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The Stokes structure in asymptotic analysis II: from differential equation to Stokes structure

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We present a method of direct derivation of the *Stokes structure* $\mathfrak{S}$ from a differential equation. We introduce and revise the related important definitions and statements using the Weber's differential equation as an example. Our technique presented in this paper will be extended later to matrix differential equations.

In honor of the 100th birthday of Naum Il'ich Akhiezer

The Stokes structure $\mathfrak{S}(2) = \mathfrak{S}\{p_1(z), p_2(z)\}$ generated by $z^\beta e^{\alpha z}$, with given constants α, β, has been defined in [2] as a pair of functions $\{p_1(z), p_2(z)\}$

(i) analytic on the Riemann surface of $\log z$ with at most exponential growth at ∞,

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(ii) bounded in closed sub-sectors of

$$S_\alpha(1) = \{z : -\frac{3\pi}{2} - \arg\alpha < \arg z < \frac{3\pi}{2} - \arg\alpha,\ 0 < |z| < \infty\}, \quad (1)$$

$$S_\alpha(2) = \{z :\ -\frac{\pi}{2} - \arg\alpha < \arg z < \frac{5\pi}{2} - \arg\alpha,\ 0 < |z| < \infty\}, \quad (2)$$

(iii) satisfying the monodromic relations

$$\begin{aligned} p_1(ze^{2\pi i}) &= p_1(z) + T_1\, z^{\beta}\, e^{\alpha z}\, p_2(ze^{2\pi i}), \\ p_2(ze^{2\pi i}) &= p_2(z) + T_2 z^{-\beta} e^{-\alpha z} p_1(\, z\,) \end{aligned} \quad (3)$$

with complex constants T_1, T_2.

The phase amplitudes P_1, P_2 of the Hankel functions $H_\nu^{(1)}$, $H_\nu^{(2)}$ form a Stokes structure, with $\alpha = 2i, \beta = 0$, with monodromic relations

$$P_1(ze^{2\pi i}) = P_1(z) + T_1\, e^{2iz}\, P_2(ze^{2\pi i}), \quad (4)$$

$$P_2(ze^{2\pi i}) = P_2(z) + T_2 e^{-2iz} P_1\,(\, z\,) \quad (5)$$

and with coefficients T_1, T_2 to be calculated later. This Stokes structure denoted by $\mathfrak{S}_{\mathfrak{B}}(2)$ has been introduced firstly in [1], see, also [3].

The Stokes structures for the incomplete Gamma, Bessel's and reduced Weber's differential equations have been derived also in [2] using the integral representations of their solutions in terms of the corresponding Gauss Hypergeometric functions $F(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; \xi)$ and the monodromic properties of $F(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; \xi)$. However, such a derivation is not possible in general. Therefore, we present an alternative way.

Traditionally, the study of differential equations can be reduced to the study of the corresponding equivalent system of integral equations. It is easy to check, for example, that the phase amplitudes of Hankel functions satisfy the following pair of integral equations:

$$P_1(z) = 1 - \frac{b}{2i}\int_l \frac{P_1(w+z)}{(w+z)^2}\, dw\ + \frac{b}{2i}\int_l e^{2iw}\frac{P_1(w+z)}{(w+z)^2}\, dw\,, \quad (6)$$

$$P_2(z) = 1 + \frac{b}{2i}\int_l \frac{P_2(w+z)}{(w+z)^2}\, dw\ - \frac{b}{2i}\int_l e^{-2iw}\frac{P_2(w+z)}{(w+z)^2}\, dw\,, \quad (7)$$

with their paths of integration being a ray l, starting at the origin of the complex plane $\mathbb{C}$ or of the Riemann surface of $\log z$, and where $b = \nu^2 - 1/4$.

Using integral equations similar to (6), (7) one can study the asymptotic properties of solutions. Moreover, the monodromic relations (4), (5) can also be derived from (6), (7). However, we prefer to show another possibility which we illustrate by means of Weber's differential equation

$$u''(z) + (2E - z^2)\,u(z) = 0. \tag{8}$$

This equation, modelling the quantum harmonic oscillator, is (like Bessel's equation) a special case of Kummer's or Whittaker's equation, see [5]. We consider it separately in order to introduce the four-element Stokes structure.

