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On Wiegerinck's support theorem

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Let continuous function $f(x), x \in \mathbb{R}^n$, tend to 0 as $\|x\| \to \infty$ faster than any negative degree of $\|x\|$. Let Radon transform $\tilde{f}(\omega, t), \omega \in \mathbb{R}^n, \|\omega\| = 1, t \in \mathbb{R}$, of f also tend to 0 as $t \to \infty$ and, besides, do it very fast on a massive enough set of ω. In the paper, we describe the additional properties that f has under these assumptions for different rates of fast decreasing. In particular, the extremal case where $\tilde{f}(\omega, t)$ has the compact support with respect to t for the open subset of unit sphere corresponds to Wiegerinck's Theorem mentioned in the title.

1. Introduction

Let $\mathbb{R}^n$ be a Euclid space of vectors $x = (x_1, \dots, x_n)$ with a scalar product

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j,$$

and let $\mathbb{S}^{n-1}$ be a unit sphere $\left\{\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n : \|\omega\|^2 = \langle \omega, \omega \rangle = 1\right\}$ in it. Denote by $\mathbf{A}$ the set of all hyperplanes in $\mathbb{R}^n$. It is possible to parametrize this set associating a point $(\omega, t) \in \mathbb{S}^{n-1}, t \in \mathbb{R}$ to a hyperplane $A(\omega, t) = \{x \in \mathbb{R}^n : \langle x, \omega \rangle = t\} \in \mathbf{A}$. Obviously, points (ω, t) and $(-\omega, -t)$ correspond to the same hyperplane.

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Let function $f : \mathbb{R}^n \to \mathbb{C}$ be integrable on each hyperplane. Then one can associate with it the function $\tilde{f}(\omega, t)$ defined on $\mathbb{S}^{n-1} \times \mathbb{R}$ as follows:

$$\tilde{f}(\omega, t) = \int\limits_{A(\omega,t)} f(x)\, dm(x), \tag{1}$$

where $dm(x) = dm_{A(\omega,t)}(x)$ is Lebesgue measure on $A(\omega, t)$. The function $\tilde{f}$ is called Radon transform of f. If f has a compact support, say, a closed subset of a ball $\{x \in \mathbb{R}^n : \|x\| \le r\}$, then the support of $\tilde{f}$ is a closed subset of the direct product of $\mathbb{S}^{n-1}$ and a closed interval $[-r, r]$ that is also compact.

The inverse statement is, generally speaking, false.

Example 1. *Let $n = 2, z = x_1 + ix_2$, and let $f(z) = z^{-2}$ for $|z| = \|(x_1, x_2)\| \ge 1$ and $f(z) = \bar{z}^2$ for $|z| < 1$. Then $\tilde{f}$ vanishes on the complement of the closed unit disk.*

L. Zalcman [13] built the following example of an entire function f whose support is $\mathbb{C}$ and Radon transform $\tilde{f}$ vanishes everywhere.

Example 2. *Let $f(z), z = x_1 + ix_2 \in \mathbb{C} = \mathbb{R}^2$, be any nonzero entire function that tends to 0 faster than $|z|^{-2}$ as z approaches infinity outside the curvilinear halfstrip*

$$H = \left\{z = x_1 + ix_2 \in \mathbb{C} : \left|x_1 - x_2^2\right| < 1, x_2 > -1\right\}.$$

The existence of such entire functions follows, for instance, from one general Arakelian's theorem [1] on the tangential approximation by entire functions. By Cauchy's integral theorem applied to appropriate halfdisks, Radon transform of f vanishes identically.

Of course, this function f does not belong to $L^1\left(\mathbb{R}^2\right)$.

Because of these examples, it seems important to have some criteria or, at least, some sufficient conditions on f that guarantee the compactness of its support whenever the support of its Radon transform is compact. (In the theory of Radon transform such statements are called support theorems; the first support theorem was proved by S. Helgason [3] in his proof of the formula for the inverse Radon transform.)

The least restrictive condition implying such compactness was obtained by J. Wiegerinck [12] in 1985. To formulate his theorem, we need the following definition:

Definition 3. *A continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is called reasonably fast decreasing if for each natural k the value of* $\sup\left\{|f(x)|\,\|x\|^k : x \in \mathbb{R}^n\right\}$ *is finite.*

Theorem 4 (Wiegerinck). *Let $f \in C(\mathbb{R}^n)$ be reasonably fast decreasing and let for each $\omega \in \mathbb{S}^{n-1}$ there exist such positive numbers $\epsilon = \epsilon(\omega)$ and $C = C(\omega)$ that for each real t the inequality*

$$\left|\tilde{f}(\omega, t)\right| \le C \exp\{-\epsilon|t|\} \tag{2}$$

is valid. If there exits such an open set $e \in \mathbb{S}^{n-1}$ that for some finite positive R

$$\forall \omega \in e : \left(\forall t \in \mathbb{R} \text{ with } |t| \ge R : \tilde{f}(\omega, t) = 0\right), \tag{3}$$

then the support of f is a compact set.

