$

where $d\chi$ is the Haar measure on G. Define

$$(\widehat{J}x)(\alpha) = J(\alpha)x(\alpha), \quad (\widehat{S}x)(\alpha) = Sx(T\alpha), \quad x \in L^2_{d\chi}(l^2(\mathbb{Z}))$$

Then (3) implies

$$(\widehat{J}\widehat{S}x)(\alpha) = J(\alpha)Sx(T\alpha) = SJ(T\alpha)x(T\alpha) = (\widehat{S}\widehat{J}x)(\alpha).$$

Further, $\widehat{S}$ is a unitary operator and $\widehat{J}$ is selfadjoint. The space $L^2_{d\chi}(l^2(\mathbb{Z}_+))$ is an invariant subspace for $\widehat{S}$. It is not invariant for $\widehat{J}$ but it does for the product $\widehat{J}\widehat{S}$. Let us put

$$\widehat{S}_+ = \widehat{S}|L^2_{d\chi}(l^2(\mathbb{Z}_+)), \quad (\widehat{J}\widehat{S})_+ = \widehat{J}\widehat{S}|L^2_{d\chi}(l^2(\mathbb{Z}_+)).$$

We are interested in a functional model, where $\widehat{S}_+$ and $(\widehat{J}\widehat{S})_+$ became operators multiplication by functions in a functional space on an appropriate Riemann surface.

We say that a pair of commuting operators $\widehat{S}_+$ and $(\widehat{J}\widehat{S})_+$ has a (local) functional model if there is a unitary embedding of the space $L^2_{d\chi}(l^2(\mathbb{Z}_+))$ in a space X_O, consisting of holomorphic in some domain O functions $F(\zeta)$, $\zeta \in O$, with a reproducing kernel ($F \mapsto F(\zeta_0)$, $\zeta_0 \in O$, is a bounded functional in X_O) and under this embedding the operators became a pair of operators multiplication by holomorphic functions, say

$$\widehat{S}_+ x \mapsto b(\zeta)F(\zeta), \quad (\widehat{J}\widehat{S})_+ x \mapsto \lambda(\zeta)F(\zeta).$$

In fact, such assumptions imply quite strong consequences. Let k_ζ be the reproducing kernel in X_O and let $\hat{k}_\zeta$ be its preimage in $L^2_{d\chi}(l^2(\mathbb{Z}_+))$. Then

$$\langle \widehat{S}^*_+ \hat{k}_\zeta, x\rangle = \langle \hat{k}_\zeta, \widehat{S}_+ x\rangle = \langle k_\zeta, bF\rangle. \tag{4}$$

By the reproducing property

$$\langle k_\zeta, bF\rangle = \overline{b(\zeta)F(\zeta)} = \langle \overline{b(\zeta)}k_\zeta, F\rangle.$$

So, we can continue (4) in the following way

$$\langle \widehat{S}_+^* \hat{k}_\zeta, x\rangle = \langle \overline{b(\zeta)}\hat{k}_\zeta, x\rangle.$$

That is $\hat{k}_\zeta$ is an eigenvector of $\widehat{S}_+^*$ with the eigenvalue $\overline{b(\zeta)}$. In the same way, $\hat{k}_\zeta$ is an eigenvector of $(\widehat{J}\widehat{S})_+^*$ with the eigenvalue $\overline{\lambda(\zeta)}$.

Thus, if a functional model exists then the spectral problem

$$\begin{cases} \widehat{S}_+^* \hat{k}_\zeta & = \overline{b(\zeta)}\hat{k}_\zeta \\ (\widehat{J}\widehat{S})_+^* \hat{k}_\zeta & = \overline{\lambda(\zeta)}\hat{k}_\zeta \end{cases}, \tag{5}$$

has a solution $\hat{k}_\zeta$ with an anti–holomorphic dependence of ζ. Viceversa, if (5) has a solution of this kind, and the system of eigenvectors $\{\hat{k}_\zeta\}_{\zeta\in O}$ is complete in $L^2_{d\chi}(l^2(\mathbb{Z}_+))$ then we put

$$F(\zeta) = \langle x, \hat{k}_\zeta\rangle, \quad ||F||^2 = ||x||^2,$$

and this provide a local functional model for the pair $\widehat{S}_+$, $(\widehat{J}\widehat{S})_+$.