It is known that any solution $u(z)$ of (8) is an entire function satisfying in the whole complex plane the following estimate

$$|u(z)| < M_\varepsilon e^{(1/2+\varepsilon)|z|^2} \tag{9}$$

for any $\varepsilon > 0$ with an appropriate choice of M_ε . The following fact is also well known.

Proposition 1. *Given a sector S of the complex plane or of the Riemann surface of $\log z$ with its apex at the origin and subtending an angle less than $\pi/2$, there exists a pair of solutions of (8) which can be represented as*

$$\begin{aligned} u_1(z) &= z^{E-\frac{1}{2}} e^{-\frac{z^2}{2}}(1+o(1)), \quad z \in S, \quad z \to \infty\,; \\ u_2(z) &= z^{-E-\frac{1}{2}} e^{\frac{z^2}{2}}(1+o(1)), \quad z \in S, \quad z \to \infty\,. \end{aligned} \tag{10}$$

This result can be derived from the corresponding system of integral equations similar to (6), (7), (see, for example, [4, 7]). The exponential factors $z^{E-\frac{1}{2}} e^{-\frac{z^2}{2}}$, $z^{-E-\frac{1}{2}} e^{\frac{z^2}{2}}$ in (10) are the so called *Green–Liouville approximants* for the corresponding solutions as $z \to \infty$. The branches of $z^{\mp E-\frac{1}{2}} = e^{(\mp E-\frac{1}{2})\log z}$ are chosen so that the value of $\log z$ is real when z belongs to the positive ray of the complex plane or of the Riemann surface of $\log z$. We refer also to [6, Theorem 12.1], for the matrix version of this result.

Our principal aim is to study the global behavior of the factors $1 + o(1)$ in (10). Therefore, we introduce a special notation for these factors

$$\begin{aligned} u_1(z) &= z^{E-\frac{1}{2}} e^{-\frac{z^2}{2}} P_1(z), \\ u_2(z) &= z^{-E-\frac{1}{2}} e^{\frac{z^2}{2}} P_2(z). \end{aligned} \tag{11}$$

Definition 1. *The functions $P_1(z), P_2(z)$ are said to be the phase amplitudes of $u_1(z), u_2(z)$, respectively.*

It follows from (11) that the functions $P_1(z), P_2(z)$

(i) are holomorphic on the Riemann surface of $\log z$,

(ii) satisfy the asymptotic relations

$$\begin{aligned} P_1(z) &= 1 + o(1), \quad z \in S, \quad z \to \infty\,; \\ P_2(z) &= 1 + o(1), \quad z \in S, \quad z \to \infty\,. \end{aligned} \tag{12}$$

Remark 1. *One should note that, although $u_1(z), u_2(z)$ are single valued (entire) functions in the whole complex plane $\mathbb{C}$, their phase-amplitudes $P_1(z), P_2(z)$ are multi-valued functions in $\mathbb{C}$ (for any non half-integer E). These functions can be regarded as single-valued on the Riemann surface of $\log z$. We are forced to factorize the single valued functions $u_1(z)$ and $u_2(z)$ into the product (11) of multi-valued factors because the leading terms of the asymptotic relations (10), the functions $z^{\pm E-1/2}e^{\mp z^2/2}$, are multi-valued functions (except for the case of half integer E).*

Remark 2. *The functions $P_1(z), P_2(z)$ are bounded not only in the initial sector S, they may also be bounded in other sectors. However, as a rule, the phase amplitude $P_j(z)$ are unbounded functions on the whole Riemann surface of $\log z$. It is worth noting that the boundedness of the phase amplitudes $P_j(z)$ in a sector does not involve the boundedness of the corresponding solutions $u_j(z)$ in the same sector.*

Let us examine the behavior of the leading term in (10), say the exponentials $e^{\pm\frac{z^2}{2}}$ in the z-plane. Introduce the following ray l_θ, in the Riemann surface of $\log z$:

$$l_\theta = \{z : 0 < |z| < \infty, \arg z = \theta\} \tag{13}$$

and consider the l_{θ_j}, $\theta_j = \frac{j\pi}{4}$, for all integers j. These rays separate the sectors in which the exponentials $e^{\pm\frac{z^2}{2}}$ are decaying as $z \to \infty$ from the sectors in which the same exponentials are growing as $z \to \infty$.

Definition 2. *The rays l_{θ_j}, $\theta_j = \frac{j\pi}{4}$, $j \in \mathbb{Z}$, are called the* separation rays *for the equation (8).*

These special rays are very important in asymptotic analysis.