Concerning this Wiegerinck's theorem, the following questions arise:

(i) Whether the statement of the theorem remains true for sets e of positive Lebesgue measure on $\mathbb{S}^{n-1}$?

(ii) If the answer to (i) is positive, is it possible to get an analogue of Wiegerinck's theorem for sets e less massive than of positive Lebesgue measure on $\mathbb{S}^{n-1}$?

(iii) May condition (2) be relaxed?

(iv) What's going on if one changes (3) to a condition of very fast decreasing along directions $\omega \in e$?

The present paper gives the answers, at least partial, to these questions .

First, recall some definitions and facts.

Definition 5. *Let $\{m_k\}_{k=0}^{\infty}, m_0 = 1$, be a sequence of positive numbers. With this sequence, it is associated the class*

$$C(\{m_k\}) = \left\{f \in C^\infty(\mathbb{R}) : \left(\exists C = C_f < \infty\ \forall t \in \mathbb{R}\ \forall k \in \mathbb{Z}_+ : \left|f^{(k)}(t)\right| \le C^{k+1} m_k\right)\right\}.$$

Class $C(m_k)$ is called Δ-quasianalytic if any two its elements that coincide together with all their derivatives at some point coincide identically.

For $C(\{m_k\})$ is a linear space, definition (5) simply means that if a function of this class has a zero of infinite order, then it must equal zero identically.

The criteria of Δ-quasianalyticity were proved independently by Carleman, Denjoy, and A. Ostrovski (for references, see [4]). We formulate only one of these criteria which we need below.

Theorem 6. *Class $C(\{m_k\})$ is Δ-quasianalytic if, and only if, the condition:*

$$\int_1^{\infty} \frac{\ln T(r)}{r^2} dr = \infty, \tag{4}$$

where

$$T(r) = \sup\left\{\frac{r^k}{m_k} : k = 0, 1, \dots\right\}$$

is so-called Ostrovski's function, is satisfied.

By the definition, any Ostrovski's function increases as r increases and is a convex function of $\log r$.

The following theorem contains answers to questions (i) and (iii).

Theorem 7. *Let $f \in C(\mathbb{R}^n)$ be reasonably fast decreasing, and let for each $\omega \in \mathbb{S}^{n-1}$ there exist such a quasianalytic class $C(\{m_k(\omega)\})$ that for its Ostrovski's function $T_\omega(r)$ the inequality*

$$\left|\tilde{f}(\omega, t)\right| \le \{T_\omega(|t|)\}^{-1} \tag{5}$$

is valid for every $t \in \mathbb{R}$. If for some set $e \subset \mathbb{S}^n$ of positive Lebesgue measure

$$\forall \omega \in e \; \exists R_\omega < \infty : \left(\forall t \in \mathbb{R} \text{ with } |t| \ge R_\omega : \tilde{f}(\omega, t) = 0\right),$$

then the support of f is a compact set.

Later we will see that there exists a quantitative version of this qualitative result.

It is also possible to relax the restriction on the size of e in Theorem 7. The price that one should pay for it is vanishing the average of f on some affine subsets of smaller dimension. To formulate the correspondent result, we need some notations and definitions.

Let us treat $\mathbb{R}^n$ as the orthogonal sum of Euclid spaces $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_p}, n_1 + \dots + n_p = n$, of vectors

$$x^{(j)} = \left(x_1^{(j)}, \dots, x_{n_j}^{(j)}\right), j = 1, \dots, p.$$

Denote by $\mathbf{x} \in \mathbb{R}^n$ a polyvector $\left(x^{(1)}, \dots, x^{(p)}\right)$. $\mathbb{S}$ stands for the polysphere, i.e., the distinguished boundary of the direct product of unit balls. So

$$\mathbb{S} = \mathbf{S}^{n_1 - 1} \times \dots \times \mathbf{S}^{n_p - 1} = \left\{\omega = \left(\omega^{(1)}, \dots, \omega^{(p)}\right) : \left\|\omega^{(j)}\right\| = 1, j = 1, \dots, p\right\}.$$

Let $\mathbb{A}$ mean the set of all $(n-p)$-dimensional affine subspaces of $\mathbb{R}^n$ whose projection on each of $\mathbb{R}^{n_j}$ is a hyperplane in it. Associate to each point

$$(\omega, t), \omega \in \mathbb{S}, t = (t_1, \dots, t_p) \in \mathbb{R}^p,$$

the following element of $\mathbb{A}$

$$A(\omega, t) = \left\{ \mathbf{x} \in \mathbb{R}^n : \left\langle x^{(j)}, \omega^{(j)} \right\rangle = t_j, j = 1, \dots, p \right\}.$$

It is a coverage of $\mathbb{A}$ of multiplicity 2^p.

