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Perturbation of soliton for Davey–Stewartson II equation

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It is known that, under a small perturbation (of order ε) of lump (soliton) for Davey–Stewartson (DS-II) equation, the scattering data become nonsoliton. As a result, the solution has the form of Fourier type integral. Asymptotical analysis, given in this work, shows that in spite of dispersion, the main term of the asymptotic expansion for the solution has the solitary wave form up to large time (of order ε^{-1}).

Introduction

The Davey–Stewartson II (DS-II) equation, describing the interaction of the gravitational and capillary waves on a surface of a liquid [1], is the example of the 2+1-dimensional equation integrable by the inverse scattering transform (IST) see [2, 3].

Presence of solitary waves in solution of the DS-II equation is determined by existence of singularities of solution for auxilary scattering problem [2, 3]. Hence, the problem of the soliton stability and of variation of its parameters is reduced to study of the dependence of the spectral data with respect to the perturbation. For 1+1-dimensional integrable equations, it is well-known [4, 5], that a small perturbation of the potential for the scattering problem implies a regular variation of spectral data and, hence, small changes of soliton parameters.

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In this work the problem of the small perturbation for the 2-dimensional soliton of the DS-II equation is considered. It would be naturally to expect results similar to results for perturbation of solitons for the 1+1-dimensional integrable equations. However, the asymptotical analysis for the DS-II equation [6] showed that the soliton structure of the scattering data disappears under a small perturbation. Therefore the soliton is unstable with respect to perturbation of the initial data [7]. Hence, the solution of the perturbed problem vanishes as t (time) tends to infinity ([8]).

In this paper we show that, in spite of the nonsoliton structure, the perturbed soliton solution conserves the soliton-like form for the principal term of the asymptotics up to long time, and calculate the evolution of the parameters for this soliton-like solution.

The structure of this paper is following. The setting of the problem and the main result are formulated in Section 1. In the next auxiliary section we remind the IST scheme for the DS II equation [3]. The asymptotic behavior of the scattering data for perturbed soliton solution is given in the third section following [6]. In Section 4 we solve the inverse scattering problem in the nonsoliton case asymptotically. The asymptotic solution allows to obtain the perturbed soliton-like solution for the DS-II equation in Section 5.

1. Setting of a problem

The DS-II equation system is considered in the form [3]:

$$i\partial_t q + 2(\partial_z^2 + \partial_{\bar{z}}^2)q + (g + \overline{g})q = 0, \quad \partial_{\bar{z}} g = \partial_z |q|^2, \tag{1}$$

where $z = x + iy$, and the overbar is complex conjugation.

The elementary solution of the equation is the lump or soliton (see [3]):

$$q(z,t) = q^0(z,t) = \frac{2\overline{\nu}_0}{|z + 4ik_0 t + \mu_0|^2 + |\nu_0|^2} \exp\{k_0 z - \overline{k_0 z} + 2i(k_0^2 + \overline{k}_0^2)t\},$$

$$g(z,t) = g^0(z,t) = \frac{-4\overline{(z + 4ik_0 t + \mu_0)}^2}{(|z + 4ik_0 t + \mu_0|^2 + |\nu_0|^2)^2},$$

where k_0, ν_0 and μ_0 are arbitrary complex numbers.

In this paper we consider the initial-boundary problem for (1) with perturbed soliton in the initial data:

$$q^\varepsilon(z,0) = q_\varepsilon(z) = q_0(z) + \varepsilon q_1(z), \tag{2}$$

where $q^0(z,0) = q_0(z)$, q_1 is a smooth function with a finite support and ε is a small positive parameter. The main result of the paper reads as follows. The solution (2) of (1), has the soliton-like asymptotics up to $t = O(\varepsilon^{-1})$:

$$q^\varepsilon(z,t) \sim \frac{2\overline{\nu_0}}{|z + 4it(k_0 + \varepsilon k_1) + \mu_0|^2 + |\nu_0|^2} \exp\{k_0 z - \overline{k_0 z} + 2i(k_0^2 + \overline{k}_0^2)t\}, \quad (3)$$

$$g^\varepsilon(z,t) \sim \frac{-4\overline{(z + 4it(k_0 + \varepsilon k_1) + \mu_0)}^2}{(|z + 4it(k_0 + \varepsilon k_1) + \mu_0|^2 + |\nu_0|^2)^2}, \quad (4)$$

where

$$k_1 = -\frac{1}{2\pi} \int dz \wedge d\bar{z} \frac{z + \mu_0}{(|z + \mu_0|^2 + |\nu_0|^2)^2} \mathrm{Im}(q_1 \nu_0 \exp\{\overline{z k_0} - z k_0\}). \quad (5)$$

Hereafter, $dz \wedge d\overline{z} = -2i\, dx\, dy$.

