Matematicheskaya fizika, analiz, geometriya
2002, v. 9, No. 2, p. 285–293

Interaction between non-uniform flows in a gas of rough spheres

V.D. Gordevskyy

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Square, Kharkov, 61077, Ukraine
E-mail:vajra@vl.kharkov.ua

Yu.A. Sysoyeva

Department of Computer Sciences, Kharkov State University of Economics
6a Lenin Ave., Kharkov, 61077, Ukraine

Received September 29, 2001
Communicated by E.Ya. Khruslov

For the model of rough spheres the bimodal distribution with inhomogeneous but stationary Maxwellians is considered. It approximately describes the interaction between two flows, which can rotate as a rigid body about the immovable axes. Conditions ensured the infinitesimality of the uniform-integral remainder for the accordant Boltzmann equation are obtained.

The model of rough spheres takes an important place among many models used for the description of the behaviour of molecules in the kinetic theory of a gas. It successfully combines comparative simplicity and sufficient physical verisimilitude [1]. Bryan [2] was the first who had taken into consideration this model; some later Pidduck [3] carried out its investigations. Recently, the works devoted to this model and some of its generalizations appeared again [4–6]. The description of the interaction between uniform flows in a gas of rough spheres was proposed in [7, 8]. The accordant explicit approximate solution of the Boltzmann equation has a form of a linear combination of two global Maxwellians with zero mass angular velocities but arbitrary mass linear velocities.

The aim of the present paper is the construction of bimodal distributions for the description of the interaction between two non-uniform flows in a gas of rough spheres. Now they must include local Maxwellians of a special type, the so-called spirals, which correspond to stationary equilibrium states of a gas [1] (the analogous approximate solutioms in case of a gas of hard spheres was studied in [9, 10]). The exact statement of the problem is the following.

Mathematics Subject Classification 2000: 76P05 (Primary), 82B40, 82C40 (Secondary).

We consider the Boltzmann equation for rough spheres [1, 6–8]:

$$D(f) = Q(f,f), \tag{1}$$

$$D(f) = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}, \tag{2}$$

$$Q(f,f) = d^2 \int\limits_{R^3} dv_1 \int\limits_{R^3} d\omega_1 \int\limits_{\Sigma} d\alpha h((v_1 - v, \alpha))$$

$$\times [f(t,x,v^*,\omega^*)f(t,x,v_1^*,\omega_1^*) - f(t,x,v,\omega)f(t,x,v_1,\omega_1)]. \tag{3}$$

Here d is a diameter of the molequle, which is connected with its moment of inertia I by the relation:

$$I = \frac{bd^2}{4}, \tag{4}$$

where the constant $b \in [0; 2/3]$ depends on the inside structure of the molecule; $f(t,x,v,\omega)$ — the distribution function we want to find; t — the time; $x \in R^3$ — the position of a molecule; $v;\omega \in R^3$ — its linear and angular velocities, respectively; $\frac{\partial f}{\partial x}$ — the spatial gradient of the function f; Σ — the unit sphere in R^3; $\alpha \in \Sigma$; the function h is of the form

$$h(u) = \frac{1}{2}(u + |u|), \tag{5}$$

and linear $(v^*; v_1^*)$ and angular $(\omega^*; \omega_1^*)$ velocities of two molecules before the collision are expressed in terms of accordant values v, v_1, ω, ω_1 after the collision by the formulae:

$$v^* = v - \frac{1}{b+1}\{b(v - v_1) + \alpha(v - v_1, \alpha) + \frac{1}{2}bd[\alpha \times (\omega + \omega_1)]\},$$

$$v_1^* = v_1 + \frac{1}{b+1}\{b(v - v_1) + \alpha(v - v_1, \alpha) + \frac{1}{2}bd[\alpha \times (\omega + \omega_1)]\},$$

$$\omega^* = \omega + \frac{2}{d(b+1)}\{[\alpha \times (v - v_1)] + \frac{1}{2}d[\alpha(\alpha, \omega + \omega_1) - \omega - \omega_1]\},$$

$$\omega_1^* = \omega_1 + \frac{2}{d(b+1)}\{[\alpha \times (v - v_1)] + \frac{1}{2}d[\alpha(\alpha, \omega + \omega_1) - \omega - \omega_1]\}. \tag{6}$$