Let us denote the sectors mentioned above by $s(j), j \in \mathbb{Z}$,

$$s\,(j) = \Big\{z : 0 < |z| < \infty,\ -\frac{\pi}{4} + (j-1)\frac{\pi}{2} < \arg z < \frac{\pi}{4} + (j-1)\frac{\pi}{2}\Big\}\,. \tag{14}$$

We will regard the four sectors

$$\begin{aligned} s\,(1) &= \{z : 0 < |z| < \infty, \quad -\tfrac{\pi}{4} < \arg z < \tfrac{\pi}{4}\}, \\ s\,(2) &= \{z : 0 < |z| < \infty, \quad \tfrac{\pi}{4} < \arg z < \tfrac{3\pi}{4}\}, \\ s\,(3) &= \{z : 0 < |z| < \infty, \quad \tfrac{3\pi}{4} < \arg z < \tfrac{5\pi}{4}\}, \\ s\,(4) &= \{z : 0 < |z| < \infty, \quad \tfrac{5\pi}{4} < \arg z < \tfrac{7\pi}{4}\} \end{aligned} \tag{15}$$

as the initial principal sectors. These sectors together with the four principal separation rays l_{θ_j}, $j = 1, 2, 3, 4$, cover the whole complex plane, except $z = 0$.

One can also derive from Proposition 1 the following result.

Proposition 2.

(i) *For each $s(j)$, $j \in \mathbb{Z}$, there exists a solution $u^{(j)}(z)$ decaying inside of $s(j)$. More precisely, there exist solutions $u^{(2k-1)}(z)$, $u^{(2k)}(z)$, $k \in \mathbb{Z}$, such that*

$$\begin{aligned} u^{(2k-1)}(z) &= z^{E-\frac{1}{2}} e^{-\frac{z^2}{2}} P_{2k-1}(z), \\ u^{(2k)}(z) &= z^{-E-\frac{1}{2}} e^{\frac{z^2}{2}} P_{2k}(z), \end{aligned} \tag{16}$$

where the functions $P_j(z)$, $j \in \mathbb{Z}$, satisfy the relations

$$P_j(z) = 1 + o(1), z \to \infty, \tag{17}$$

in every closed sub-sector* of the sector $s(j)$.

(ii) *The solution of the equation (8), bounded in the sector $s(j)$, is essentially unique for if $v(z)$ is another solution of (8), bounded inside the sector $s(j)$, then $v(z)$ is proportional to $u^{(j)}(z)$: $v(z) = Cu^{(j)}(z)$ where C is a complex constant.*

Definition 3. *We will regard the solutions $u^{(j)}(z)$ of the equation (8) as normalized solutions. The four solutions $u^{(j)}(z)$, $j = 1, 2, 3, 4$, will be regarded as the principal normalized solutions of the equation.*

Further, the sector $s(j)$ is the *maximal* sector (in general) in which the solution $u^{(j)}(z)$ is bounded. The *maximality* means: if a sector s is wider than the sector $s(j)$, i.e., $s \supset s(j)$, $s \neq s(j)$, then the solution $u^{(j)}(z)$ is unbounded in s. Nevertheless, it is possible to obtain the precise behavior of $u^{(j)}(z)$ in some extended sector $S(j)$, $s(j) \subset S(j)$. By definition, for each j the sector $S(j)$ is the union of the closure of $s(j)$ and of a pair of its adjacent open sectors on the Riemann surface of $\log z$.

We introduce the four principal extended sectors

$$\begin{aligned} S(1) &= \left\{z : 0 < |z| < \infty, \ -\tfrac{3\pi}{4} < \arg z < \tfrac{3\pi}{4} \right\}, \\ S(2) &= \left\{z : 0 < |z| < \infty, \ -\tfrac{\pi}{4} < \arg z < \tfrac{5\pi}{4} \right\}, \\ S(3) &= \left\{z : 0 < |z| < \infty, \ \tfrac{\pi}{4} < \arg z < \tfrac{7\pi}{4} \right\}, \\ S(4) &= \left\{z : 0 < |z| < \infty, \ \tfrac{3\pi}{4} < \arg z < \tfrac{9\pi}{4}. \right\}. \end{aligned} \tag{18}$$

Note that $S(j)$ subtends an angle three times as large as that of $s(j)$.