Let function $f : \mathbb{R}^n \to \mathbb{C}$ be integrable on each affine subspace of $\mathbb{R}^n$ of dimension $(n-p)$. Let us define the function $\tilde{f}(\omega, t)$ on $\mathbb{S} \times \mathbb{R}^p$ by the same equation (1):

$$\tilde{f}(\omega, t) = \int\limits_{A(\omega,t)} f(\mathbf{x})\, dm(\mathbf{x}),$$

where $A(\omega, t) \in \mathbb{A}$ and $dm(\mathbf{x}) = dm_{A(\omega,t)}(\mathbf{x})$ is $(n-p)$-dimensional Lebesgue measure on it. For $p = 1$, this function is the standard Radon transform. For $p > 1$, it is natural to call this function the block Radon transform.

The following result, which is a generalization of Theorem 7, provides the positive answer to question (ii):

Theorem 8. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a continuos reasonably fast decreasing function, and let for any ω of the polysphere $\mathbb{S}$ there exist such a quasianalytic class $C(\{m_k(\omega)\})$ with Osrovski's function $T_\omega(r)$ that the inequality*

$$\left| \tilde{f}(\omega, t) \right| \le \{T_\omega(\|t\|)\}^{-1}$$

is valid for block Radon transform $\tilde{f}(\omega, t)$ of f at each $t \in \mathbb{R}^p$. If there exists such a set $\mathbf{e} \subset \mathbb{S}$ of positive Lebesgue measure that

$$\forall \omega \in \mathbf{e}\ \exists R_\omega < \infty : \left(\forall t \in \mathbb{R}^p \text{ with } \|t\| \ge R_\omega : \tilde{f}(\omega, t) = 0 \right),$$

then the supp*ort of f is compact.*

For $n_1 = n_2 = \cdots = n_p = 2$, the restriction on the massiveness of $\mathbf{e}$ can be relaxed further. As we will see, this restriction can be formulated in terms of positive capacity instead of the terms of positive Lebesgue measure.

To better understand the answer to question (iv), we have to reformulate Theorem 7 and Theorem 8. By Wiener–Paley–Schwartz Theorem, these theorems can be reformulated as follows: If conditions of Theorem 7 (Theorem 8) are satisfied, then f is Fourier transform of some entire function of finite exponential type* in $\mathbb{C}^n$ whose restriction on $\mathbb{R}^n$ belongs to $L^2(\mathbb{R}^n)$.

Here is the answer itself.

Theorem 9. *Let $f:\mathbb{R}^n\to\mathbb{C}$ be a continuous reasonably fast decreasing function, and let for each $\omega\in\mathbb{S}$ there exist such a quasianalytic class $C(\{m_k(\omega)\})$ with Ostrovski's function $T_\omega(r)$ that the inequality*

$$\left|\tilde{f}(\omega,t)\right|\le\{T_\omega(\|t\|)\}^{-1}$$

is valid for the block Radon transform $\tilde{f}(\omega,t)$ of f at each $t\in\mathbb{R}^p$. If there exists such a number $\rho>1$ and such a set $\mathbf{e}\subset\mathbb{S}$ of positive Lebesgue measure on $\mathbb{S}$ that

$$\forall\omega\in\mathbf{e}\exists C_\omega<\infty:\left(\forall t\in\mathbb{R}^p:\left|\tilde{f}(\omega,t)\right|\le C_\omega\exp\{-C_\omega\|t\|^\rho\}\right),$$

then f is the Fourier transform of an entire function of finite type in $\mathbb{C}^n$ with respect to order $\frac{\rho}{\rho-1}$ whose restriction on $\mathbb{R}^n$ belongs to $L^2(\mathbb{R}^n)$.

2. Proof of Theorem 7

Let $\omega\in\mathbb{S}^n$ and let $\lambda\in\mathbb{R}$. It is well known (see, for instance, Helson's monography cited above) that Fourier transform $\hat{f}$ and Radon transform $\tilde{f}$ of f relate to each other as follows:

$$\hat{f}(\lambda\omega)=\left(\int\limits_{\mathbb{R}^n}f(x)\exp\{i\lambda\langle\omega,x\rangle\}\,dx\right)=\int\limits_{\mathbb{R}}\tilde{f}(\omega,t)\exp\{i\lambda t\}\,d,\qquad(6)$$

which is crucial for the present reasoning. For instance, it means that for $\omega\in e$ the function $\hat{f}(\lambda\omega)$ can be extended onto the complex λ-plane as an entire function of exponential type.