The only class of the exact solutions of the equations (1)–(6) is known by now, namely, the Maxwellians:

$$M(t,x,v,\omega) = \rho I^{3/2} (\frac{\beta}{\pi})^3 e^{-\beta((v-\bar{v})^2 + I(\omega-\bar{\omega})^2)}, \tag{7}$$

where the hydrodynamical parameters ρ (the density), $\beta = \frac{1}{2T}$ (the inverse temperature), $\bar{v}$ (the mass linear velocity of molecules) and $\bar{\omega}$ (their mass angular velocity) can, generally, depend on t and x in a special way [1]. If $\bar{\omega} = 0$, then $\rho, \beta, \bar{v}$ are constant (i.e., $M = M(v,\omega)$ is the global Maxwellian considered in [7, 8]). Now we suppose, that $\bar{\omega} \neq 0$ is an arbitrary constant, then as it is shown in [1], the local Maxwellian (7) has a following form (stationary, inhomogeneous, equilibrium state of a gas of rough spheres, analogous to spiral from [9, 10]):

$$M(x,v,\omega) = \rho e^{\beta[\bar{\omega}\times(x-x_0)]^2} I^{3/2} (\frac{\beta}{\pi})^3 e^{-\beta[(v-\tilde{v})^2 + I(\omega-\bar{\omega})^2]}, \tag{8}$$

$$\tilde{v} = \tilde{v}(x) = \bar{v} + [\bar{\omega} \times x], \tag{9}$$

$$x_0 = \frac{1}{\bar{\omega}^2} [\bar{\omega} \times \bar{v}]. \tag{10}$$

From the physical point of view, the distribution (8)–(10) describes the rotation of a gas in whole (as a rigid body) with the angular velocity $\bar{\omega}$ about the axis which pass through the point x_0, and, besides that, its translational movement along this axis (the density of a gas now depends on x, too).

Let us seek an explicit approximate solution of equations (1)–(6) in a form of bimodal distribution:

$$f = \sum_{i=1}^{2} \varphi_i(t,x) M_i, \tag{11}$$

where M_i, $i = 1,2$, are of the form (8)–(10) with arbitrary corresponding values of parameters $\rho_i, \beta_i, \bar{v}_i, \bar{\omega}_i, x_{0i}$, $i = 1,2$, and φ_i, $i = 1,2$, are some non-negative coefficient functions from $C^1(R^1 \times R^3)$.

The problem is: to find $\varphi_i, i = 1,2$ such, that the following uniform-integral remainder [7, 8]

$$\Delta = \sup_{(t,x)\in R^1\times R^3} \int_{R^3} dv \int_{R^3} d\omega \, |D(f) - Q(f,f)| \tag{12}$$

tends to zero with corresponding behaviour of all parameters of the distribution.

For more convenient formulation of main results let us accept such a definition.

Definition. *We will write:*

$$Lim\Delta = 0, \tag{13}$$

if it exists such a value Δ', *that*

$$\Delta \leq \Delta' \tag{14}$$

and

$$\lim_{\beta_i \to +\infty, i=1,2} \Delta' \longrightarrow 0 \tag{15}$$

with some choice of the functions $\varphi_i, i = 1,2$ *and special behaviour of all parameters, except* $\beta_i, i = 1,2$.

Theorem 1. *Let*

$$\varphi_i(t,x) = \psi_i(t,x)e^{-\beta_i[\bar{\omega_i}\times(x-x_{0i})]^2},\ i = 1,2, \tag{16}$$

where the functions ψ_i *are independent of* β_i, $i = 1,2$, *and the following expressions are bounded on* $R^1 \times R^3$*:*

$$\psi_i;\ \frac{\partial\psi_i}{\partial t};\ \left|\frac{\partial\psi_i}{\partial x}\right|;\ |[\bar{\omega_{0i}} \times x]\psi_i|\,;\ \left([\bar{\omega_{0i}} \times x], \frac{\partial\psi_i}{\partial x}\right), \tag{17}$$

where $\bar{\omega}_{0i} \in R^3$ *— arbitrary fixed vectors, and*

$$\bar{\omega_i} = \bar{\omega_{0i}} s_i \beta_i^{-m_i}, \tag{18}$$

where $s_i, m_i > 0$ *— any constants,* $i = 1,2$.