* If S is a sector: $S = \{z : \alpha < \arg z < \beta, 0 < |z| < \infty\}$, then by definition the closed sub-sector of S is a sector S' of the form $S' = \{z : \alpha' < \arg z < \beta', 0 < |z| < \infty\}$ where $\alpha < \alpha' < \beta' < \beta$.

Proposition 3. *In every closed sub-sector of the sector $S(j)$ the phase amplitudes $P_j(z)$, $j \in \mathbb{Z}$, satisfy the asymptotic relation*

$$P_j(z) = 1 + o(1), z \to \infty. \tag{19}$$

To prove this statement, consider all the pairs $(u^{(j)}(z), u^{(j+1)}(z))$, $j \in \mathbb{Z}$, of neighboring solutions. The elements of every pair are linearly independent because of their different behaviors in the respective sectors $s(j)$ and $s(j+1)$. Therefore, for each pair $(u^{(j)}(z), u^{(j+1)}(z))$, every solution of the equation can be represented as a linear combination of $u^{(j)}(z)$, $u^{(j+1)}(z)$. Consider then an open sector s with an angle less than $\pi/4$, and assume that s contains the separation ray l_θ, $\theta = \frac{\pi}{4}(2j-1)$, which separate sectors $s(j)$ and $s(j+1)$. Due a Proposition 1 there is a pair (v_1, v_2) of solutions of (8) whose leading factors are equal to those for $u^{(j)}$ and $u^{(j)}$, respectively. We have

$$\begin{aligned} v^{(1)}(z) &= a\, u^{(j)}(z) + b\, u^{(j+1)}(z), \\ v^{(2)}(z) &= c\, u^{(j)}(z) + d\, u^{(j+1)}(z) \end{aligned} \tag{20}$$

with complex constants a, b, c, d.

Considering the first equation of (20) in $s(j)$ and the second equation in $s(j+1)$ yields immediately $b = 0, a = 1$ and $c = 0, d = 1$, respectively. This argument and a similar one for $s(j-1)$ and $s(j)$ finish the proof.

It is important to note the following property of the extended sectors. In contrast to the sectors $s(j)$, the sectors $S(j)$ overlap. In what follows later, this property allows us to relate the asymptotic expansions of the same solution in distant sectors. But first we show a simple example of how this property works.

As follows from the above proof, four relations

$$\begin{aligned} u^{(1)}(z) &= a\, u^{(2)}(z) + b\, u^{(3)}(z), \\ u^{(2)}(z) &= c\, u^{(3)}(z) + d\, u^{(4)}(z), \\ u^{(3)}(z) &= f\, u^{(4)}(z) + g\, u^{(5)}(z), \\ u^{(4)}(z) &= h\, u^{(5)}(z) + k\, u^{(6)}(z) \end{aligned} \tag{21}$$

are valid with a, b, c, d, f, g, h, k complex constants. The following statement is true.

Proposition 4. *Assume that the solution $u^{(j)}(z)$, $j \in \mathbb{Z}$, is defined by (16), (17). Then the relations (21) hold with*

$$b = d = g = k = 1. \tag{22}$$

Indeed, for each relation of (21), there is a sector where two of the functions $u^{(j)}$ are growing, with the same normalization (16)–(17), and the third one is decaying as $z \to \infty$ within this sector. The equalities of the form (22) are related not only to this specific case (of Weber's equation) but can be extended to more general matrix differential equations. Thus, we essentially rely on the fact that the relations $P_j(z) = 1 + o(1), z \to \infty$ are valid not only inside the sectors $s(j)$, but also inside the extended sectors $S(j)$.

To be more precise, let us examine the relation $u^{(1)}(z) = a\,u^{(2)}(z) + b\,u^{(3)}(z)$, the first of the relations (21), in the sector $s(2)$. As follows from (16) and (17) for $j = 2$, inside this sector $u^{(2)}(z) = z^{-E-\frac{1}{2}}e^{\frac{z^2}{2}}(1 + o(1))$ as $z \to \infty$. Thus, $u^{(2)}(z) = o(1)$ as $z \to \infty$. Two other solutions $u^{(1)}(z)$, $u^{(3)}(z)$ are exponentially growing and have the same asymptotic behavior inside $s(2)$: $u^{(1)}(z)$, $u^{(3)}(z) = z^{E-\frac{1}{2}}e^{-\frac{z^2}{2}}(1 + o(1))$ as $z \to \infty$, which yields immediately $b = 1$. We emphasize that for $u^{(1)}(z)$ the sector $s(2)$ is considered part of the sector $S(1)$, while for $u^{(3)}(z)$ the same sector $s(2)$ is considered part of the sector $S(3)$.