From equation (6) and estimate (2) it follows that $\hat{f}(\lambda\omega)$ can be extended from the real axis to the strip $\{\lambda\in\mathbb{C}:|\mathrm{Im}\,\lambda|<\epsilon\}$ as an analytic function. In

* An entire function $\varphi:\mathbb{C}^n\to\mathbb{C}$ is called a function of exponential type σ if

$$\sup\{|\varphi(z)|\exp\{-A(|z_1|+\cdots+|z_n|)\}:z=(z_1,\dots,z_n)\in\mathbb{C}^n\}$$

is finite for each $A>\sigma$ and infinite for each $A<\sigma$. Similarly, an entire function $\varphi:\mathbb{C}^n\to\mathbb{C}$ is called a function of finite type σ with respect to order ρ if

$$\sup\{|\varphi(z)|\exp\{-A(|z_1|+\cdots+|z_n|)^\rho\}:z\in\mathbb{C}^n\}$$

is finite for each $A>\sigma$ and infinite for each $A<\sigma$.

other words, in Wiegerinck's theorem the Fourier transform of f is slice analytic. Upon the conditions of Theorem 7, $\hat{f}(\lambda\omega)$ belongs, as a function of λ, to the Δ-quasianalytic class $C(\{m_{k+2}(\omega)\})$, and that property may be called slice quasianalyticity. Indeed, for all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}_+$ the inequality

$$\begin{aligned} \left|\frac{\partial^k \hat{f}}{\partial \lambda^k}(\lambda\omega)\right| &= \left|\int_{-\infty}^{\infty} (it)^k \tilde{f}(\omega,t) \exp\{i\lambda t\}\, dt\right| \le C + C\int_1^{\infty} \frac{t^k}{T_\omega(t)} dt \\ &\le C\int_1^{\infty} \frac{t^{k+2}}{T_\omega(t)} \frac{dt}{1+t^2} \le C m_{k+2} \end{aligned}$$

is valid. Here and in what follows C stands for different constants.

Since f is a reasonably fast decreasing function, its Radon transform $\tilde{f}(\omega,t)$ also decreases fast. It means that Fourier transform $\hat{f}(x) \in C^\infty(\mathbb{R}^n)$. We can associate with $\hat{f}(x)$ its formal expansion into Taylor series of homogeneous polynomials:

$$\hat{f}(x) \sim \sum_{k\in\mathbb{Z}_+} \left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \cdots + x_n\frac{\partial}{\partial x_n}\right)^k \hat{f}(0)/k! = \sum_{k\in\mathbb{Z}_+} A_k(x). \quad (7)$$

If $x = \lambda\omega$ and $\omega \in e$, this expansion becomes informal

$$\hat{f}(\lambda\omega) = \sum_{k\in\mathbb{Z}_+} A_k(\omega)\lambda^k$$

and the series converges for all complex λ.

It is easy to see that a function $\omega \mapsto R_\omega$ us semicontinous from above. In particular, it is measurable on e. So, there exist such a set $\tilde{e} \subset e$ of positive Lebesgue measure and such a positive finite number R that

$$\sup\{R_\omega : \omega \in \tilde{e}\} = R.$$

For $\omega \in \tilde{e}$

$$\begin{aligned} |A_k(\omega)| &= \left|\int_{-\infty}^{\infty} \tilde{f}(\omega,t)\left(\frac{\partial^k}{\partial\lambda^k}\exp\{i\lambda t\}\,|_{\lambda=0}\right) dt\right| /k! \\ &= \left|\int_{-R}^{R} \tilde{f}(\omega,t)\left(\frac{\partial^k}{\partial\lambda^k}\exp\{i\lambda t\}\,|_{\lambda=0}\right) dt\right| /k! \le CR^{k+1}/k! = CR^k/k!. \end{aligned}$$