Then the statement (13) holds true in the following situations:

I. $$m_i > \frac{1}{2},\ i = 1,2, \tag{19}$$

and at least one of the suppositions is valid:

1) $$\bar{v_1} = \bar{v_2} = 0, \tag{20}$$

$$\psi_i = \psi_i(x). \tag{21}$$

2) $$\bar{v_1} = \bar{v_2} \neq 0, \tag{22}$$

$$[\bar{\omega_{0i}} \times \bar{v_i}] = 0, \tag{23}$$

$$\psi_i = C_i([x \times \bar{v_i}]), \tag{24}$$

or

$$\psi_i = C_i(x - \bar{v_i}t), \tag{25}$$

where $C_i \geq 0$ *are arbitrary finite or fast decreasing functions on the specified vector arguments.*

3) $$\bar{v_1} = 0, \;\; \bar{v_2^1} \neq 0, \tag{26}$$

$$\bar{v_2} \parallel \bar{\omega_1} \parallel \bar{\omega_2}, \tag{27}$$

$$d \to 0, \tag{28}$$

$$\lambda\psi_1 + \mu\psi_2 = g([x \times \bar{v_2}]), \tag{29}$$

$$\psi_1 = g([x \times \bar{v_2}])\{\lambda + C([x \times \bar{v_2}])$$

$$\times \exp\left[-\pi d^2 \left|\bar{v_2}\right| g([x \times \bar{v_2}])\left(\frac{x^1}{\bar{v_2}^1}\left(\frac{\rho_2}{\mu} + \frac{\rho_1}{\lambda}\right) - \frac{\rho_2}{\mu}t\right)\right]\}^{-1}, \tag{30}$$

where $\lambda, \mu > 0$ *— arbitrary constants, and the functions* g, C *have the same properties, as* $C_i, i = 1, 2$ *in (24) or (25).*

4) $$\bar{v_1} \neq 0; \; \bar{v_2} \neq 0; \; \bar{v_1} \neq \bar{v_2}, \tag{31}$$

ψ_i *has a form of (25), and*

$$supp\psi_1 \cap supp\psi_2 = \emptyset, \tag{32}$$

and (23) is satisfied.

5) ψ_i *have a form of (24) or (25), and the requirements (31),(28) and (23) are satisfied.*

II. $$m_i = \frac{1}{2}, \; i = 1, 2, \tag{33}$$

one of the suppositions 1)–5) *of the point I holds true, and, in addition to that,*

$$s_i \to 0, \; i = 1, 2, \tag{34}$$

or

$$s_i \to 0;\ [\bar{\omega_{0j}} \times \bar{v_j}] = 0,\ i \neq j, \tag{35}$$

or (23) is fulfilled.

III. *(23) holds true together with one of the suppositions* 1)–5) *of the point I, and*

$$m_i \in (\frac{1}{4}, \frac{1}{2}),\ i = 1, 2, \tag{36}$$

or

$$m_i = \frac{1}{4},\ i = 1, 2 \tag{37}$$

and (34) is fulfilled.

P r o o f. After the substitution of (11) into (1)–(3), with taking into account of the technique developed in [8, 9], the integral from (12) can be estimated from above in such a way:

$$\int\limits_{R^3} dv \int\limits_{R^3} d\omega\, |D(f) - Q(f, f)| \le I^{3/2} \sum_{i=1}^{2} \rho_i \pi^{-3} \beta_i^{3/2} \int\limits_{R^3} du \int\limits_{R^3} d\omega$$

$$\times e^{-u^2 - \beta_i I(\omega - \bar{\omega_i})^2} \left[A_i(u, t, x) + \left| \frac{\partial \psi_i}{\partial t} + A_i(u, t, x) + B_i(u, t, x) \right| \right], \tag{38}$$

where

$$A_i = \psi_1 \psi_2 \rho_j \frac{d^2}{\sqrt{\pi}} \int\limits_{R^3} dw e^{-w^2} \left| \frac{u}{\sqrt{\beta_i}} + \bar{v_i} - \bar{v_j} + [(\bar{\omega_i} - \bar{\omega_j}) \times x] - \frac{w}{\sqrt{\beta_j}} \right|, \tag{39}$$

$$B_i(u, t, x) = \frac{\partial \psi_i}{\partial x} \left(\frac{u}{\sqrt{\beta_i}} + \bar{v_i} + [\bar{\omega_i} \times x] \right)$$

$$+ 2\psi_i \sqrt{\beta_i} \{ -[\bar{\omega_i} \times u] \cdot [\bar{\omega_i} \times x] + (u, [\bar{\omega_i} \times \bar{v_i}]) \}. \tag{40}$$