At least, under some assumptions on the spectrum of an almost periodic Jacobi matrix we can present *a global* functional model of this kind. To this end we need some definitions and notation.

A real compact E is homogeneous if there is an $\eta > 0$ such that

$$|(x-\delta, x+\delta)\cap E| \ge \eta\delta \quad \text{for all } 0<\delta<1 \quad \text{and all } x\in E. \tag{6}$$

By $J(E)$ we denote the class of all almost periodic Jacobi matrices with absolutely continuous spectrum E.

Let $z(\zeta): \mathbb{D}\to\Omega$ be a uniformization of the domain $\Omega = \bar{\mathbb{C}}\setminus E$. Thus there exists a discrete subgroup Γ of the group $SU(1,1)$ consisting of elements of the form

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22}\end{bmatrix}, \ \gamma_{11} = \overline{\gamma_{22}},\ \gamma_{12} = \overline{\gamma_{21}},\ \det\gamma = 1,$$

such that $z(\zeta)$ is automorphic with respect to Γ, i.e., $z(\gamma(\zeta)) = z(\zeta)$, $\forall\gamma\in\Gamma$, and any two preimages of $z_0\in\Omega$ are Γ–equivalent. We normalize $z(\zeta)$ by the conditions $z(0)=\infty$, $(\zeta z)(0) > 0$.

Condition (6) implies that Γ acts dissipatively on $\mathbb{T}$ with respect to the Lebesgue measure dm, that is there exists a measurable (fundamental) set $\mathbb{E}$, which does not contain any two Γ–equivalent points, and the union $\cup_{\gamma\in\Gamma}\gamma(\mathbb{E})$ is a set of full measure. For the space of square summable functions on $\mathbb{E}$ (with respect to dm), we use the notation $L^2_{dm|\mathbb{E}}$.

A character of Γ is a complex–valued function $\alpha:\Gamma\to\mathbb{T}$, satisfying

$$\alpha(\gamma_1\gamma_2)=\alpha(\gamma_1)\alpha(\gamma_2),\quad \gamma_1,\gamma_2\in\Gamma.$$

The characters form an Abelian compact group denoted by Γ^*.

Let f be an analytic function in $\mathbb{D}$, $\gamma\in\Gamma$. Then we put

$$f|[\gamma]_k=\frac{f(\gamma(\zeta))}{(\gamma_{21}\zeta+\gamma_{22})^k}\quad k=1,2.$$

Notice that $f|[\gamma]_2=f$ for all $\gamma\in\Gamma$, means that the form $f(\zeta)d\zeta$ is invariant with respect to the substitutions $\zeta\to\gamma(\zeta)$ ($f(\zeta)d\zeta$ is an Abelian integral on $\mathbb{D}/\Gamma$). Analogically, $f|[\gamma]=\alpha(\gamma)f$ for all $\gamma\in\Gamma$, $\alpha\in\Gamma^*$, means that the form $|f(\zeta)|^2\,|d\zeta|$ is invariant with respect to these substitutions.

We recall, that a function $f(\zeta)$ is of Smirnov class, if it can be represented as a ratio of two functions from H^∞ with an outer denominator. The following spaces related to the Riemann surface $\mathbb{D}/\Gamma$ are counterparts of the standard Hardy spaces H^2 (H^1) on the unit disk.

Definition 2.1. *The space $A^2_1(\Gamma,\alpha)$ ($A^1_2(\Gamma,\alpha)$) is formed by functions f, which are analytic on $\mathbb{D}$ and satisfy the following three conditions:*

$$\begin{aligned}&1) f \text{ is of Smirnov class},\\ &2) f|[\gamma]=\alpha(\gamma)f\quad (f|[\gamma]_2=\alpha(\gamma)f),\quad \forall\gamma\in\Gamma,\\ &3)\int_{\mathbb{E}}|f|^2\,dm<\infty\quad (\int_{\mathbb{E}}|f|\,dm<\infty).\end{aligned}$$

