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An algorithm of nonlinear approximation by piecewise polynomials

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We present a newly-developed version of the Up-and-Down Algorithm (UDA) designed for nonlinear approximation by piecewise polynomials, and establish the order of approximation by this algorithm in weighted L_q-spaces.

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1. Introduction

The UDA was firstly developed by the authors for nonlinear approximation by compactly supported refinable functions, see the survey [B1] and the forthcoming paper [BK]. In this paper we present its version intended for approximation by piecewise polynomials. We believe that this modification of the UDA will have important applications to Numerical Analysis and deserves to be presented to experts in this field. On the other hand, an approximation theorem to be proved in the present paper has important applications to Approximation Theory. In particular, one can derive from it the corresponding optimal approximation results for functions from Besov spaces (see [B2]).

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The algorithm considered in this paper makes use of a collection $\mathcal{T} := \{\mathcal{T}_j;\ j \in \mathbb{Z}_+\}$ of subsequent subdivisions of measurable set $\Omega \subset \mathbb{R}^d$. This collection is equipped with the structure of *ordered tree.* The *input* of the algorithms consists of an integer $N \geq 1$ and a set function

$$F : \mathcal{T} \to X,$$

where X is a subspace of polynomials in $\mathbb{R}^d$. The *output* is a function

$$F_N : \mathcal{T} \to X$$

such that

$$\operatorname{supp} F_N := \{\omega \in \mathcal{T};\ F_N(\omega) \neq 0\} \leq 4N.$$

Using this we then introduce an approximation aggregate

$$T_N(F) := \sum_{\omega} F_N(\omega)\chi_\omega, \tag{1.1}$$

where χ_ω here and below stands for the characteristic function of $\omega \subset \mathbb{R}^d$.

If, in particular, $X := \mathcal{P}_{s,d}$, the space of polynomials of degree s in $\mathbb{R}^d$, the aggregate $T_N(F)$ becomes a piecewise polynomial of degree s with $4N$ "pieces". However, $\operatorname{supp} T_N(F)$ does not form a subdivision of Ω and therefore these pieces relate to subsets that may differ from subsets $\omega \in \mathcal{T}$.

We present the description of the algorithm in Section 2. In the next section, we apply the algorithm to establish a general approximation theorem for functions $f \in L_p(\Omega)$, $\ 0 < p < \infty$, presented in a form

$$f = \sum_{\omega \in \mathcal{T}} f\omega\chi_\omega \quad \text{(convergence in } L_p).$$

In this case the aforementioned function F is defined by $F(\omega) := f_\omega$, $\omega \in \mathcal{T}$, and we estimate the rate of approximation of f by $T_N(F)$ in a weighted L_q-norm, $p < q \leq \infty$. In the subsequent paper [B2], this theorem is applied to derive the corresponding approximation results for Besov spaces. The aggregate (1.1) in this case yields an optimal in order rate of approximation showing the efficiency of the algorithm.

2. The algorithm

2.1. Tree of subdivisions

We begin with the introduction of a *tree of subdivisions* $\mathcal{T}$. Let Ω be a subset of $\mathbb{R}^d$ with finite d-measure $|\Omega|$, and $\mathcal{T}$ is a collection of subsets of Ω. $\mathcal{T}$ is called a *tree of subdivisions* for Ω, if the following holds.

(i) Every $\omega', \omega'' \in \mathcal{T}$ either *nonoverlap*, i.e.,

$$|\omega' \cap \omega''| = 0,$$

or one of them is contained in the other. This condition introduces an ordered tree structure on $\mathcal{T}$. Actually, we regard subsets of $\mathcal{T}$ as *vertices* and connect $\omega', \omega'' \in \mathcal{T}$ by the *edge directed from* ω' *to* ω'' (written $\omega' \to \omega''$) if $\omega' \subset \omega''$ and there is no set of $\mathcal{T}$ situated between them different from ω' and ω''.

Assume that $\Omega \in \mathcal{T}$. Then $\mathcal{T}$ is an ordered tree with the *root* Ω. Hence each $\omega \in \mathcal{T}$ can be connected with Ω by a unique *array*. In other words, there is a collection $\{\omega_j : 1 \le j \le n\} \subset \mathcal{T}$ such that $\omega_1 \to \omega_2 \to \cdots \to \omega_n$ (i.e., this collection is an *array*), and $\omega_1 = \omega$ and $\omega_n = \Omega$. Because of uniqueness of this array one can correctly define a *height* $h : \mathcal{T} \to \mathbb{Z}_+$ letting

$$h(\omega) := (\operatorname{card} A) - 1, \tag{2.1}$$

where A is the array connecting ω and Ω. Specially, $h(\Omega) = 0$.