Next, the value Δ' for (14) we will obtain, if take supremums with respect to t, x of both summands under the integral sign in the right-hand side of (38) (note, that the existence of such Δ' ensured by (17), (39), (40)).

Now, integration in the expression for Δ' with respect to ω yields:

$$\Delta' = \sum_{i=1}^{2} \rho_i \pi^{-3/2} \int\limits_{R^3} du\, e^{-u^2} \Bigg[\sup_{(t,x) \in R^1 \times R^3} A_i(u, t, x)$$

$$+\sup_{(t,x)\in R^1\times R^3}\left|\frac{\partial\psi_i}{\partial t}+A_i(u,t,x)+B_i(u,t,x)\right|\Bigg]. \tag{41}$$

If we substitute (18) into (39), (40) and use the Lemma 1 from [9], we can pass to the limit in (41). The result will be different:

$$\lim_{\beta_i\to+\infty, i=1,2}\Delta' = \left[\begin{array}{lll} L, & if & (19),\ or\ (36)\ and\ (23), \\ L+K, & if & (33), \\ L+N, & if & (37)\ and\ (23), \end{array}\right. \tag{42}$$

where:

$$L=\sum_{i,j=1,i\neq j}^{2}\rho_i\sup_{(t,x)\in R^1\times R^3}\left|\frac{\partial\psi_i}{\partial t}+\bar{v_i}\frac{\partial\psi_i}{\partial x}+\rho_j\pi d^2\psi_1\psi_2\left|\bar{v_i}-\bar{v_j}\right|\right|$$

$$+2\pi d^2\rho_1\rho_2\left|\bar{v_1}-\bar{v_2}\right|\sup_{(t,x)\in R^1\times R^3}(\psi_1\psi_2), \tag{43}$$

$$K=\frac{4}{\sqrt{\pi}}\sum_{i=1}^{2}\rho_i s_i\left|[\bar{\omega_{0i}}\times\bar{v_i}]\right|\sup_{(t,x)\in R^1\times R^3}\psi_i, \tag{44}$$

$$N=\frac{4}{\sqrt{\pi}}\sum_{i=1}^{2}\rho_i s_i^2\left|\bar{\omega_{0i}}\right|\sup_{(t,x)\in R^1\times R^3}([\bar{\omega_{0i}}\times x]\psi_i). \tag{45}$$

Finally, from (43)–(45) it can be checked directly, that the expression (42) tends to zero (i.e., (15) holds true), if one of the suppositions of points I–III of the Theorem in each accordant case is fulfilled, so, we obtain (13).The Theorem is proved.

R e m a r k 1. The function of a form (24) or (25) can satisfy the conditions (17) only if (23) holds true. Hence, the requirement (32) contradict (24) but not (25) because (23) means that $x_{0i}=0$ (see (10)), i.e., the axes of spirals intersect with each other, so in the stationary case (32) and (23) cannot fulfilled together.

R e m a r k 2. The pair of functions (29),(30) was described in [9]; it is one of the non-trivial solutions of the system:

$$\frac{\partial\psi_i}{\partial t}+\bar{v_i}\frac{\partial\psi_i}{\partial x}=-\rho_j\pi d^2\psi_1\psi_2\left|\bar{v_2}\right|,\ i,j=1,2;\ i\neq j, \tag{46}$$

and, at the same time, satisfies the condition (17), if (26),(27) are true.