2. Preliminary

The solution with smooth and decreasing as $z \to \infty$ initial data for (1) is given by the IST method ([2, 3]). The auxiliary scattering problem for (1) has the following form [2, 3]:

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \psi(k,z) = \begin{pmatrix} 0 & \frac{q(z,0)}{2} \\ -\frac{\overline{q(z,0)}}{2} & 0 \end{pmatrix} \psi(k,z), \quad (6)$$

$$E(-kz)\psi(k,z) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad z \to \infty, \quad (7)$$

where $E(kz) = \mathrm{diag}(\exp(kz), \exp(\overline{kz}))$. The scattering data $\mathcal{L}$ of the problem (6), (7) consists of the continuous and discrete parts. The first of them is defined by the equality

$$b(k) = -\frac{i}{4\pi} \int dz \wedge d\overline{z}\, q(z,0)\, \overline{\psi^{(1)}(z)} \exp\{-kz\}, \quad (8)$$

where $\psi^{(1)}$ is the corresponding component of the vector ψ.

If ψ has no singularities with respect to k, then the discrete part $\mathcal{L}^d = \emptyset$ and the solution is nonsoliton, i.e., it vanishes as $t \to \infty$ ([8]). In the nonsoliton case the solution of the DS II equation has form

$$g(z,t) = -\frac{1}{2\pi i} \int \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - z)^2} |q(\zeta,t)|^2, \quad (9)$$

$$q(z,t) = \frac{i}{\pi} \int dp \wedge d\overline{p}\, b(p)\, \phi^{(1)}(z,t,p) \exp\{is(z,t,p)\}, \quad (10)$$

where $s(z,t,p) = 2t(p^2+\bar{p}^2) - i(pz - \overline{pz})$ and $\phi^{(1)}(z,t,k)$ is the first component of the following boundary value problem

$$\overline{\mathcal{D}}\phi \equiv \begin{pmatrix} \partial_{\bar{k}} & \overline{b(k)}\exp\{-is(z,t,k)\} \\ -b(k)\exp\{is(z,t,k)\} & \partial_k \end{pmatrix}\phi = 0,$$
$$\phi \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad k\to\infty. \tag{11}$$

The discrete part $\mathcal{L}^d$ of the scattering data consists of the singularities of ψ and of some their characterizing constants. Namely, this part corresponds to the existence of the solitons and describes their parameters. The case, when the continuous component vanishes, corresponds to the pure soliton solution. The 1-soliton solution is the example of the pure soliton solution. In this case

$$\psi(k,z) = \psi_0(k,z) = E_1(kz) - \tfrac{\exp\{(k-k_0)z\}}{k-k_0}A_1(z), \tag{12}$$
$$\mathcal{L}^d = \mathcal{L}^d_0 = \{k_0, \mu_0, \nu_0\},$$

where $E_1(kz)$ is the first column of the matrix $E(kz)$ and

$$A_1(z) = \frac{1}{|z+\mu_0|^2 + |\nu_0|^2}\begin{pmatrix} \overline{(z+\mu_0)}\exp\{zk_0\} \\ \nu_0\exp\{\overline{zk_0}\} \end{pmatrix}. \tag{13}$$

So, for the unperturbed initial condition, spectral data $\mathcal{L} = \mathcal{L}_0$ has form $\mathcal{L}_0 = \{\mathcal{L}^d_0; \mathcal{L}^c_0\} = \{\{k_0,\mu_0,\nu_0\}; 0\}$.

On the other hand, asymptotical analysis in [6] showed, that for the perturbed initial condition, spectral data $\mathcal{L} = \mathcal{L}_\varepsilon$ reads as $\mathcal{L}_\varepsilon = \{\mathcal{L}^d_\varepsilon; \mathcal{L}^c_\varepsilon\} = \{\emptyset; b_\varepsilon(k)\}$.

