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On independence of characteristics of asymptotic behavior of entire functions

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It is known [6, Ch. 3, Theorem 18] that using any *total* family of asymptotic characteristics, we have a possibility to learn whether an entire function of finite order is a function of completely regular growth. However, the main result of this paper shows that all these characteristics do not give a possibility to know whether a function has an H-multiplicator [4].

Dedicated to the 100th anniversary of the birth of Naum Il'ich Akhiezer

1. Multiplier problem

Let Φ be an entire function of order ρ and normal type; we write $\Phi \in A(\rho)$. Its *Phragmén–Lindelöf indicator* is defined by the equality

$$h(\Phi, \phi) := \limsup_{r \to \infty} r^{-\rho} \log |\Phi(re^{i\phi})|.$$

The function $h(\Phi, \phi)$ is 2π-periodic and ρ-trigonometrically convex. This means that on the unit circle $\mathbb{T}$ it satisfies a differential inequality in the sense of distributions:

$$h'' + \rho^2 h := \nu \geq 0. \tag{1.1}$$

Let $H(\phi)$ be a fixed ρ-trigonometrically convex function. The function $g \in A(\rho)$ is called an *H-multiplicator* of Φ if the Phragmén–Lindelöf indicator $h_{g\Phi}$ of the product $g\Phi$ satisfies the condition

$$h(g\Phi, \phi) \leq H(\phi).$$

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The property of having an H-multiplicator depends only on the asymptotic behavior of Φ. It is important for some problems in analysis to know whether the function Φ has an H-multiplicator (for details, see [4]).

2. Completely regular growth

A function $f \in A(\rho)$ is called a *function of completely regular growth* in sense of Levin–Pfluger (CRG-function) if

$$|\lambda|^{-\rho} \log |f(\lambda)| - h(f, \arg \lambda) \to 0$$

when $\lambda \to \infty$ outside some small set (a C_0-set) (for details, see [7, Ch. 2, 3]). To be a CRG-function is an asymptotic property of an entire function. This class of entire functions has been studied completely and plays an important role in applications [6, Ch. 2].

3. Total families

There exist numerous asymptotic characteristics of growth of entire functions. Now we recall some of them and their connection to the CRG-functions.

The *lower indicator* in sense of A. Gol'dberg is defined by the equality

$$\underline{h}(f, \phi) := \sup_{E \in \mathcal{E}} \left\{ \liminf_{r \to \infty, r \notin E} \frac{\log |f(re^{i\phi}|}{r^\rho} \right\} \tag{3.1}$$

where $\mathcal{E}$ is a class of "small" sets $E \subset [0, \infty)$ satisfying the condition

$$mes\{E \cap [0, R)\} = o(R)$$

as $R \to \infty$.

The sets E serve for excluding the influence of zeros of the function f. If the function has no zeros in some open angle containing the ray $\{\arg \lambda = \phi\}$, then the right side of (3.1) can be replaced by a standard *lower limit.*

The family $\mathcal{H} := \{h(f, \phi), \underline{h}(f, \phi) : \phi \in [0, 2\pi)\}$ is *total*, i.e., it gives a possibility to learn whether a function $f \in A(\rho)$ is a CRG-function.

Theorem A (A. Gol'dberg). *(See [5]). A function $f \in A(\rho)$ is a CRG-function iff*

$$h(f, \phi) = \underline{h}(f, \phi), \ \forall \phi \in [0, 2\pi). \tag{3.2}$$

The family $\mathcal{H}$ is also "minimal" in some sense. Namely, if we claim that the equalities (3.2) hold for $\phi \in [0, 2\pi) \setminus U$ where U is an open set, then the function f

need not be CRG-function. Moreover, for any U there exists a function $f \in A(\rho)$ such that the equalities (3.2) are broken exactly for $\phi \in U$ ([2, 3]).

Consider another family of asymptotic characteristics. Set

$$c_k(f,r) := \begin{cases} \frac{1}{2\pi} \int\limits_0^{2\pi} \log |f(re^{i\phi})| \cos k\phi d\phi, & \text{for } k = 0,1,2,\dots, \\ -\frac{1}{2\pi} \int\limits_0^{2\pi} \log |f(re^{i\phi})| \sin k\phi d\phi, & \text{for } k = -1,-2,\dots, \end{cases}$$

the Fourier coefficients of $\log |f|$.

The corresponding asymptotic characteristics are defined by

$$\overline{c}_k(f) := \limsup_{r\to\infty} \frac{c_k(f,r)}{r^\rho}; \underline{c}_k(f) := \liminf_{r\to\infty} \frac{c_k(f,r)}{r^\rho},\ k \in \mathbb{Z}.$$

The family $\mathcal{F} := \{\underline{c}_k(\cdot), \overline{c}_k(\cdot) : k \in \mathbb{Z}\}$ is also total:

Theorem B (V. Azarin, A. Gol'dberg). *(See [1], [6, Ch. 3, Theorem 11]). A function $f \in A(\rho)$ is a CRG-function iff*

$$\overline{c}_k(f) = \underline{c}_k(f) \tag{3.3}$$

for all $k \in \mathbb{Z}$.

The family $\mathcal{F}$ is also "minimal". If we claim the equalities (3.3) not for all $k \in \mathbb{Z}$ the function f need not be a CRG-function. Moreover, for any decomposition $\mathbb{Z}$ into two subsets A and $\mathbb{Z} \setminus A$, there exists $f \in A(\rho)$ such that the equalities (3.3) hold for $k \in A$ and do not hold for $k \in \mathbb{Z} \setminus A$ (see [1]).

4. The main result

The author was going to study a connection between the property of a function Φ of having a multiplicator and the described above asymptotic characteristics but it turned out, that the result is negative as the following theorem shows

Theorem 4.1. *For any smooth, strictly** *ρ-trigonometrically convex function $H(\phi)$ there exist two functions $\Phi_1, \Phi_2 \in A(\rho)$, $\rho > 1/2$ such that*

$$h(\Phi_1,\phi) = h(\Phi_2,\phi),\ \underline{h}(\Phi_1,\phi) = \underline{h}(\Phi_2,\phi),\ \forall\, \phi \in [0,2\pi) \tag{4.1}$$

$$\overline{c}_k(\Phi_1) = \overline{c}_k(\Phi_2),\ \underline{c}_k(\Phi_1) = \underline{c}_k(\Phi_2),\ \forall\, k \in \mathbb{Z}, \tag{4.2}$$

but Φ_1 has an H-multiplicator and Φ_2 does not have anyone.

* It means that ν from (1.1) is summable and strictly positive.

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