$A^2_1(\Gamma,\alpha)$ is a Hilbert space with the reproducing kernel $k^\alpha(\zeta,\zeta_0)$, moreover

$$0<\inf_{\alpha\in\Gamma^*}k^\alpha(\zeta_0,\zeta_0)\le\sup_{\alpha\in\Gamma^*}k^\alpha(\zeta_0,\zeta_0)<\infty. \tag{7}$$

Put

$$k^\alpha(\zeta)=k^\alpha(\zeta,0)\quad\text{and}\quad K^\alpha(\zeta)=\frac{k^\alpha(\zeta)}{\sqrt{k^\alpha(0)}}.$$

We need one more special function. The Blaschke product

$$b(\zeta)=\zeta\prod_{\gamma\in\Gamma,\gamma\neq 1_2}\frac{\gamma(0)-\zeta}{1-\overline{\gamma(0)}\zeta}\frac{|\gamma(0)|}{\gamma(0)}$$

is called the *Green's function* of Γ with respect to the origin. It is a character–automorphic function, i.e., there exists $\mu \in \Gamma^*$ such that $b(\gamma(\zeta)) = \mu(\gamma)b(\zeta)$. Note, if $G(z) = G(z, \infty)$ denotes the Green's function of the domain Ω, then

$$G(z(\zeta)) = -\log|b(\zeta)|.$$

Let Γ' be the commutator of the group Γ. Evidently, $b(\zeta)$ and $(zb)(\zeta)$ are functions on the surface $\mathbb{D}/\Gamma'$. It is convenient to accept the normalization condition $(zb)(0) = 1$.

Theorem 2.2. *Let E be a homogeneous set. The set $J(E)$ can be represent in the form* (2) *with $G = \Gamma^*$, $T\alpha = \mu^{-1}\alpha$ and*

$$p(\alpha) = \left(\frac{K^\alpha}{K^{\alpha\mu}}\right)(0), \quad b'(0)q(\alpha) = (zb)'(0) + \left(\log\frac{K^\alpha}{K^{\alpha\mu}}\right)'(0). \tag{8}$$

The operators $\widehat{S}_+$ and $(\widehat{J}\widehat{S})_+$ are unitary equivalent to multiplication by b and (bz) in $A^2_1(\Gamma')$ respectively. This unitary map is given by the formula

$$\sum_{\{\gamma\}\in\Gamma/\Gamma'} f|[\gamma]\alpha^{-1}(\gamma) = \sum_{n\in\mathbb{Z}_+} x_n(\alpha)b^n K^{\alpha\mu^{-n}},$$

where $f \in A^2_1(\Gamma')$ and the vector function $x(\alpha) = \{x_n(\alpha)\}$ belongs to $L^2_{d\chi}(l^2(\mathbb{Z}_+))$.

P r o o f. The statement follows immediately from the *individual* functional model for $J(\alpha)$ that was given in [6] and the decomposition

$$A^2_1(\Gamma') = \int_{\Gamma^*} A^2_1(\Gamma, \alpha)\, d\chi \tag{9}$$

that was proved in [8]. In fact, this decomposition means that for any $f \in A^2_1(\Gamma')$ the series

$$F(\zeta, \alpha) = \sum_{\{\gamma\}\in\Gamma/\Gamma'} (f|[\gamma])(\zeta)\alpha^{-1}(\gamma) \tag{10}$$

define a function which belongs to $A^2_1(\Gamma, \alpha)$ for a.e. α and

$$\|f(\zeta)\|^2 = \int_{\Gamma^*} \|F(\zeta, \alpha)\|^2_{A^2_1(\Gamma,\alpha)} d\chi(\alpha),$$

moreover

$$f(\zeta) = \int_{\Gamma^*} F(\zeta, \alpha)\, d\chi(\alpha). \tag{11}$$

Conversely, we can start from any $F(\zeta,\alpha) \in \int_{\Gamma^*} A_1^2(\Gamma,\alpha)\, d\chi$; then we get an element $f(\zeta) \in A_1^2(\Gamma')$ by (11) and we can restore $F(\zeta,\alpha)$ by (10).