Let τ be the standard parametrization of $\mathbb{S}^{n-1}$ by spherical coordinates $\theta = (\theta_1, \ldots, \theta_{n-1})$ extended onto the cube $\{\theta_j \in [-\pi,\pi) : j = 1, \ldots, n-1\}$. (There exist several such parametrizations but it does not matter for us which one is chosen.) τ realizes the coverage of $\mathbb{S}^{n-1}$ of essentially finite multiplicity. Besides, it maps the homogeneous polynomial $A_k(\omega)$ onto the trigonometric polynomial

$$G_k(\theta) = H_k(\exp\{i\theta_1\}, \exp\{-i\theta_1\}, \dots, \exp\{-i\theta_{n-1}\}) = A_k(\omega)$$

of degree k with respect to vector variable $\theta \in \mathbb{R}^{n-1}$. According to our estimate, this trigonometric polynomial is bounded by $CR^k/k!$ on the relatively dense (with respect to Lebesgue measure) set *

$$E = \cup_{l=(l_1,\dots,l_{n-1})\in(\mathbb{Z}_+)^{n-1}} \left(2\pi l + \tau^{-1}(\tilde{e})\right) \subset \mathbb{R}^{n-1}.$$

To proceed further, we need the following statement that was proved by A. Schaeffer [11] for $m = 1$ and by B. Levin [6] for the general case:

Theorem 10 (Levin-Schaeffer). *Let E be a relatively dense subset of $\mathbb{R}^m$ with the dense characteristics L and δ, and let $g(z)$ be an entire function of exponential type σ in $\mathbb{C}^m$. Then*

$$\sup\{|g(x)| : x \in \mathbb{R}^m\} \le \exp\left\{C\sigma L^{m+1}/\delta\right\} \sup\{|g(y)| : y \in E\},$$

where $C = C(m)$ depends only on dimension m.

Applying Theorem 10 to trigonometric polynomials $G_k(\theta)$, we get that

$$\begin{aligned} \max\left\{|A_k(\omega)| : \omega \in \mathbb{S}^{n-1}\right\} &= \max\left\{|G_k(\theta)| : \theta \in \mathbb{R}^{n-1}\right\} \\ \le C\exp\{CkL^n/\delta\} R^k/k! &= C\exp\{\gamma k\}/k! \end{aligned}$$

where positive finite γ does not depend on k.

This estimate allows us to draw two inferences. First, for each $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n, z_j = x_j + iy_j,\ j = 1, \dots, n$, the inequality

$$\ln|A_k(z)| \le \int_{\mathbb{R}^n} \mathbb{P}(t,z) \ln|A_k(t)|\, dt,$$

where

$$\mathbb{P}(t,z) = \prod_{j=1}^{n} \frac{|y_j|}{\pi\left[(t_j - x_j)^2 + y_j^2\right]}$$

* A measurable set $E \subset \mathbb{R}^m$ is called relatively dense (with respect to Lebesgue measure) if there exist such positive constants L and δ that for any $x \in \mathbb{R}^m$ Lebesgue measure of the portion of E in the closed ball $\mathbf{B}_L(x) = \{y \in \mathbb{R}^m : \|y - x\| \le L\}$ is at least δ. L and δ are called the density characterisitics of E.

is the product of n Poisson's kernels for the halfplane, is valid. By the homogeneity of A_k, we have

$$\begin{aligned}\ln|A_k(z)| &\le \max\{\ln|A_k(\omega)| : \omega\in\mathbb{S}^n\} + k\int_{\mathbb{R}^n}\mathbb{P}(t,z)\ln\|t\|\,dt \\ &\le C+\gamma k-\ln(k!)+Ck.\end{aligned}$$

It means that the series

$$\sum_{k\in\mathbb{Z}_+} A_k(z)$$

converges to an entire function $g(z)$. The following theorem of A. Goldberg [2] (see also [5]) implies that $g(z)$ is of finite exponential type:

Theorem 11 (Goldberg). *Let*

$$F(z)=\sum_{k\in\mathbb{Z}_+} P_k(z),\, z\in\mathbb{C}^n,$$

be the expansion of entire function $F(z)$ into Taylor series of homogeneous polynomials $P_k(z)$, and let $C_k=\max\{|P_k(z)| : \|z\|\le 1\}, k\in\mathbb{Z}_+$. Then $F(z)$ is of order $\rho>0$ if, and only if,

$$\limsup_{k\to\infty}\frac{\ln C_k}{k\ln k}=-\frac{1}{\rho}.$$

$F(z)$ is of type σ with respect to order ρ if, and only if,

$$\limsup_{k\to\infty}\left(\frac{\rho}{k}\ln C_k+\ln k\right)=\ln(\sigma e\rho).$$

According to Goldberg's theorem, the estimate of $|A_k(z)|$ we have just stated implies that $g(z)$ is of finite exponential type in $\mathbb{C}^n$.

Second, for each $\omega\in\mathbb{S}^n$ functions $g(\lambda\omega)$ and $\hat{f}(\lambda\omega)$ coincide as functions of λ identically. It is a consequence of the following lemma of L. Ronkin [9]:

Lemma 12 (Ronkin). *If function f that belongs to a Δ-quasianalytic class $C(\{m_k\})$ coincides (together with all its derivatives) at some point a with function g that is analytic in some disk centered at a, then they coincide on the correspondent diameter of the disk.*

So, $\hat{f}(x)=g(x)$ for all $x\in\mathbb{R}^n$. By Wiener–Paley–Schwartz theorem, f vanishes outside some ball.