Theorem 2. *Let the distribution f be of the form (11), where the functions $\varphi_i, i = 1,2$, are independent of $\beta_i, i = 1,2$, and the products of the expressions*

$$e^{\beta_i[\bar{\omega_i}\times(x-x_{0i})]^2} \tag{47}$$

for each $i = 1,2$ on the values

$$\varphi_i;\ \frac{\partial\varphi_i}{\partial t};\ \left|\frac{\partial\varphi_i}{\partial x}\right|;\ \varphi_i\left|[\bar{\omega_{0i}}\times x]\right|;\ \left|\frac{\partial\varphi_i}{\partial x}[\bar{\omega_{0i}}\times x]\right|;\ \varphi_i[\bar{\omega_{0i}}\times x]^2 \tag{48}$$

are bounded on $R^1\times R^3$ when $\beta_i\to+\infty, i = 1,2$, and the conditions (18),(23) and (19) or (33) are fulfilled. Let the functions

$$\xi_i = \varphi_i\exp\{s_i^2[\bar{\omega_{0i}}\times x]^2(1 - sign(m_i - \frac{1}{2}))\} \tag{49}$$

satisfy one of the suppositions 1)–5) of the point I of the Theorem 1 with the substitution of ξ_i instead of ψ_i. Then the statement (13) holds true.

P r o o f. If we substitute (11) into (1)–(3), transform and estimate the integral from (12) in the manner, analogous to (38)–(41), and introduce the value Δ', like in the proof of the Theorem 1 (its existense follows from (47), (48)), with taking into account, that, because of (10), (18), (23) and (19) or (33),

$$\lim_{\beta_i\to+\infty, i=1,2} e^{\beta_i[\bar{\omega_i}\times(x-x_{0i})]^2} = \exp\{s_i^2[\bar{\omega_{0i}}\times x]^2(1 - sign(m_i - \frac{1}{2}))\}, \tag{50}$$

then after the limiting passage we will obtain the expression analogous to (43), which will be include the functions ξ_i and $\varphi_i, i = 1,2$. After the utilising of the Lemma 2 from [9] this expression reduces to (43) with ξ_i instead of $\psi_i, i = 1,2$, so, the last supposition of the Theorem 2 leads to (13) in the same way as in the proof of the point I of the Theorem 1 (note, that ξ_i satisfy (17) too because of (47)–(50)). The Theorem is prooved.

R e m a r k 3. In the Theorems 1, 2 we had put the values $\bar{v_i}, i = 1,2$, are fixed. The next theorem is one of the possible results deals with the case, when $\bar{\omega_i}, \bar{v_i}, i = 1,2$, tend to zero concordantly, i.e., by some "trajectory" in the space of parameters.

Theorem 3. *Let (16)–(18) with (37) are valid, and, in addition to that,*

$$\bar{v_i} = \sigma_i\bar{v_{0i}}\beta_i^{-1/4},\ i = 1,2, \tag{51}$$

where $\sigma_i \geq 0; \bar{v_{0i}} \in R^3$ are arbitrary and fixed. Then (13) holds true, if the suppositions (21) and (34) are fulfilled.

P r o o f is analogous to one of the Theorem 1 up to (39)–(41), but now because of (18) with (37) and (51) the limiting passage gives some other result, than (42) (the Lemma 1 from [9] is used once more):

$$\lim_{\beta_i\to+\infty, i=1,2} \Delta' = \sum_{i=1}^{2} \rho_i \left\{ \sup_{(t,x)\in R^1\times R^3} \left|\frac{\partial\psi_i}{\partial t}\right| \right. \tag{52}$$
$$\left. + \frac{4s_i}{\sqrt{\pi}} [s_i\, |\bar{\omega_{0i}}| \sup_{(t,x)\in R^1\times R^3} (|[\bar{\omega_{0i}}\times x]|\,\psi_i) + \sigma_i\, |[\bar{\omega_{0i}}\times \bar{v_{0i}}]| \sup_{(t,x)\in R^1\times R^3} \psi_i] \right\}.$$

Evidently, (52) together with (21),(34) leads to (15). The Theorem 3 is proved.

R e m a r k 4. The physical sense of the results, obtained above, is analogous to one from the paper [9], where its detailed analysis was done. Therefore, we will only note here, that the process of interaction between the spiral flows of a form (8)–(10) in a gas of rough spheres can be described, in principle, in the same way, as for a gas of hard spheres, despite of the fact, that the Boltzmann equation (1)–(6) now is more complex, and some technical difficulties arise, as in [7, 8] too.

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