Set now for $j \in \mathbb{Z}_+$

$$\mathcal{T}_j := \{\omega \in \mathcal{T} : h(\omega) = j\}. \tag{2.2}$$

These form a *partition* of $\mathcal{T}$:

$$\mathcal{T}_j \cap \mathcal{T}_{j'} = \phi, \quad \text{if} \quad j \ne j' \quad \text{and} \quad \mathcal{T} = \bigcup_{j \in \mathbb{Z}_+} \mathcal{T}_j. \tag{2.3}$$

(ii) For every $j \in \mathbb{Z}_+$

$$\operatorname{supp} \mathcal{T}_j := \bigcup_{\omega \in \mathcal{T}} \omega = \Omega. \tag{2.4}$$

In other words, $\{\mathcal{T}_j\}$ is a sequence of *consequent subdivisions* of Ω.

To formulate the last condition one set

$$S(\omega) := \{\omega' \in \mathcal{T} : \omega' \to \omega\}. \tag{2.5}$$

In accordance with the terminology of Graph Theory, each element of this set is a *son* of ω (and ω is its *father*).

(iii) There is a constant $C(\mathcal{T})$ such that for every $\omega \in \mathcal{T}$

$$1 < \operatorname{card} S(\omega) \le C(\mathcal{T}). \tag{2.6}$$

Definition 2.1. *A collection $\mathcal{T}$ of subsets of Ω is said to be a tree of subdivision, if it meets the conditions (i)–(iii).*

2.2. Weighted ℓ_p-spaces on $\mathcal{T}$.

Let $w : \mathcal{T} \to \mathbb{R}_+$ be a weight, and $0 < p \leq \infty$. Introduce a space $\ell_p^w(\mathcal{T}; X)$ of functions $F : \mathcal{T} \to \mathbb{R}$ defined by the quasinorm

$$\|F\|_{p,w} := \left\{ \sum_{\omega \in \mathcal{T}} \left(w(\omega) \sup_{\omega} |F(\omega)| \right)^p \right\}^{\frac{1}{p}}. \tag{2.7}$$

Note that the ω-term of this sum with unbounded ω is finite, only if the polynomial $F(\omega)$ is constant (or $w(\omega) = 0$). To avoid unnecessary complications we *assume that Ω is bounded.*

The input of the algorithm comprises a fixed $F \in \ell_p^w(\mathcal{T}; X)$ and integer $N \geq 1$. Because of homogeneity of (2.7) we can and do assume that

$$\|F\|_{p,w} = 1. \tag{2.8}$$

2.3. Basic arrays

Given F, we introduce a *cost function* $\mathcal{I} : 2^{\mathcal{T}} \longrightarrow \mathbb{R}_+$ by

$$\mathcal{I}(S) = \mathcal{I}(F; S) := \left\{ \sum_{\omega \in S} \left(w(\omega) \sup_{\omega} |F(\omega)| \right)^p \right\}. \tag{2.9}$$

Specially, for the subset

$$\mathcal{T}(\omega) := \{\omega' \in \mathcal{T} : \omega' \subset \omega\} \tag{2.10}$$

we simplify this notation by setting

$$\mathcal{I}(\omega) := \mathcal{I}(\mathcal{T}(\omega)). \tag{2.11}$$

Note that $\mathcal{I}(\omega) \neq \mathcal{I}(\{\omega\}) := \big(w(\omega)|f(\omega)|\big)^p$, and

$$\mathcal{I}(\Omega) = 1, \tag{2.12}$$

see (2.7) and (2.8).