Theorem 7 is proved.

Remark 13. *It is easy to see that Theorem 7 can be reformulated in quantitative terms. Indeed, if Lebesgue measure of* $e \subset \mathbb{S}^n$ *equals* δ *and* $R_\omega = R$ *for all* $\omega \in e$*, then for each* $\varphi \in \mathbb{R}^{n-1}$

$$m\left(E \cap \left\{\theta \in \mathbb{R}^{n-1} : |\theta_j - \varphi_j| \le \pi, j = 1, \dots, n-1\right\}\right) \ge C\delta,$$

where $m(A)$ *is Lebesgue measure of* $A \subset \mathbb{R}^{n-1}$ *and* $C > 0$ *depends on* n *only. Choosing* $L = 2\pi$ *in Levin–Schaeffer's theorem, we easily verify that the support of* f *is contained in the ball*

$$\{x \in \mathbb{R}^n : \|x\| \le CR\},$$

where $C = C(n, \delta) > 0$ *does not depend on* f *and* $R = \max\{dist\,(x, 0) : x \in \ supp\ \tilde{f}\}$.

Wiegerinck derives his theorem from the so-called Korevaar–Wiegerinck lemma [12] on mixed derivatives. This lemma allows to estimate the order n partial derivatives of a function in terms of the estimate of its n^{th} differential for all vectors of an open subset of $\mathbb{S}^n$. Using Levin–Schaeffer's theorem, one can sharpen the statement of this lemma using the estimate of the n^{th} differential only on a subset of positive Lebesgue measure.

3. Proof of Theorem 8

Reasonably fast decreasing of f implies that Fourier transform $\hat{f}$ of f is an element of $C^\infty(\mathbb{R}^n)$ and therefore has the following formal expansion

$$\hat{f}(\mathbf{x}) \sim \sum_{\mathbf{k} \in (\mathbb{Z}_+)^p} \frac{1}{\mathbf{k}!} \prod_{j=1}^{p} \left(x_1^{(j)} \frac{\partial}{\partial x_1^{(j)}} + \dots + x_{n_p}^{(j)} \frac{\partial}{\partial x_{n_p}^{(j)}} \right)^{k_j} \hat{f}(0) = \sum_{\mathbf{k} \in (\mathbb{Z}_+)^p} A_{\mathbf{k}}(\mathbf{x}).$$

Here $\mathbf{k} = (k_1, \dots, k_p) \in (\mathbb{Z}_+)^p, \mathbf{k}! = k_1! \dots k_p!$, and $A_{\mathbf{k}}(\mathbf{x})$ is an algebraic polynomial that is homogeneous of degree k_j with respect to $x^{(j)}, j = 1, \dots, p$. For $\omega \in \mathbb{S}$ and $\lambda \in \mathbb{R}^p$ denote by $\lambda \cdot \omega$ polyvector

$$\left(\lambda_1 \omega^{(1)}, \lambda_2 \omega^{(2)}, \dots, \lambda_p \omega^{(p)}\right) \in \mathbb{R}^n.$$

If $\mathbf{x} = \lambda \cdot \omega, \omega \in \mathbf{e}$, the expansion above becomes informal.

$$\hat{f}(\lambda \cdot \omega) = \sum_{\mathbf{k} \in (\mathbb{Z}_+)^p} A_{\mathbf{k}}(\omega) \lambda^{\mathbf{k}}, \lambda^{\mathbf{k}} = \lambda^{k_1} \dots \lambda^{k_p}.$$

Once again, the Fourier transform of f admits the following representation in terms of its block Radon transform:

$$\begin{aligned}\hat{f}(\lambda\cdot\omega) &= \int\limits_{\mathbb{R}^n} f\left(x^{(1)}, x^{(2)}, \dots, x^{p}\right) \\ \times \exp&\left\{i\left[\lambda_1\left\langle\omega^{(1)}, x^{(1)}\right\rangle + \cdots + \lambda_p\left\langle\omega^{(p)}, x^{(p)}\right\rangle\right]\right\} dx \\ &= \int\limits_{\mathbb{R}^p} \tilde{f}(\omega, t)\exp\{i\langle\lambda, t\rangle\}\, dt,\end{aligned} \tag{8}$$