Equation (8), as an equation with single valued coefficients, is invariant with respect to the transformation $z \to ze^{-2i\pi}$, hence the function $u^{(j)}(ze^{-2i\pi})$, $j \in \mathbb{Z}$, also is a solution of equation (8). Clearly, solutions $u^{(1)}(ze^{-2i\pi})$, $u^{(2)}(ze^{-2i\pi})$ are bounded in sectors $s(5), s(6)$, respectively, introduced in (15). Moreover, as it follows from (16), (17) and (19), these solutions satisfy the following estimates when $z \to \infty$ and $z \in S(5), S(6)$, respectively:

$$\begin{aligned} u^{(1)}(ze^{-2i\pi}) &= e^{-2i\pi(E-\frac{1}{2})}z^{E-\frac{1}{2}}e^{-\frac{z^2}{2}}(1 + o(1)), \\ u^{(2)}(ze^{-2i\pi}) &= e^{2i\pi(E+\frac{1}{2})}z^{-E-\frac{1}{2}}e^{\frac{z^2}{2}}(1 + o(1)). \end{aligned} \tag{23}$$

It follows immediately from Proposition 2,(ii) that

$$\begin{aligned} u^{(5)}(z) &= e^{2\pi i(E-\frac{1}{2})}u^{(1)}(ze^{-2\pi i}), \\ u^{(6)}(z) &= e^{-2\pi i(E+\frac{1}{2})}u^{(2)}(ze^{-2\pi i}). \end{aligned} \tag{24}$$

The matrix version of (24) can be found, for example, in [6].

Substituting the right-hand sides of (24) for $u^{(5)}(z)$, $u^{(6)}(z)$, respectively, into (21) using (22) from Proposition 4, and renaming a, c, f, h as T_1, T_2, T_3, T_4 we rewrite (21) as

$$\begin{aligned} u^{(1)}(z) &= u^{(3)}(z) + T_1\,u^{(2)}(z), \\ u^{(2)}(z) &= u^{(4)}(z) + T_2\,u^{(3)}(z), \\ u^{(3)}(z) &= e^{2\pi i(E-\frac{1}{2})}\,u^{(1)}(ze^{-2\pi i}) + T_3\,u^{(4)}(z), \\ u^{(4)}(z) &= e^{-2\pi i(E+\frac{1}{2})}\,u^{(2)}(ze^{-2\pi i}) + e^{2\pi i(E-\frac{1}{2})}T_4\,u^{(1)}(ze^{-2\pi i}). \end{aligned} \tag{25}$$

Finally, replacing the functions $u^{(j)}(z), j=1,\,2,\,3,\,4$, in (25) by their phase-amplitudes $P_j(z)$, defined by (16), we obtain

$$\begin{aligned} P_1(z) &= P_3(z) + T_1\, z^{-2E} e^{z^2}\, P_2(z),\\ P_2(z) &= P_4(z) + T_2\, z^{2E} e^{-z^2}\, P_3(z),\\ P_3(z) &= P_1(ze^{-2\pi i}) + T_3\, z^{-2E} e^{z^2}\, P_4(z),\\ P_4(z) &= P_2(ze^{-2\pi i}) + T_4\, z^{2E} e^{-z^2}\, P_1(ze^{-2\pi i}). \end{aligned} \tag{26}$$

Definition 4. *The coefficients $T_k = T_k(E)$ in (25) and (26) are called the connection coefficients for the equation (8) and for the normalized principal solutions (16).*

Eliminating $u^{(3)}(z)\,, u^{(4)}(z)$ from the relations (25) or $P_3(z)$, $P_4(z)$ from the relations (26) obviously yields relations which can be regarded (similarly to (4), (5)) as the monodromic relations for the first pair of principal solutions or their phase amplitudes, respectively.

The relations (26) provide the basis for the four-element Stokes structure.