The individual functional model looks as follows. The systems of functions $\{b^n K^{\alpha\mu^{-n}}\}_{n\in\mathbb{Z}_+}$ and $\{b^n K^{\alpha\mu^{-n}}\}_{n\in\mathbb{Z}}$ form an orthonormal basis in $A_1^2(\Gamma,\alpha)$ and in $L^2(\Gamma,\alpha) \simeq L^2_{dm|\mathbb{E}}$, respectively, for any $\alpha \in \Gamma^*$. With respect to this basis, the operator multiplication by $z(t)$ is a three–diagonal almost periodic Jacobi matrix, moreover

$$zb^n K^{\alpha\mu^{-n}} = p_n(\alpha) b^{n-1} K^{\alpha\mu^{-n+1}} + q_n(\alpha) b^n K^{\alpha\mu^{-n}} + p_{n+1}(\alpha) b^{n+1} K^{\alpha\mu^{-n-1}},$$

with $p(\alpha)$ and $q(\alpha)$ given by (8). And conversely, every almost periodic Jacobi matrix of $J(E)$ can be represented in this form with some $\alpha \in \Gamma^*$. ■

Remark 2.3. *It seems very interesting to give an independent proof of this theorem.*

3. The result

First, we need some information on properties of $\mathbb{D}/\Gamma'$ and reproducing kernels on it.

For a given character $\alpha \in \Gamma^*$, as usual let us define

$$H^\infty(\Gamma,\alpha) = \{f \in H^\infty : \ f(\gamma(\zeta)) = \alpha(\gamma) f(\zeta), \ \forall \gamma \in \Gamma\}.$$

Generally, a group Γ is said to be of *Widom type* if for any $\alpha \in \Gamma^*$ the space $H^\infty(\Gamma,\alpha)$ is not trivial (contains a non-constant function).

The *Direct Cauchy theorem* [3] holds on $\mathbb{D}/\Gamma$ if

$$\int_{\mathbb{E}} \frac{f}{b}(\zeta)\, \frac{d\zeta}{2\pi i} = \frac{f}{b'}(0), \quad \forall f \in A_2^1(\Gamma,\mu). \tag{12}$$

Lemma 3.1. *Let Γ be a Fuchsian group of Widom type with (DCT). Then (DCT) holds on $\mathbb{D}/\Gamma'$.*

P r o o f. Let $b(\zeta) = b(\zeta,\Gamma)$ and $b_0(\zeta) = b(\zeta,\Gamma')$ be the Green's functions (with respect to the origin) of the groups Γ and Γ' respectively. Let us show that for any automorphic function $f \circ \gamma = f,\ \gamma \in \Gamma'$, integrable on $\mathbb{T}$, $f \in L^1$, we have the identity:

$$\int_{\mathbb{T}} f\, dm = \int_{\mathbb{T}} \left\{ \sum_{\{\gamma\}\in\Gamma/\Gamma'} \left(f \frac{|b_0'|}{|b'|} \right) \circ \gamma \right\} dm. \tag{13}$$

Recall that for a group of Widom type $b'(\zeta)$ is a function of bounded characteristic and that $\sum_{\gamma\in\Gamma}|\gamma'(\zeta)| = |b'(\zeta)|$ for a.e. ζ on $\mathbb{T}$. Then

$$\int_{\mathbb{T}} f\,dm = \sum_{\gamma_0\in\Gamma'} \int_{\gamma_0(\mathbb{E}_0)} f\,dm = \int_{\mathbb{E}_0} \sum_{\gamma_0\in\Gamma'} f(\gamma_0(\zeta))|\gamma_0'(\zeta)|\,dm(\zeta) = \int_{\mathbb{E}_0} f|b_0'|\,dm,$$

where $\mathbb{E}_0$ is a fundamental set for Γ'. Let $\tilde\Gamma$ be a system of representatives of the classes Γ/Γ'. Let us choose $\mathbb{E}_0 = \cup_{\tilde\gamma\in\tilde\Gamma}\tilde\gamma(\mathbb{E})$, in this case

$$\int_{\mathbb{E}_0} f|b_0'|\,dm = \int_{\mathbb{E}} \sum_{\tilde\gamma\in\tilde\Gamma}(f|b_0'|)(\tilde\gamma(\zeta))|\tilde\gamma'(\zeta)|\,dm(\zeta).$$