It means that

$$\begin{aligned}\sup\left\{\left|\frac{\partial^\nu}{\partial\lambda_j^\nu}\hat{f}(\lambda\cdot\omega)\right| : \lambda_j\in\mathbb{R}\right\} \quad &\leq \quad \int\limits_{\mathbb{R}^p}\left|\tilde{f}(\omega,t)\right| |t_j|^\nu\, dt \\ \leq \int\limits_{\mathbb{R}^p}\frac{\|t\|^\nu\left(1+\|t\|^{p+1}\right)}{T_\omega(\|t\|)}\frac{dt}{\left(1+\|t\|^{p+1}\right)} \quad &\leq \quad C\left\{m_\nu(\omega) + m_{\nu+p+1}(\omega)\right\}.\end{aligned}$$

By Carleman–Denjoy–Ostrovski's Tteorem, classes $C\left(\{m_\nu(\omega) + m_{\nu+p+1}(\omega)\}\right)$ and $C\left(\{m_\nu(\omega)\}\right)$ are Δ-quasianalytic or not at the same time.

Now we need to extend the concept of quasianalytic classes onto functions of several variables. Let

$$\{m_{\mathbf{k}}\}_{\mathbf{k}=(k_1,\dots,k_p)\in(\mathbb{Z}_+)^p}, m_{(0,\dots,0)} = 1,$$

be a sequence of positive numbers. This sequence generates the class

$$\begin{aligned}&C\left(\{m_{\mathbf{k}}\}\right) \\ = \Big\{f\in C^\infty(\mathbb{R}) : \Big(\exists C = C_f < \infty\ \forall t\in\mathbb{R}^p\ &\forall\mathbf{k}\in(\mathbb{Z}_+)^p : \left|D^{\mathbf{k}}f(t)\right| \\ &\leq C^{k_1+\cdots k_p+1}m_{\mathbf{k}}\Big)\Big\},\end{aligned}$$

where

$$D^{\mathbf{k}}f(t) = \frac{\partial^{k_1+\cdots k_p}f}{\partial t_1^{k_1}\dots\partial t_p^{k_p}}(t).$$

As before, this class is called Δ-quasianalytic if any two its functions coinciding together with all its partial derivatives at some point of $\mathbb{R}^p$ coincide identically.

The following criterion of Δ-quasianalyticity of the class $C\left(\{m_k\}\right)$ of functions of $p \geq 2$ variables was obtained by V. Matsaev and L. Ronkin [8]:

Theorem 14 (Matsaev–Ronkin). *Class* $C\left(\left\{m_{(k_1,\dots,k_p)}\right\}\right)$ *of functions of* p *variables is* Δ*-quasianalytic if, and only if,* p *classes*

$$C\left(\left\{m_{(k_1,0,\dots,0)}\right\}\right), C\left(\left\{m_{(0,k_2,0,\dots,0)}\right\}\right), \dots, C\left(\left\{m_{(0,0,\dots,0,k_p)}\right\}\right)$$

of functions of one variable are Δ*-quasianalytic.*

By this theorem, defining

$$m_{(j,0,\dots,0)}(\omega) = m_{(0,j,\dots,0)}(\omega) = \dots = m_{(0,\dots,0,j)}(\omega) = m_j(\omega) + m_{j+p+1}(\omega), \quad j = 0, 1, \dots,$$

and $m_{\mathbf{k}}(\omega) = \sup\left\{\left|D^{\mathbf{k}}_{(\lambda)}\hat{f}(\lambda\cdot\omega)\right| : \lambda \in \mathbb{R}^p\right\}$ otherwise, we get a Δ-quasianalytic class that contains $\hat{f}(\lambda\cdot\omega)$ as a function of $\lambda \in \mathbb{R}^p$. Let us assume, for the sake of simplicity, that for all $\omega \in \mathbf{e}$ we have $R_\omega = R < \infty$. So, for $\omega \in \mathbf{e}$ we have

$$\begin{aligned} &\left|\frac{1}{\mathbf{k}!}\frac{\partial^{k_1+\dots+k_p}}{\partial\lambda_1^{k_1}\dots\partial\lambda_p^{k_p}}\hat{f}(\lambda\cdot\omega)_{|\lambda=0}\right| \\ &= \frac{1}{\mathbf{k}!}\left|\int\limits_{\mathbb{R}^p}(it_1)^{k_1}\cdots(it_p)^{k_p}\tilde{f}(\omega,t)\exp\{i\langle\lambda,t\rangle\}\,dt\right| \\ &\le \frac{CR^{k_1+\dots+k_p+p}}{\mathbf{k}!} = \frac{CR^{k_1+\dots+k_p}}{\mathbf{k}!}. \end{aligned}$$