We first introduce the *subtree*

$$\mathcal{G}_N := \{\omega \in \mathcal{T} : \mathcal{I}(\omega) \geq N^{-1}\}. \tag{2.13}$$

$\mathcal{G}_N$ is nonempty and has the root Ω by (2.12). Since $\mathcal{T}$ is an ordered set, the set $\mathcal{M}_N$ of *minimal elements* of $\mathcal{G}_N$ is well-defined. Hence for each $\omega \in \mathcal{M}_N$ and every its son ω'

$$\mathcal{I}(\omega) \geq N^{-1}, \quad \text{while} \quad \mathcal{I}(\omega') < N^{-1}. \tag{2.14}$$

Numerate the elements of $\mathcal{M}_N$ in some order

$$\mathcal{M}_N := \left\{\omega_j^{\min} : 1 \leq j \leq m_N\right\}. \tag{2.15}$$

Since the subsets of $\mathcal{M}_N$ nonoverlap, we have

$$1 = \mathcal{I}(\Omega) \geq \sum_j \mathcal{I}\left(\omega_j^{\min}\right) \geq m_N/N,$$

whence

$$m_N \leq N. \tag{2.16}$$

Our goal is to partition $\mathcal{G}_N$ in order to obtain a collection of (*basic*) arrays $\mathcal{B}_N$. An algorithm fulfilling this operation is the main part of our construction. In its description we will use the notation

$$[\omega, \omega'] := \{\omega_1 \to \omega_2 \to \ldots \to \omega_n\} \tag{2.17}$$

for the array connecting $\omega(=\omega_1)$ and $\omega'(=\omega_n)$. We also introduce an "open from the top" subarray of this array setting

$$[\omega, \omega') := [\omega, \omega'] \backslash \{\omega'\}. \tag{2.18}$$

At the first stage we introduce a partition of $\mathcal{G}_N$ into a collection $\mathcal{A} := \{A_j : 0 \leq j \leq m_N\}$ of ("big") arrays satisfying the following conditions:

(a) $\{A_j : 0 \leq j \leq i\}$ is a *partition* of the set

$$\mathcal{G}_N^i := \bigcup_{s \leq i} [\omega_s^{\min}, \Omega], \quad 0 \leq i \leq m_N;$$

(b) each A_i has a form

$$A_i = [\omega_i^{\min}, \omega),$$

where ω belongs to some $A_{i'}$ with $i' < i$. This ω is called a *contact element* and is denoted by ω_i^c; hence

$$A_i = \left[\omega_i^{\min}, \omega_i^c\right), \quad 1 \le i \le m_N. \tag{2.19}$$

Since $\mathcal{G}_N^i = \mathcal{G}_N$, if $i = m_N$, the collection $\mathcal{A} = \{A_j : 0 \le j \le m_N\}$ forms the desired partition of the subtree $\mathcal{G}_N$. Besides, $\mathcal{A}$ determines the *set of contact elements*

$$C_N := \{\omega_i^c\} \cup \{\Omega\}. \tag{2.20}$$

As we shall see, some of these may coincide and therefore

$$\operatorname{card} C_N \le m_N + 1, \tag{2.21}$$

where the inequality may be strict.

In order to introduce $\mathcal{A}$ we use induction on j starting with

$$A_0 := \{\Omega\} \text{ and } A_1 := \left[\omega_1^{\min}, \Omega\right] \backslash A_0 = \left[\omega_1^{\min}, \Omega\right).$$

Assume now that we have determined the arrays A_i, $i = 0, 1, \ldots, j$, satisfying the conditions (a) and (b) with $i \le j$. Define

$$A_{j+1} := \left[\omega_{j+1}^{\min}, \Omega\right] \backslash \left(\bigcup_{i \le j} A_i\right).$$

Then $\{A_i : 0 \le i \le j+1\}$ is clearly a partition of $\mathcal{G}_N^{j+1}$. Show that A_{j+1} has a form (2.19). In fact, consider the intersection of $\left[\omega_{j+1}^{\min}, \Omega\right]$ with each $\left[\omega_i^{\min}, \Omega\right]$, $i \le j$. Since $\mathcal{G}_N$ is a tree with the root Ω, this intersection is of a form $[\omega_i, \Omega]$, and $\{\omega_i : 1 \le i \le j\}$ is a subset of the array $\left[\omega_{j+1}^{\min}, \Omega\right]$. Hence this subset inherits the linear order of the last array. If ω_{i_0} is the smallest element of $\{\omega_i\}$ with respect to this order, then

$$A_{j+1} := \left[\omega_{j+1}^{\min}, \Omega\right] \backslash \left(\bigcup_{i \le j} A_i\right) = \left[\omega_{j+1}^{\min}, \Omega\right] \backslash \left(\bigcup_{i \le j} \left[\omega_i^{\min}, \Omega\right]\right) = \left[\omega_{j+1}^{\min}, \omega_{i_0}\right).$$

Moreover, $\omega_{i_0} \in \bigcup_{i \le j} A_i$. Hence the induction is complete.