As $|b'(\tilde\gamma(\zeta)|\,|\tilde\gamma'(\zeta)| = |b'(\zeta)|$, we have

$$\int_{\mathbb{T}} f\,dm = \int_{\mathbb{E}} \left\{\sum_{\tilde\gamma\in\tilde\Gamma}\left(\frac{f|b_0'|}{|b'|}\right)(\tilde\gamma(\zeta))\right\}|b'(\zeta)|\,dm(\zeta).$$

Since the function $\left(\frac{f|b_0'|}{|b'|}\right)$ is Γ'-automorphic the sum in curly braces does not depend on a choice of representatives $\tilde\Gamma$ and, therefore, it is a Γ-automorphic function. Thus the integral of this function on a fundamental set with weight $|b'|$ and its integral on $\mathbb{T}$ with respect to dm are the same. (13) is proved.

Let us put

$$b'(\zeta,\Gamma) = \Delta(\zeta,\Gamma)\phi(\zeta,\Gamma),$$

where $\Delta(\zeta,\Gamma)$ is an inner factor $\phi(\zeta,\Gamma)$ is an outer factor of $b'(\zeta,\Gamma)$. Note that $\Delta(\zeta,\Gamma)$ is a character-automorphic function and denote by α_Δ its character. Let $H^1(\Gamma,\alpha)$ be a set of α-automorphic functions from H^1. In this terms (DCT) holds iff

$$\int_{\mathbb{T}} \frac{h}{\Delta}\,dm = \frac{h}{\Delta}(0),\ \forall\, h\in H^1(\Gamma,\Delta).$$

Thus we are going to prove the same property for the group Γ'.

Let $f = h/\Delta_0$, where $\Delta_0(\zeta) = \Delta(\zeta,\Gamma_0)$ and $h\in H^1(\Gamma_0,\alpha_{\Delta_0})$. Since

$$|b'(\zeta,\Gamma)| = \frac{\zeta b'(\zeta,\Gamma)}{b(\zeta,\Gamma)},\ \zeta\in\mathbb{T},$$

we have by (13)

$$\int_{\mathbb{T}} \frac{h}{\Delta_0} dm = \int_{\mathbb{T}} \left\{ \sum_{\{\gamma\}\in\Gamma/\Gamma'} \left(\frac{h}{\Delta_0} \frac{b_0'}{b_0} \frac{b'}{b} \right) (\gamma(\zeta)) \right\} dm$$
$$= \int_{\mathbb{T}} \frac{\sum_{\{\gamma\}\in\Gamma/\Gamma'} \left(h \frac{\phi_0}{\phi} \frac{b}{b_0} \right) (\gamma(\zeta)) \alpha_\Delta^{-1}(\gamma)}{\Delta} dm.$$

We claim that $\sum_{\{\gamma\}\in\Gamma/\Gamma'} \left(h \frac{\phi_0}{\phi} \frac{b}{b_0} \right) (\gamma(\zeta)) \alpha_\Delta^{-1}(\gamma) \in H^1(\Gamma, \alpha_\Delta)$. The automorphic property is evident. Putting $f = |h|$ in (13), we get that the series converges absolutely ($\sum_{\{\gamma\}\in\Gamma/\Gamma'} \left(|h| \frac{|\psi_0|}{|\psi|} \right) (\gamma(\zeta)) \in L^1$) and each term is of H^1.

Due to (DCT) in $\mathbb{D}/\Gamma$ we have

$$\int_{\mathbb{T}} \frac{h}{\Delta_0} dm = \frac{1}{\Delta_0(0)} \left(h \frac{\phi_0}{\phi} \frac{b}{b_0} \right) (0) = \frac{h(0)}{\Delta(0)} \frac{\phi_0(0)}{\phi(0)} \frac{b'(0)}{b_0'(0)} = \frac{h(0)}{\Delta_0(0)}.$$

■

Lemma 3.2. *Let $k^\alpha(\zeta, \zeta_0) = k^\alpha(\zeta, \zeta_0; \Gamma)$ be the reproducing kernel of $A_1^2(\Gamma, \alpha)$ and $w(\zeta)$ be an automorphic function with a positive real part. Then the kernel*

$$(w(\zeta_1) + \overline{w(\zeta_2)}) k^\alpha(\zeta_1, \zeta_2) \tag{14}$$

is positive definite.