3. Asymptotics of the scattering data

Here we remind the result of [6] for scattering data of perturbed soliton in the initial condition (2). In this case the scattering data contain the continuous part only. This continuous scattering data are $O(\varepsilon)$ on almost complex plain, but have a big magnitude $O(\varepsilon^{-1})$ near point k_0. This point is associated with pole of the solution for the Dirac equation (6) for a nonperturbed case.

$$b_\varepsilon(k) \sim \varepsilon b_1(k) \quad \text{for } |k-k_0| > \varepsilon^{1/2}, \tag{14}$$

$$b_\varepsilon(k) \sim \varepsilon^{-1}B_{-1}\left(\frac{k-k_0}{\varepsilon}\right) + B_0\left(\frac{k-k_0}{\varepsilon}\right) \quad \text{for } |k-k_0| < 2\varepsilon^{1/2}, \tag{15}$$

$$B_{-1}(\kappa) = -\frac{\overline{Q_1}}{|Q_1|^2 + |Q_2+\kappa|^2}, \tag{16}$$

$$b_1(k) = -\frac{i}{4\pi}\int dz\wedge d\overline{z}\,(q_1\,\overline{\psi_0^{(1)}}\psi_0^{*,(1)} + \overline{q_1}\,\overline{\psi_0^{(2)}}\psi_0^{*,(2)}). \tag{17}$$

Hereafter

$$Q_j = \frac{1}{4\pi i}\int dz\wedge d\overline{z}\,(\overline{q_1}\,A_1^{(1)}\overline{C_j^{(1)}} + q_1\,A_1^{(2)}\overline{C_j^{(2)}}), \tag{18}$$

and $A_1^{(m)}$, $\psi_0^{(m)}$, $\psi_0^{*,(m)}$ and $C_j^{(m)}$ are the components of the vectors A_1, ψ_0, ψ_0^* and C_j defined as

$$\psi_0^*(k,z) = E_1(-kz) + \frac{\exp\{(-k+k_0)z\}}{k-k_0}C_1(z), \tag{19}$$

$$C_1(z) = \frac{1}{|z+\mu_0|^2+|\nu_0|^2}\begin{pmatrix} \overline{(z+\mu_0)}\exp\{-zk_0\} \\ \overline{\nu}_0\exp\{-\overline{zk_0}\}\end{pmatrix}, \tag{20}$$

$$C_2 = \sigma\overline{C_1}, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{21}$$

Here $\psi_0^*(k,z)$ is the solution for formal adjoint equation for (6) with respect to sesquilinear form:

$$(f,w)_q = \int dz\wedge d\overline{z}\,(\overline{q}\,f^{(1)}\overline{w^{(1)}} + q\,f^{(2)}\overline{w^{(2)}}),$$

where $f^{(i)}$, $w^{(i)}$ are components of vectors f, w.

R e m a r k 1. One can see that $\varepsilon b_1(k)|_{k\to k_0} = O(\varepsilon|k-k_0|^{-2})$. It leads to nonapplicability of external asymptotics (14) near k_0. Therefore near k_0 we must construct internal asymptotics (16). The domains of applicability of external (14) and internal (16) asymptotics are defined by applicability of corresponded asymptotic solutions for (6) (see [6]).

4. Asymptotics of solution for inverse problem in nonsoliton case

From the results which are reminded in the above sections one can see the solution of the initial-boundary problem for (1) is reduced to investigating the perturbation of the $\bar{D}$-problem for (11) with $b(k)\equiv b_\varepsilon(k)$.

For $b(k) = b_\varepsilon(k)$ and $|k-k_0| > \varepsilon^{1/2}$, the asymptotic solution $\phi = \phi^\varepsilon$ of the problem (11) is constructed in the form:

$$\phi^\varepsilon(z,t,k) = \phi_0(z,t,k) + \varepsilon\phi_1(z,t,k) + \dots \tag{22}$$

Putting (22) and (14) in (11), we obtain the following problems for ϕ_0:

$$\begin{pmatrix} \partial_{\overline{k}} & 0 \\ 0 & \partial_k \end{pmatrix} \phi_0 = 0, \qquad \phi_0 \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad k \to \infty. \tag{23}$$

Obviously, the function

$$\phi_0 = \begin{pmatrix} 1 + \frac{\beta^{(1)}}{k-k_0} \\ \frac{\beta^{(2)}}{\overline{k-k_0}} \end{pmatrix} \tag{24}$$

satisfies (23) for any $\beta^{(m)}$ independent of k.