Definition 5. *A set of functions $p_1(z)$, $p_2(z)$, $p_3(z)$, $p_4(z)$*

(i) *analytic on the Riemann surface of* $\log z$ *satisfying*

$$|p_j(z)| < Ae^{B|z|^2},\ z\to\infty,\quad A,\ B>0,\ j=1,\,2,\,3,\,4,$$

(ii) *bounded in closed sub-sectors of $S(j)$*

$$\begin{aligned} S(1) &= \left\{z: 0<|z|<\infty,\ -\tfrac{3\pi}{4} - \tfrac{\arg\alpha}{2} < \arg z < \tfrac{3\pi}{4} - \tfrac{\arg\alpha}{2}\right\},\\ S(2) &= \left\{z: 0<|z|<\infty,\ -\tfrac{\pi}{4} - \tfrac{\arg\alpha}{2} < \arg z < \tfrac{5\pi}{4} - \tfrac{\arg\alpha}{2}\right\},\\ S(3) &= \left\{z: 0<|z|<\infty,\ \tfrac{\pi}{4} - \tfrac{\arg\alpha}{2} < \arg z < \tfrac{7\pi}{4} - \tfrac{\arg\alpha}{2}\right\},\\ S(4) &= \left\{z: 0<|z|<\infty,\ \tfrac{3\pi}{4} - \tfrac{\arg\alpha}{2} < \arg z < \tfrac{9\pi}{4} - \tfrac{\arg\alpha}{2}\right\}, \end{aligned} \tag{27}$$

respectively, and

(iii) *satisfying the monodromic relations with connection coefficients T_1, $T_2,\ T_3,\ T_4$*

$$\begin{aligned} p_1(z) &= p_3(z) + T_1\ z^{\gamma}\ e^{\,\alpha z^2+\beta z}\ \ p_2(z),\\ p_2(z) &= p_4(z) + T_2 z^{-\gamma} e^{-(\alpha z^2+\beta z)} p_3(z),\\ p_3(z) &= p_1(ze^{-2\pi i}) + T_3\ z^{\gamma}\ e^{\,\alpha z^2+\beta z}\ \ p_4(z),\\ p_4(z) &= p_2(ze^{-2\pi i}) + T_4 z^{-\gamma} e^{-(\alpha z^2+\beta z)} p_1(ze^{-2\pi i}) \end{aligned} \tag{28}$$

form a four-element Stokes structure $\mathfrak{S}(4)$ *generated by* $z^{\gamma}e^{\alpha z^2+\beta z}$*:*

$$\mathfrak{S}(4) = \mathfrak{S}\{p_1(z),\, p_2(z),\, p_3(z),\, p_4(z)\}.$$

Thus, using (26), we have shown that the phase amplitudes $P_j(z)$ of the principal solutions $u^{(j)}(z)$ in (16) form a four-element Stokes structure $\mathfrak{S}_{\mathfrak{W}}(4)$ with $\alpha = 1, \beta = 0, \gamma = -2E$ and complex coefficients T_1, T_2, T_3, T_4. These coefficients will be determined later in a subsequent paper using our method which follows.

Remark 3. *For the sake of uniformity one could rewrite the relations (3) as*

$$\begin{aligned} p_1(z) &= p_1(ze^{-2\pi i}) + T_1\, z^{\beta}\, e^{\alpha z}\, p_2(z), \\ p_2(z) &= p_2(ze^{-2\pi i}) + T_2 z^{-\beta} e^{-\alpha z} p_1(\, ze^{-2\pi i}\,), \end{aligned} \tag{29}$$

which better agrees with (28). However, we prefer to work with relations (3) or (4) and (5) presented in their original Hankel form, see, for example, relations (15.6) in [6], to emphasize also the peculiarity of the two-element Stokes structure.

Lemma 1. *Due to the additional symmetry of Weber's equation (the coefficients of (8) are even functions), its four-element Stokes structure can be reduced to a two-element Stokes structure.*

Equation (8) is also invariant with respect to the transformation $z \to ze^{-i\pi}$, hence the functions $u^{(1)}(ze^{-i\pi}), u^{(2)}(ze^{-i\pi})$ are also solutions of (8).