P r o o f. It is well known that for a contractive automorphic function $u(\zeta), |u(\zeta)| \le 1$, the kernel $(1 - u(\zeta_1)\overline{u(z_2)}) k^\alpha(\zeta_1, \zeta_2)$ is positive (see e.g. [4]). Making linear–fractional transformation $w(\zeta) \mapsto u(\zeta) = \frac{w(\zeta) - w(0)}{w(\zeta) + \overline{w(0)}}$, we get (14). ■

Lemma 3.3. *Assume that the real part of an automorphic function $v(\zeta)$ is bounded, $C_1 \le Re\ v(\zeta) \le C_2$. Then $v k_{\zeta_0}^\alpha \in A_1^2(\Gamma, \alpha)$.*

P r o o f. According to the previous lemma we have the inequalities for kernels

$$C_1 k^\alpha(\zeta_1, \zeta_2) \le \frac{v(\zeta_1) + \overline{v(\zeta_2)}}{2} k^\alpha(\zeta_1, \zeta_2) \le C_2 k^\alpha(\zeta_1, \zeta_2).$$

This implies that the operator V^* is well define by $V^* k_{\zeta_0}^\alpha = \overline{v(\zeta_0)} k_{\zeta_0}^\alpha$ on a dense in $A_1^2(\Gamma, \alpha)$ set consisting of finite linear combinations of reproducing kernels

$$\mathcal{D}_{V^*} = \{x = \sum c_j k_{\zeta_j}^\alpha\}.$$

Moreover, the quadratic form $\frac{1}{2}\{\langle V^*x, x\rangle + \langle x, V^*x\rangle\}$, $x \in \mathcal{D}_{V^*}$, is bounded and hence it is the quadratic form of a bounded self adjoint operator, say A. Thus, in particular, we have

$$v(\zeta)k^{\alpha}_{\zeta_0}(\zeta) + \overline{v(\zeta_0)}k^{\alpha}_{\zeta_0}(\zeta) = 2(Ak^{\alpha}_{\zeta_0})(\zeta).$$

The lemma is proved. ∎

Remark 3.4. *Lemma 3.3 shows that the reproducing kernels possess some "additional smoothness". It is highly interesting to clarify an exact meaning of this words.*

Lemma 3.5. *Let $k^{\alpha}_{\zeta_0;\Gamma}(\zeta) = k^{\alpha}(\zeta, \zeta_0; \Gamma)$ be the reproducing kernel of $A^2_1(\Gamma, \alpha)$. Then*

$$k_{\zeta_0;\Gamma'} = \int\limits_{\Gamma^*} k^{\alpha}_{\zeta_0;\Gamma}\, d\chi(\alpha), \tag{15}$$

and

$$k^{\alpha}_{\zeta_0;\Gamma} = \sum_{\{\gamma\}\in\Gamma/\Gamma'} k_{\zeta_0;\Gamma'}|[\gamma]\alpha^{-1}(\gamma). \tag{16}$$

P r o o f. We use decomposition (9) and (10). For any $\zeta_0 \in \mathbb{D}$ and $f \in A^2_1(\Gamma')$

$$f(\zeta_0) = \langle f(\zeta), k_{\zeta_0;\Gamma'}(\zeta)\rangle = \int\limits_{\Gamma^*} \langle F(\zeta, \alpha), \sum_{\{\gamma\}\in\Gamma/\Gamma'} (k_{\zeta_0;\Gamma'}|[\gamma])(\zeta)\alpha^{-1}(\gamma)\rangle\, d\chi(\alpha).$$