We proceed the refinement of $\mathcal{G}_N$ subdividing each array A_j by the elements of the set $A_j \cap C_N,\ j \ge 1$. In this way, we introduce a collection of "open from the top" subarrays $[\omega', \omega'')$ where ω' is either a minimal or contact element, and ω'' is a contact one. The set of these subarrays one denotes by $\mathcal{R}_N$. According to its definition

$$\operatorname{supp} \mathcal{R}_N := \bigcup_{R \in \mathcal{R}_N} R = \mathcal{G}_N \backslash \{\Omega\} \tag{2.22}$$

and different elements of $\mathcal{R}_N$ do not overlap, i.e., $\mathcal{R}_N$ is a partition of (2.22).

At the final stage we complete the partition algorithm subdividing each subarray $R \in \mathcal{R}_N$ into *"basic" arrays* as follows.

Let $\omega_-(R)$ and $\omega_+(R)$ be, respectively, the bottom and top endpoints of R, i.e.,

$$R = \big[\omega_-(R),\ \omega_+(R)\big]. \tag{2.23}$$

One defines inductively a collection $\{\omega_\ell^R : 1 \le \ell \le \ell^R\}$ beginning with $\omega_1^R := \omega_-(R)$. If ω_ℓ^R has been determined, we choose $\omega_{\ell+1}^R$ as an element from $\big(\omega_\ell^R, \omega_+(R)\big]$ satisfying the conditions

$$\mathcal{I}\left(\left[\omega_\ell^R, \omega_{\ell+1}^R\right]\right) \ge N^{-1} \ \text{ and } \ \mathcal{I}\big(\big[\omega_\ell^R, \omega_{\ell+1}^R\big)\big) < N^{-1},$$

and then set

$$B_\ell^R := \big[\omega_\ell^R,\ \omega_{\ell+1}^R\big). \tag{2.24}$$

This element may not exist in the next cases:

(a) $$\omega_\ell^R = \omega_+(R) \ \text{ or } \ \mathcal{I}\big(\big[\omega_\ell^R,\ \omega_+(R)\big]\big) < N^{-1}.$$

We define $\omega_{\ell+1}^R$ as the father of $\omega_+(R)$ and set

$$B_\ell^R := \big[\omega_\ell^R,\ \omega_{\ell+1}^R\big)\big(= \big[\omega_\ell^R,\ \omega_+(R)\big]\big).$$

(b) $$\omega_\ell^R \ne \omega_+(R) \text{ and } \mathcal{I}(\{\omega_\ell^R\}) \ge N^{-1}.$$

We define $\omega_{\ell+1}^R$ as the father of ω_ℓ^R and introduce B_ℓ^R by (2.24).
In this case $\omega_{\ell+1}^R \in R$, and the procedure can be continued. Note also that now B_ℓ^R consists of a single point, $B_\ell^R = \{\omega_\ell^R\}$.

Completing the procedure one obtains the partition $\{B_\ell^R : 1 \le \ell \le \ell^R\}$ of R into *the basic arrays* B_ℓ^R. By their definition

$$\mathcal{I}\left(B_\ell^R \backslash \{\omega_\ell^R\}\right) < N^{-1}. \tag{2.25}$$

Note that the argument in (2.25) is an empty set, if B_ℓ^R is a singleton. Besides, for $\ell < \ell^R$ and $\operatorname{card}(B_\ell^R) > 1$

$$\mathcal{I}(\hat{B}_\ell^R) \ge N^{-1}, \tag{2.26}$$

provided that $\hat{B}_\ell^R := B_\ell^R \cup \{\omega_{\ell+1}^R\}$, if B_ℓ^R is not a singleton, and $\hat{B}_\ell^R = B_\ell^R$, otherwise.

Collecting all the basic arrays for all $R \in \mathcal{R}_N$, we lastly obtain the desired set of the basic arrays

$$\mathcal{B}_N := \left\{B_\ell^R : 1 \le \ell \le \ell^R, \quad R \in \mathcal{R}_N\right\}.$$

Proposition 2.2. *(a)* $\mathcal{B}_N$ *is a partition of the set* $\mathcal{G}_N \backslash \{\Omega\}$.