Since, for $|k-k_0| < 2\varepsilon^{1/2}$, b_ε has the form (15), it is naturally to come in (11) to the variable of the same scale $\kappa = (k-k_0)\varepsilon^{-1}$. In this domain, the asymptotics of ϕ^ε we construct by using the method of matched expansions [9]. Following this method, we rewrite the asymptotics of (22), (24) as $k \to k_0$ in the "inner" variable κ and obtain that:

$$\phi^\varepsilon = \varepsilon^{-1} \begin{pmatrix} \frac{\beta^{(1)}}{\kappa} \\ \frac{\beta^{(2)}}{\overline{\kappa}} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dots .$$

The latter equality implies that, for $|k - k_0| < 2\varepsilon^{1/2}$ (or $|\kappa| < 2\varepsilon^{-1/2}$), the asymptotics of ϕ^ε has the structure

$$\phi^\varepsilon = \varepsilon^{-1}\Phi_{-1} + \Phi_0 + \dots , \tag{25}$$

where the coefficients satisfy the following boundary conditions

$$\Phi_{-1} = \begin{pmatrix} \frac{\beta^{(1)}}{\kappa} \\ \frac{\beta^{(2)}}{\overline{\kappa}} \end{pmatrix} (1 + o(1)) \quad \text{and} \quad \Phi_0 \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{as } \kappa \to \infty. \tag{26}$$

Denote, $s_0 = s(z,t,k_0)$, $l = 4ik_0t + z$, $T = \varepsilon 2t$. In this variables we have, that

$$s(z,t,k) = s_0 - \varepsilon\, i(l\kappa - \overline{l\kappa}) + \varepsilon T(\kappa^2 + \bar{\kappa}^2). \tag{27}$$

Putting (25) and (15) in (11) and taking into account (27), we obtain the following equations for Φ_j:

$$\begin{pmatrix} \partial_{\overline{\kappa}} & 0 \\ 0 & \partial_\kappa \end{pmatrix} \Phi_{-1} = \begin{pmatrix} 0 & -\overline{\Omega_0} \\ \Omega_0 & 0 \end{pmatrix} \Phi_{-1}, \tag{28}$$

$$\begin{pmatrix} \partial_{\overline{\kappa}} & 0 \\ 0 & \partial_\kappa \end{pmatrix} \Phi_0 = \begin{pmatrix} 0 & -\overline{\Omega_0} \\ \Omega_0 & 0 \end{pmatrix} \Phi_0 + \begin{pmatrix} 0 & -(\overline{\Omega_1 + Z}) \\ \Omega_1 + Z & 0 \end{pmatrix} \Phi_{-1}, \tag{29}$$

where

$$\Omega_0(\kappa,s_0) = -\frac{\overline{Q_1}\exp\{is_0\}}{|Q_1|^2+|Q_2+\kappa|^2}, \qquad \Omega_1(\kappa,s_0) = B_0(\kappa)\exp\{is_0\},$$
$$Z(\kappa,l,s_0,T) = \Omega_0(\kappa,s_0)\left(\left(\kappa l - \overline{\kappa l}\right) + iT(\kappa^2+\bar{\kappa}^2)\right).$$

R e m a r k 2. These equations are obtained by neglecting of terms order by $O(\varepsilon^2T^2)$, then this approach is valid up to $T \equiv 2\varepsilon t < T_0 = Const > 0$.

The equation (28) has two linearly independent solutions decreasing as $\kappa \to \infty$, only:

$$\mathcal{A}_1 = \frac{1}{|\kappa+Q_2|^2+|Q_1|^2}\begin{pmatrix} \overline{Q_2+\kappa} \\ \overline{Q_1}\exp\{is_0\} \end{pmatrix}, \qquad \mathcal{A}_2 = \sigma\overline{\mathcal{A}_1}.$$

Hence, the solution of (28), (26) reads as follows:

$$\Phi_{-1} = \mathcal{A}\beta, \tag{30}$$

here $\mathcal{A}$ is the matrix with the column $\mathcal{A}_j$ and β is vector.

For calculation β, we consider the problem (29), (26) for Φ_0. Using the Cauchy–Green formula, we rewrite the boundary value problem (29), (26) in the form of the integral equation:

$$(I - \mathcal{H}[\Omega_0])\Phi_0 = F, \tag{31}$$

where I is the unit matrix,

$$\left(\mathcal{H}[h]w\right)(\kappa) = -\frac{i}{2\pi}\int dm\wedge d\overline{m}\begin{pmatrix} 0 & -\frac{\overline{h}(m)}{m-\kappa} \\ \frac{h(m)}{m-\kappa} & 0 \end{pmatrix} w(m),$$

$$F = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{H}[(\Omega_1+Z)]\Phi_{-1}. \tag{32}$$