On the other hand by (11),

$$f(\zeta_0) = \int\limits_{\Gamma^*} F(\zeta_0, \alpha)\, d\chi(\alpha) = \int\limits_{\Gamma^*} \langle F(\zeta, \alpha), k^{\alpha}(\zeta, \zeta_0)\rangle\, d\chi(\alpha).$$

Thus, we get (16). Then (11) implies (15). ∎

Now we are in position to prove the Main Theorem

P r o o f. We consider a symbol a as a real–valued symmetric function on $\mathbb{T}$, $a(\zeta) = a(\bar{\zeta})$ possessing the automorphic property $a(\gamma(\zeta)) = a(\zeta)$. For any α it defines a bounded self–adjoint operator $M_a(\alpha)$ as the multiplication operator in $L^2(\Gamma, \alpha)$ with respect to the basis $\{e_n(\alpha, \zeta)\}_{n\in\mathbb{Z}}$, $e_n(\alpha, \zeta) = b^n(\zeta)K^{\alpha\mu^{-n}}(\zeta)$.

Under isomorphism

$$L^2(\Gamma') \simeq \int\limits_{\Gamma^*} L^2(\Gamma, \alpha)\, d\chi(\alpha) \simeq L^2_{d\chi}(l^2(\mathbb{Z})), \tag{17}$$

the operator multiplication by a defines an operator $A : L^2_{d\chi}(l^2(\mathbb{Z})) \to L^2_{d\chi}(l^2(\mathbb{Z}))$ of the form

$$(Ax)(\alpha) = M_a(\alpha)x(\alpha), \quad x \in L^2_{d\chi}(l^2(\mathbb{Z})).$$

Define

$$(a + \tilde{a})(\zeta) = v(\zeta) := \int_{\mathbb{T}} \frac{t+\zeta}{t-\zeta} a(t)\, dm(t),$$

where $a(\zeta)$ and $\tilde{a}(\zeta)$ are respectively real and imaginary parts of $v(\zeta)$, $\zeta \in \mathbb{D}$. Thus $\tilde{a}$ is the harmonically conjugated function to a. The boundary values of $\tilde{a}(\zeta)$ define an imaginary–valued function on $\mathbb{T}$, generally, it does not belong to L^∞, but, for sure, $\tilde{a} \in L^2$. Also, this function has the following property:

$$\tilde{a}(\gamma(\zeta)) = \tilde{a}(\zeta) + \xi(\gamma),$$

where $\xi : \Gamma \to i\mathbb{R}$ is an additive function on Γ. Thus $\tilde{a}$ and v are automorphic with respect to Γ'.

Using Lemma 3.3 we define an (unbounded) operator multiplication by v in $L^2(\Gamma')$. Note that $k_{\zeta_0;\Gamma'} \in \mathcal{D}_v$, moreover

$$v(\zeta)k_{\zeta_0;\Gamma'}(\zeta) = -\overline{v(\zeta_0)}k_{\zeta_0;\Gamma'}(\zeta) + 2P_+(ak_{\zeta_0;\Gamma'})(\zeta). \tag{18}$$

Using Lemma 3.5 we obtain from (18)

$$\begin{gathered}\sum_{\{\gamma\}\in\Gamma/\Gamma'} (v(\zeta) + \xi(\gamma))k_{\zeta_0;\Gamma'}|[\gamma](\zeta)\alpha^{-1}(\gamma) = \\ v(\zeta)k^\alpha_{\zeta_0;\Gamma}(\zeta) + \sum_{\{\gamma\}\in\Gamma/\Gamma'} \xi(\gamma)k_{\zeta_0;\Gamma'}|[\gamma](\zeta)\alpha^{-1}(\gamma) = -\overline{v(\zeta_0)}k^\alpha_{\zeta_0;\Gamma}(\zeta) + 2\langle ak^\alpha_{\zeta_0;\Gamma}, k^\alpha_{\zeta;\Gamma}\rangle.\end{gathered} \tag{19}$$

Note that the sum in the last line of (19) is the formal derivative of the series (16) in the direction $-\xi$ that is

$$\partial_{-\xi} f(\alpha) := \frac{\mathrm{d}}{\mathrm{d}t} f(e^{-t\xi}\alpha)\Big|_{t=0}.$$