(b) For each $B := \left[\omega_-(B),\ \omega_+(B)\right]$ *from* $\mathcal{B}_N$

$$\mathcal{I}\big(\big(\omega_-(B),\ \omega_+(B)\big]\big) < N^{-1}. \tag{2.27}$$

(c) It is true that

$$\operatorname{card} \mathcal{B}_N \le 4N + 1. \tag{2.28}$$

P r o o f. (a) follows from (2.22) and the definition of B_R^ℓ.

(b) follows from (2.25), since the argument in (2.27) is $B \backslash \{\omega_-(B)\}$.

(c) Using (2.26) and noting that the mutiplicity of the cover of R by $\{\hat{B}_\ell^R\}$ is at most 2, one has

$$N^{-1}(\ell_R - 1) \le \sum_{\ell=1}^{\ell_R - 1} \mathcal{I}\,(\hat{B}_\ell^R) < 2\,\mathcal{I}(R).$$

This implies, see (2.12),

$$\sum_{R \in \mathcal{R}_N} (\ell_R - 1) < 2N \sum_{R \in \mathcal{R}_N} \mathcal{I}\,(R) \le 2N\mathcal{I}(\mathcal{G}_N) \le 2N,$$

whence

$$\operatorname{card}(\mathcal{B}_N) = \sum_{R\in\mathcal{R}_N} \ell_R < 2N + \operatorname{card}(\mathcal{R}_N).$$

By the definition of $\mathcal{R}_N$

$$\operatorname{card}(\mathcal{R}_N) \le \operatorname{card}(C_N) + \operatorname{card}(M_N) \le 2N+1,$$

see (2.16) and (2.21).

Combining the last estimates we get (2.28). ■

2.4. The output of the algorithm

The output of the algorithm is a function F_N on $\mathcal{T}$ defined as follows.

If $\omega := \omega_-(B)$, the bottom endpoint of a *basic array* $B \in \mathcal{B}_N$, then

$$F_N(\omega) := \left(\sum_{\omega'\in B} F(\omega')\right)\chi_\omega. \tag{2.29}$$

We also let $F_N(\Omega) := G(\Omega)\chi_\Omega$.

For all other $\omega \in \mathcal{T}$ we let

$$F_N(\omega) := 0. \tag{2.30}$$

Hence $F_N(\omega)(x)$ is a polynomial from X, if $x \in \omega$, and

$$\operatorname{supp} F_N \subset \{\omega_-(B) : B \in \mathcal{B}_N\} \cup \{\Omega\}. \tag{2.31}$$

3. A general approximation theorem

Let $\mathcal{T}$ and X be defined as above. We introduce, first, a subspace of $L_p(\Omega)$, $0<p<\infty$, consisting of functions f that can be presented in a form

$$f = \sum_{\omega\in\mathcal{T}} f_\omega\chi_\omega \text{ (convergence in } L_p) \tag{3.1}$$

with suitable $f_\omega \in X$.

Then we define the space $B_p^w(\mathcal{T})$ by finiteness of the Banach norm (quasinorm, if $p<1$)

$$|f|_{B_p^w(\mathcal{T})} := \inf\left\{\sum_{\omega\in\mathcal{T}}\left(w(\omega)\sup_\omega|f_\omega|\right)^p\right\}^{\frac{1}{p}}, \tag{3.2}$$

where the infimum is taken over all decompositions (3.1).

Here $w : \mathcal{T} \to \mathbb{R}_+$ is a given weight.

Assume now that for some $p < q \leq \infty$ the following embedding*

$$B_p^w(\mathcal{T}) \subset L_q(d\mu) \tag{3.3}$$

holds with embedding constant C_{em}. Here μ is a Borel measure supported by Ω. Under this assumption the following is true.

Theorem 3.1. *Given $f \in B_p^w(\mathcal{T})$ and integer $N \geq 1$, there is an N-term linear combination*

$$T_N(f) := \sum_{\omega} f_\omega \chi_\omega$$

with suitable $f_\omega \in X$ and $\omega \in \mathcal{T}$ such that

$$\left\|f - T_N(f)\right\|_{L_q(d\mu)} \leq C N^{\frac{1}{q}-\frac{1}{p}} |f|_{B_p^w(\mathcal{T})}. \tag{3.4}$$

Besides,

$$\left\|T_N(f)\right\|_{L_q(d\mu)} \leq C|f|_{B_p^w(\mathcal{T})}. \tag{3.5}$$

Here the constant C depends only on C_{em}, $C(\mathcal{T})$, see (2.6), and $p^* := \min(1,p)$.