Denote

$$\langle w,v\rangle_h = \int d\kappa\wedge d\overline{\kappa}\left(\bar{h}w^{(1)}\bar{v}^{(1)} + hw^{(2)}\bar{v}^{(2)}\right),$$

where $w^{(i)}$, $v^{(i)}$ are the components of w and v. One can see that

$$\langle\mathcal{H}[h]w,v\rangle_{\bar{h}} = \langle w,\mathcal{H}[\bar{h}]v\rangle_{\bar{h}}. \tag{33}$$

Hence,

$$\mathcal{B}_1 = \frac{1}{|\kappa+Q_2|^2+|Q_1|^2}\begin{pmatrix} \overline{Q_2+\kappa} \\ Q_1\exp\{-is_0\} \end{pmatrix}, \quad \mathcal{B}_2 = \sigma\overline{\mathcal{B}_1}$$

are the decreasing solutions of the adjoint equation

$$(I - \mathcal{H}[\overline{\Omega_0}])\mathcal{B}_j = 0 \tag{34}$$

and the solvability conditions for (31) have the form:

$$\langle F, \mathcal{B}_i \rangle_{\overline{\Omega_0}} = 0, \quad i = 1,\, 2. \tag{35}$$

Putting (32) in (35), taking into account (33), (34) and evaluating integrals one can see, that the solvability conditions are reduced to the following equations for definition of the components $\beta^{(1,2)}$:

$$\begin{gathered} a_1 \exp\{is_0\}\beta^{(1)} + \left(a_2 - 2\pi\left(2T\bar{Q}_2 - il\right)\right)\beta^{(2)} = 0, \\ \left(\bar{a}_2 - 2\pi\left(2TQ_2 + il\right)\right)\beta^{(1)} - \bar{a}_1 \exp\{-is_0\}\beta^{(2)} = 2i\pi, \end{gathered} \tag{36}$$

where a_j are some constants. They are defined below.

From (36) we deduce that:

$$\beta^{(1)}(l, T, s_0) \quad = \frac{2\pi i\left(a_2 - 2\pi\left(2T\bar{Q}_2 - il\right)\right)}{|a_1|^2 + |a_2 - 2\pi\left(2T\bar{Q}_2 - il\right)|^2}, \tag{37}$$

$$\beta^{(2)}(l, T, s_0) \quad = -\frac{2i\pi a_1 \exp\{is_0\}}{|a_1|^2 + |a_2 - 2\pi\left(2T\bar{Q}_2 - il\right)|^2}. \tag{38}$$

Thus, the asymptotical solution of (11) has the form (22), (24), for $|k - k_0| > \varepsilon^{1/2}$, and the form (25), (30), (38) for $|k - k_0| < 2\varepsilon^{1/2}$.

5. Asymptotics of solution for DS-II equation

As above mentioned, in the nonsoliton case, the solution of the DS-II equation can be constructed by the formula (10). Putting the asymptotics (14), (15) of the scattering data and the asymptotical representation (22), (25) of ϕ^ε in (10), we see that

$$q^\varepsilon \sim I^{ex} + I^{in} \qquad \text{as } \varepsilon \to 0, \tag{39}$$

where

$$I^{ex} = \varepsilon \frac{i}{\pi} \int\limits_{|k-k_0| > \varepsilon^{1/2}} dk \wedge d\bar{k}\, \phi_0^{(1)}(k) b_1(k), \tag{40}$$

$$I^{in} = \varepsilon^{-2} \frac{i}{\pi} \int\limits_{|k-k_0| < \varepsilon^{1/2}} dk \wedge d\bar{k}\, \Phi_{-1}^{(1)}(\kappa) B_{-1}(\kappa). \tag{41}$$

Here, $\phi_0^{(1)}$ and $\Phi_{-1}^{(1)}$ are the first components of the vectors ϕ_0 and Φ_{-1}, respectively.

Putting (17) and (24) in (40) and putting (16) and (30) in (41) we obtain:

$$I^{ex} = O(\varepsilon^{1/2}), \quad I^{in} \sim -2\beta^{(2)}. \tag{42}$$

This get formula for the leading term of $q^\varepsilon(z,t)$:

$$q^\varepsilon(z,t) \sim \frac{4i\pi a_1 \exp\{is_0\}}{\left(|a_1|^2 + |a_2 - 2\pi\left(2T\overline{Q_2} - il\right)|^2\right)}. \tag{43}$$

For definition of the values of a_1 and a_2 we take into account that $q^\varepsilon(z,0) \sim q_0(z)$. This gives $a_1 = -i2\pi\overline{\nu_0}$, and $a_2 = i2\pi\mu_0$. This formulas and (9) imply (4).

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