Since all other terms in this line are continuous functions on Γ^* (for fixed ζ and ζ_0 in $\mathbb{D}$), the series represent a continuous function. Therefore $k^\alpha_{\zeta_0;\Gamma}$ is differentiable in the direction ξ and we can rewrite (19) into the form

$$(v(\zeta) - \partial_\xi)k^\alpha_{\zeta_0;\Gamma}(\zeta) = -\overline{v(\zeta_0)}k^\alpha_{\zeta_0;\Gamma}(\zeta) + 2\langle ak^\alpha_{\zeta_0;\Gamma}, k^\alpha_{\zeta;\Gamma}\rangle. \tag{20}$$

Let $\{a_{n,m}(\alpha)\}$ be matrix entries of the operator $M_a(\alpha)$ with respect to the basis $\{e_n(\alpha)\}$. We put $\zeta = \zeta_0 = 0$ in (20). Then we get

$$(a(0) - \partial_\xi)e_0^2(\alpha, 0) = -a(0)e_0^2(\alpha, 0) + 2a_{0,0}(\alpha)e_0^2(\alpha, 0)$$

or

$$(a(0) - \partial_\xi)e_0(\alpha, 0) = a_{0,0}(\alpha)e_0(\alpha, 0). \tag{21}$$

Putting $\zeta = 0$ in (20) we get

$$(a(0) - \partial_\xi)\overline{e_0(\alpha, \zeta_0)}e_0(\alpha, 0) = -\overline{v(\zeta_0)e_0(\alpha, \zeta_0)}e_0(\alpha, 0) \\ +2\sum_{n=0}^{\infty} a_{0,n}(\alpha)\overline{e_n(\alpha, \zeta_0)}e_0(\alpha, 0).$$

or making use (21) we obtain

$$-\partial_\xi\overline{e_0(\alpha, \zeta_0)} = (a_{0,0}(\alpha) - \overline{v(\zeta_0)})\overline{e_0(\alpha, \zeta_0)} + 2\sum_{n=1}^{\infty} a_{0,n}(\alpha)\overline{e_n(\alpha, \zeta_0)}. \tag{22}$$

Note that since $e_m(\alpha, \zeta_0) = b^m(\zeta_0)e_0(\alpha\mu^{-m}, \zeta_0)$ simultaneously with (22) we proved

$$-\partial_\xi\overline{e_m(\alpha, \zeta_0)} = (a_{m,m}(\alpha) - \overline{v(\zeta_0)})\overline{e_m(\alpha, \zeta_0)} + 2\sum_{n=m+1}^{\infty} a_{m,n}(\alpha)\overline{e_n(\alpha, \zeta_0)}. \tag{23}$$

Now we may claim that operator multiplication by v under isomorphism (17) becomes the operator

$$V = -\partial_\xi + \{M(\alpha) - \widetilde{M}(\alpha)\}. \tag{24}$$

Indeed, according to (18) for $m \geq 0$ we have

$$(V\hat{k}_{\zeta_0})_m = -\overline{v(\zeta_0)e_m(\alpha, \zeta_0)} + 2\sum_{n=0}^{\infty} a_{m,n}(\alpha)\overline{e_n(\alpha, \zeta_0)}, \tag{25}$$

where as in Section 2

$$\hat{k}_{\zeta_0} \in L^2_{d\chi}(l^2(\mathbb{Z}_+)), \quad (\hat{k}_{\zeta_0})_m = \overline{e_m(\alpha, \zeta_0)}.$$

Substituting $\overline{v(\zeta_0)}\hat{k}_{\zeta_0}$ from (23) into (25) we get (24) on the vector $\hat{k}_{\zeta_0}$.

Finally, the operators $\hat{J}$ and V commute and this commutant relation yields

$$\partial_{-\xi}J(\alpha) = [\widetilde{M}_a(\alpha), J(\alpha)].$$

■

Remark 3.6. *Note that stationary solutions of the hierarchy corresponds to those "directions" a for which the associated character* $\xi = 0$. *In the other words* $\tilde{a}$ *should be a* Γ*-automorphic function.*

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