P r o o f. Assume that (3.1) is an ε-optimal decomposition for f, i.e.,

$$\left(\sum_{\omega\in\mathcal{T}} \left(w(\omega) \sup_{\omega} |f_\omega|\right)^p\right)^{\frac{1}{p}} \leq (1+\varepsilon)|f|_{B_p^w(\mathcal{T})}. \tag{3.6}$$

Without loss of generality we assume that

$$\sum_{\omega\in\mathcal{T}} \left(w(\omega) \sup_{\omega} |f_\omega|\right)^p = 1. \tag{3.7}$$

Define now a function $F : \mathcal{T} \to X$ by letting

$$F(\omega) := f_\omega, \quad \omega \in \mathcal{T}. \tag{3.8}$$

By (3.7), this F satisfies (2.8) and we take it and an integer $N \geq 1$ as the input of the algorithm. As the output we obtain the function F_N, see (2.29) and (2.30) for $F(\omega) := f_\omega$. In turn, F_N gives rise to required approximation aggregate

$$T_{4N+1}(f) := f_\Omega \chi_\Omega + \sum_{B\in\mathcal{B}_N} \left(\sum_{\omega\in B} f_\omega\right) \chi_{\omega_-(B)}. \tag{3.9}$$

* See [B2] for assumptions on $\mathcal{T}$ and w providing this embedding.

Let us show that (3.9) provides the desired rate of approximation to the function f in $L_q(d\mu)$. Set

$$\phi(S) := \sum_{\omega \in S} f_\omega \chi_\omega, \quad f \subset \mathcal{T}, \tag{3.10}$$

and simplify this notation for $S := \mathcal{T}(\omega)$, see (2.10), by setting

$$\phi(\omega) := \phi\big(\mathcal{T}(\omega)\big), \quad \omega \in \mathcal{T}. \tag{3.11}$$

Note that $\phi(\omega) \neq \phi(\{\omega\}) := f_\omega \chi_\omega$.

Proposition 2.2 and (3.10) imply

$$f - T_{4N+1}(f) = \sum_{B \in \mathcal{B}_N} \phi^*(B) + \phi\,(\mathcal{T} \backslash \mathcal{G}_N), \tag{3.12}$$

where we let

$$\phi^*(B) := \phi(B) - \left(\sum_{\omega \in B} f_\omega\right) \chi_{\omega_-(B)} = \sum_{\omega \in B} f_\omega \chi_{\omega \backslash \omega_-(B)}. \tag{3.13}$$

Applying to (3.12) the $L_q(d\mu)$-norm, we get for $C := \max(1, 2^{\frac{1}{q}-1})$

$$\big\| f - T_{4N+1}(f) \big\|_q \le C(J_1 + J_2), \tag{3.14}$$

where

$$J_1 := \left\| \sum_{B \in \mathcal{B}_N} \phi^*(B) \right\|_q, \quad J_2 := \big\| \phi(\mathcal{T} \backslash G_N) \big\|_q. \tag{3.15}$$

In order to obtain the required estimate for J_1, show that for different B, B' from $\mathcal{B}_N$

$$\big| \operatorname{supp} \phi^*(B) \cap \operatorname{supp} \phi^*(B') \big| = 0. \tag{3.16}$$

Let, first, their top endpoints $\omega_+(B)$ and $\omega_+(B')$ nonoverlap. Since by (3.13)

$$\operatorname{supp} \phi^*(B) \subset \omega_+(B) \backslash \omega_-(B) \tag{3.17}$$

and the similar is true for $\operatorname{supp} \phi^*(B')$, these supports nonoverlap.

In the remaining case the biggest set of one of them, say $\omega_+(B)$, embeds in the smallest set of the other $\omega_-(B')$. Hence $\operatorname{supp} \phi^*(B) \subset \omega_+(B) \subset \omega_-(B')$, while by (3.17) $\operatorname{supp} \phi^*(B') \subset \omega_+(B') \backslash \omega_-(B')$. Thus in this case (3.16) holds, as well.