Using a technique similar to that shown above one can prove that

$$\begin{aligned} u^{(1)}\left(ze^{-\pi i}\right) &= e^{-\pi i\left(E-\frac{1}{2}\right)} u^{(3)}(z)\,, \\ u^{(2)}\left(ze^{-\pi i}\right) &= e^{\pi i\left(E+\frac{1}{2}\right)} u^{(4)}(z) \end{aligned} \tag{30}$$

or

$$\begin{aligned} u^{(3)}(z) &= e^{\pi i(E-\frac{1}{2})} u^{(1)}(ze^{-\pi i}), \\ u^{(4)}(z) &= e^{-\pi i(E+\frac{1}{2})} u^{(2)}(ze^{-\pi i}). \end{aligned} \tag{31}$$

Eliminating $u^{(3)}(z), u^{(4)}(z)$ from the relations (25), yields

$$\begin{aligned} u^{(1)}(z) &= e^{\pi i(E-\frac{1}{2})} u^{(1)}(ze^{-\pi i}) + T_1\, u^{(2)}(z), \\ u^{(2)}(z) &= e^{-\pi i(E+\frac{1}{2})} u^{(2)}(ze^{-\pi i}) + T_2\, e^{\pi i(E-\frac{1}{2})} u^{(1)}(ze^{-\pi i}), \\ e^{\pi i(E-\frac{1}{2})} u^{(1)}(ze^{-\pi i}) &= e^{2\pi i(E-\frac{1}{2})} u^{(1)}(ze^{-2\pi i}) + T_3\, e^{-\pi i(E+\frac{1}{2})} u^{(2)}(ze^{-\pi i}), \\ e^{-\pi i(E+\frac{1}{2})} u^{(2)}(ze^{-\pi i}) &= e^{-2\pi i(E+\frac{1}{2})} u^{(2)}(ze^{-2\pi i}) + e^{2\pi i(E-\frac{1}{2})} T_4\, u^{(1)}(ze^{-2\pi i}). \end{aligned} \tag{32}$$

which after the change of variable $ze^{-\pi i} \to z$ in the last pair of relations of (32) obviously yields

$$\begin{aligned} u^{(1)}(z) &= e^{\pi i(E-\frac{1}{2})}u^{(1)}(ze^{-\pi i}) + T_1\, u^{(2)}(z), \\ u^{(2)}(z) &= e^{-\pi i(E+\frac{1}{2})}u^{(2)}(ze^{-\pi i}) + T_2\, e^{\pi i(E-\frac{1}{2})}u^{(1)}(ze^{-\pi i}), \\ u^{(1)}(z) &= e^{\pi i(E-\frac{1}{2})}u^{(1)}(ze^{-\pi i}) + T_3\, e^{-2\pi i E}u^{(2)}(z), \\ u^{(2)}(z) &= e^{-\pi i(E+\frac{1}{2})}u^{(2)}(ze^{-\pi i}) + e^{\pi i(E-\frac{1}{2})}e^{2\pi i E}T_4\, u^{(1)}(ze^{-\pi i}). \end{aligned} \tag{33}$$

It can be seen that with an appropriate choice of T_3, T_4 the second pair of relations (33) is identical to the first pair of relations (33). This gives

$$\begin{aligned} T_3 &= e^{2\pi i E}T_1, \\ T_4 &= e^{-2\pi i E}T_1. \end{aligned} \tag{34}$$

Thus, the four elements Stokes structure for Weber's equations can be reduced to the two-element Stokes structure for the reduced Weber's equations, which has been derived in [2].

Of course, one could also express $u^{(j)}(z), j = 1,\ 2,\ 3,\ 4$, in terms of the Weber functions $D_{E-\frac{1}{2}}(z)$,
$D_{-E-\frac{1}{2}}(z)$. Comparing (16) and (17) with formulae (9) in [2] and using the Proposition II(ii) yields

$$\begin{aligned} u^{(1)}(z) &= 2^{-\frac{E}{2}+\frac{5}{4}}D_{E-\frac{1}{2}}\big(\sqrt{2}z\big), \\ u^{(2)}\,(z) &= 2^{\frac{E}{2}+\frac{5}{4}}e^{-\frac{\pi i}{2}(-E+\frac{1}{2})}D_{-E-\frac{1}{2}}\left(\sqrt{2}ze^{-\frac{\pi i}{2}}\right) \end{aligned} \tag{35}$$

and similar formulae for $u^{(3)}\,(z)\,, u^{(4)}\,(z)$ from the above.

It is worth emphasizing again that for Weber's equation, due to (35), we could also obtain the Stokes structure $\mathfrak{S}\,(4)$ (with explicit connection coefficients) using (17), the integral representations (13), (14) in [2] and the monodromic property of the Gauss Hypergeometric function which we have used already in [2] for the incomplete Gamma and Bessel's equations . However, we will omit this possibility in our present investigation. Moreover, it is better in this context to ignore the relations similar to (35) and the relation (14) in [2].