Applying (3.16), we get

$$J_1 = \left\{ \sum_{B\in\mathcal{B}_N} \|\phi^*(B)\|_q^q \right\}^{\frac{1}{q}}.$$

Using now embedding (3.3) and remembering the definition of the cost function $\mathcal{I}$, see (2.9), we have

$$\begin{aligned} \|\phi^*(B)\|_q &\le \left\| \sum_{\omega\in B\backslash\{\omega_-(B)\}} |f_\omega|\chi_\omega \right\|_q \\ &\le C_{em} \left\{ \sum_{\omega\in B\backslash\{\omega_-(B)\}} \left(w(\omega) \sup_\omega |f_\omega| \right)^p \right\}^{\frac{1}{p}} \\ &= C_{em}\mathcal{I}\big(B\backslash\{\omega_-(B)\}\big). \end{aligned}$$

Combining this and (2.27) and (2.28), we have

$$\begin{aligned} J_1 &\le C_{em} \left\{ \sum_{B\in\mathcal{B}_N} \mathcal{I}\big(B\backslash\{\omega_-(B)\}\big)^{-\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\le C_{em}N^{-\frac{1}{p}}\,\mathrm{card}\,(\mathcal{B}_N)^{\frac{1}{q}} \le 4^{\frac{1}{q}} C_{em} N^{\frac{1}{q}-\frac{1}{p}}. \end{aligned}$$

According to (3.6) and (3.7) this can be rewritten as

$$J_1 \le 4^{\frac{1}{q}}(1+\varepsilon)^{-1}\, C_{em} N^{\frac{1}{q}-\frac{1}{p}}. \tag{3.18}$$

To carry out the similar estimate for J_2 we introduce a *collection* $\{H_j\}$ *of subsets of the set*

$$T_0 := \mathcal{T}\backslash\mathcal{G}_N \tag{3.19}$$

which meets the following conditions.

(a) For every j

$$\mathcal{I}(H_j) < \frac{C(\mathcal{T})}{N}. \tag{3.20}$$

(b) It is true that

$$\mathrm{card}\,(\{H_j\}) \le N+1. \tag{3.21}$$

(c) $\{H_j\}$ is a *partition* of $T_0(:= \mathcal{T}\backslash G_N)$.

We introduce the required collection by induction. In this part of proof we use the following notation: for every $T \subset \mathcal{T}$ and $\omega \in \mathcal{T}$

$$T(\omega) := \{\omega' \in T : \omega' \subset \omega\}.$$

We begin with the set

$$\{\omega \in \mathcal{T} : \mathcal{I}\big(T_0(\omega)\big) \geq N^{-1}\}.$$

Since $\mathcal{I}\big(T_0(\omega)\big) \leq \mathcal{I}(\omega) \to 0$ as $|\omega| \to 0$, see (2.11) and (2.12), this set is either empty or finite. In the former case we obtain the desired (trivial) partition putting $H_1 := T_0$. Then $\mathcal{I}(H_1) = \mathcal{I}\big(T_0(\Omega)\big) < N^{-1}$, and (3.20) is true. Otherwise, T_0 contains an element ω_1 of minimal measure. Since for each $\omega \in T_0$

$$\mathcal{I}\big(T_0(\omega)\big) \leq \mathcal{I}(\omega) < N^{-1},$$

this $\omega_1 \notin T_0$. Hence we have the disjoint decomposition of $T_0(\omega_1)$:

$$T_0(\omega_1) = \bigcup_{\omega \in S(\omega_1)} T_0(\omega);$$

recall that $S(\omega_1)$ is the set of the sons of ω_1, see (2.5). Besides, minimality of ω_1, implies for each $\omega \in S(\omega_1)$,

$$\mathcal{I}\big(T_0(\omega)\big) < N^{-1}.$$

Hence it is true that

$$\mathcal{I}\big(T_0(\omega_1)\big) = \sum_{\omega \in S(\omega_1)} \mathcal{I}\big(T_0(\omega)\big) < \frac{\operatorname{card}\,(S(\omega_1))}{N} \leq \frac{C(\mathcal{T})}{N},$$

see (2.6). Introduce now H_1 by

$$H_1 := T_0(\omega_1).$$

Then H_1 satisfies (3.20). To introduce the next set we put $T_1 := T_0\backslash H_1$ and consider the set

$$\{\omega \in \mathcal{T} : T_1(\omega) \geq N^{-1}\}.$$

If it is empty, put $H_2 := T_1$ to obtain the desired partition $\{H_1, H_2\}$ of T_0. Otherwise, this set contains an element ω_2 of minimal measure. As before $\omega_2 \notin T_0(:= \mathcal{T}\backslash G_N)$ and therefore

$$\mathcal{I}\big(T_1(\omega_2)\big) < \frac{C(\mathcal{T})}{N}.$$

Letting $H_2 := T_1(\omega_2)$, we obtain the desired subset satisfying (3.20) and not intersecting H_1. Besides,

$$\mathcal{I}(H_i) := \mathcal{I}\big(T_{i-1}(\omega_i)\big) \geq N^{-1}, \quad i = 1, 2.$$

Proceeding in this way, we lastly obtain the partition $\{H_j : 1 \leq j \leq n+1\}$ of T_0 satisfying the condition (3.20). Besides, $H_i := T_{i-1}(\omega_i)$, $1 \leq i \leq n$, and therefore

$$\mathcal{I}(H_i) \geq \frac{1}{N}, \quad 1 \leq i \leq n.$$

This implies

$$\frac{n}{N} \leq \sum_{i=1}^{n} \mathcal{I}(H_i) \leq \mathcal{I}(F_0) \leq \mathcal{I}(\Omega) = 1,$$

and the condition (3.21) holds as well.

Using now the partition introduced, we estimate J_2 as follows. By the definition of H_j their supports do not overlap:

$$\big|(\operatorname{supp} H_j) \cap (\operatorname{supp} H_{j'})\big| = 0, \quad j \neq j'.$$

Recall that $\operatorname{supp} H := \bigcup_{\omega \in H} \omega$, $H \subset \mathcal{T}$. Besides, $\operatorname{supp} H_j = \operatorname{supp} \phi(H_j)$, see (3.10). Hence

$$J_2 := \big\|\phi(\mathcal{T} \backslash \mathcal{G}_N)\big\|_q = \Big\| \sum_{j \leq n+1} \phi(H_j) \Big\|_q = \left\{ \sum_{j \leq n+1} \|\phi(H_j)\|_q^q \right\}^{\frac{1}{q}}.$$

By the embedding (3.3) and the inequality (3.20), and the definitions (2.9) and (3.2) of, respectively, $\mathcal{I}$ and the quasinorm of $B_p^w(\mathcal{T})$ we then have

$$\|\phi(H_j)\|_q \leq C_{em} \mathcal{I}(H_j)^{\frac{1}{p}} < C_{em} C(\mathcal{T})^{\frac{1}{p}} N^{-\frac{1}{p}}.$$

Together with the previous identity and (3.21) this yields

$$J_2 \leq C_{em} C(\mathcal{T})^{\frac{1}{p}} N^{-\frac{1}{p}} (n+1)^{\frac{1}{q}} \leq 2^{\frac{1}{q}} C_{em} C(\mathcal{T})^{\frac{1}{p}} N^{\frac{1}{q} - \frac{1}{p}}.$$

Combining this with (3.6), (3.17) and (3.14), we get the inequality

$$\big\| f - T_{4N+1}(f) \big\|_q \leq C N^{\frac{1}{p} - \frac{1}{q}} |f|_{B_p^w(\mathcal{T})}.$$

This clearly implies the required assertion (3.4).

It remains to establish the second assertion of the theorem, see (3.5). By (3.9) and Proposition 2.2

$$\left\|T_{4N+1}(f)\right\|_q = \left\| f_\Omega \chi_\Omega + \sum_{B \in \mathcal{B}_N} \left(\sum_{\omega \in B} f_\omega \right) \chi_{\omega_-(B)} \right\|_q \leq \left\| \sum_{\omega \in \mathcal{G}_N} |f_\omega| \chi_\omega \right\|_q .$$

Estimating the right hand side by the embedding inequality in (3.3) and then making use of the inequality (3.6) we have

$$\left\|T_{4N+1}(f)\right\|_q \leq C_{em} \left\{ \sum_{\omega \in \mathcal{G}_N} \left(w(\omega) \sup_\omega |f_\omega| \right)^p \right\}^{\frac{1}{p}} \leq C_{em}(1+\varepsilon)^{-1} |f|_{B_p^w(\mathcal{T})} .$$

The proof of Theorem 3.1 is completed. ∎

References

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