The above analysis is applicable to a wide class of differential equations. For example, the quartic anharmonic oscillator equation

$$\frac{d^2u}{dz^2} - \big(z^2 + \varepsilon z^4 - 2E\big)u = 0 \tag{36}$$

with a complex coupling parameter $0 < |\varepsilon| < 1$ yields six solutions, analytic in the whole complex plane, with exponential growth of order 3:

$$\begin{aligned}
u^{(1)}(z) &= z^{-1}e^{-\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_1(z),\\
u^{(2)}(z) &= z^{-1}\,e^{\;\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_2(z),\\
u^{(3)}(z) &= z^{-1}e^{-\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_3(z),\\
u^{(4)}(z) &= z^{-1}\,e^{\;\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_4(z),\\
u^{(5)}(z) &= z^{-1}e^{-\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_5(z),\\
u^{(6)}(z) &= z^{-1}\,e^{\;\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_6(z),
\end{aligned}\tag{37}$$

where $P_j(z) = 1 + o(1)$, $z \to \infty$ in closed sub-sectors of

$$\left\{z : 0 < |z| < \infty, -\frac{\pi}{6} + (j-1)\,\frac{\pi}{3} < \arg z < \frac{\pi}{6} + (j-1)\,\frac{\pi}{3}\right\},\ j = 1, 2, \ldots, 6\,.$$

Using the method above, this similarly imply the six relations

$$\begin{aligned}
P_1(z) &= P_3\,(z) + T_1\,e^{\;2\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_2(z),\\
P_2(z) &= P_4\,(z) + T_2\,e^{-2\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_3(z),\\
P_3(z) &= P_5\,(z) + T_3\,e^{\;2\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_4(z),\\
P_4(z) &= P_6\,(z) + T_4\,e^{-2\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_5(z),\\
P_5(z) &= P_1\left(ze^{-2\pi i}\right) + T_5\,e^{\;2\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_6(z),\\
P_6(z) &= P_2\left(ze^{-2\pi i}\right) + T_6\,e^{-2\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}\,P_1(ze^{-2\pi i}),
\end{aligned}\tag{38}$$

whence the six-element Stokes structure $\mathfrak{S}_{\mathfrak{A}}(6)$ generated by $e^{2\sqrt{\varepsilon}\left(\frac{z^3}{3}+\frac{z}{2\varepsilon}\right)}$:

$$\mathfrak{S}_{\mathfrak{A}}(6) = \mathfrak{S}\big\{P_1(z),\, P_2(z),\, P_3(z),\, P_4(z),\, P_5(z),\, P_6(z)\big\}.$$

Let us ignore the differential equations for the time being and consider the Stokes structure as an independent object. One could regard the Stokes structure as a system of linear equations in a certain class of analytic functions. This system of equations is homogeneous and it always possesses zero solutions. A question arises immediately:

Are there non-trivial functions that are non-vanishing and that satisfy this system of equations ?

For the two-element Stokes structure $\mathfrak{S}(2)$ the corresponding system of linear equations has been defined by (4) and (5). For the special case $\alpha = 2i, \beta = 0$ and $T_1 = T_2 = T$, the phase amplitudes P_1, P_2 of the Hankel functions $H_\nu^{(1)}$, $H_\nu^{(2)}$ clearly satisfy the system of equations (4) and (5) if the parameter ν is defined by the equation $\cos \nu\pi = \frac{T}{2i}$.

The following more general statement is true.

Theorem 1. *Consider the Stokes structures* $\mathfrak{S}(2)$, $\mathfrak{S}(4)$, $\mathfrak{S}(6)$ *with corresponding system of linear equations defined by (4) and (5), (28) and (38), respectively. Then there exists a non-trivial solution of these equations in the respective class of analytic functions.*

This theorem with subsequent generalizations will be discussed in the next papers in this series. In particular, for the special case of $\mathfrak{S}(2)$, we will consider the question:

Are there solutions of (4) and (5) in addition to the ones above and if such solutions do exist what are they?

Similar questions arise in more general matrix cases.

In our subsequent papers we will present a unified Fourier transform method by applying Fourier transforms to *the elements of the Stokes structure* $\mathfrak{S}$. Given $\mathfrak{S}$ we will properly define Fourier (Borel)-like transforms and study their analytic properties, extract the formal power series associated with the elements of $\mathfrak{S}$ and relate them to each other and to the elements of $\mathfrak{S}$.

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