Let us define the trigonometric polynomials $G_{\mathbf{k}}(\theta) = A_{\mathbf{k}}(\omega)$ as follows

$$\begin{aligned} G_{\mathbf{k}} = (\theta) = H_{\mathbf{k}}\Big(&\exp\left\{i\theta_1^{(1)}\right\}, \exp\left\{-i\theta_1^{(1)}\right\}, \dots, \exp\left\{-i\theta_{n_1-1}^{(1)}\right\}, \dots, \\ &\exp\left\{-i\theta_{n_p-1}^{(p)}\right\}\Big) = \frac{1}{\mathbf{k}!}\frac{\partial^{k_1+\dots+k_p}}{\partial\lambda_1^{k_1}\dots\partial\lambda_p^{k_p}}\hat{f}(\lambda\cdot\omega)_{|\lambda=0}. \end{aligned}$$

Here $\omega = \tau(\theta)$, where

$$\begin{aligned} \theta &= \left(\theta^{(1)},\dots,\theta^{(p)}\right), \theta^{(j)} = \left(\theta_1^{(j)},\dots,\theta_{n_j-1}^{(j)}\right),\ j = 1,\dots,p, \\ \theta_k^{(j)} &\in [0.2\pi),\ k = 1,\dots,n_{j-1}, \end{aligned}$$

and $\tau = (\tau_1,\dots,\tau_p)$, where each τ_j is the standard parametrization of $\mathbb{S}^{n_j-1}$ expanded onto the cube $\left\{\theta^{(j)}=\left(\theta_1^{(j)},\dots,\theta_{n_j-1}^{(j)}\right) : \theta_k^{(j)} \in [0.2\pi), k = 1,\dots,n_{j-1}\right\}$. (Again, it does not matter which particular standard parametrization we use.)

Applying Levin–Schaeffer's theorem as in the proof of theorem 7, we get that for the polynomial

$$A_{\mathbf{k}}(\omega) = \frac{1}{\mathbf{k}!}\frac{\partial^{k_1+\cdots+k_p}}{\partial\lambda_1^{k_1}\ldots\partial\lambda_p^{k_p}}\hat{f}(\lambda\cdot\omega)_{|\lambda=0}$$

the following inequality

$$|A_{\mathbf{k}}(\omega)| \le C\exp\{\gamma(k_1+\cdots k_p)R\}/\mathbf{k}! \tag{9}$$

is valid for all $\omega\in\mathbb{S}$ and some finite constants C and γ that do not depend on $\mathbf{k}$ and ω. By means of the multiple Poisson integral and homogeneity of $A_{\mathbf{k}}$, this estimate can be extended (perhaps, with different C and γ) onto complex vectors $\mathbf{z}\in\mathbb{C}^n, \|\mathbf{z}\|\le 1$. It means that the series

$$\sum_{\mathbf{k}\in(\mathbb{Z}_+)^p} A_{\mathbf{k}}(\mathbf{z})$$

converges in $\mathbb{C}^n$ to an entire function, say, $g(z)$. Since estimate (9) is valid, possibly with different C and γ, for polynomials

$$B_k(\mathbf{z}) = \sum_{k_1+\cdots k_p=k} A_{\mathbf{k}}(\mathbf{z}),$$

$g(z)$ is an entire function of finite exponential type by Goldberg's theorem mentioned in the previous section. By Matsaev–Ronkin's theorem and Ronkin's Lemma cited above, two functions of λ, $\hat{f}(\lambda\cdot\omega)$ and $g(\lambda\cdot\omega)$, coincide on each straight line containing the origin for each fixed $\omega\in\mathbb{S}$. Indeed, assuming the opposite, we get by Ronkin's lemma that for at least for one such line, say, l, the restrictions of the correspondent class of functions of p variables is not Δ-quasianalytic. Selecting new orthogonal axes so that one of them is l, we see that by Mstsaev–Ronkin's theorem our class of functions of p variables neither is Δ-quasianalytic. Since it is impossible, the restrictions we started with must coincide for each $\lambda\in\mathbb{R}^p$. For $\omega\in\mathbb{S}$ is arbitrary, these functions coincide on $\mathbb{R}^n$, and to finish the proof, it is enough to apply Wiener–Paley–Schwartz Theorem.

In the case where $n_1=\cdots=n_p=2$, we can prove a stronger result. It is based on the possibility to apply instead of Levin–Schaeffer's theorem the following particular case of the general result of B. Levin, V. Logvinenko, and M. Sodin [7]:

Theorem 15 (Levin–Logvinenko–Sodin). *Let $E\subset\mathbb{C}$ be such a closed set that for every $x\in\mathbb{R}$ and some $\delta>0$ the logarithmic capacity*

$$cap\left(E\cap\left\{\varsigma=\xi+i\eta\in\mathbb{C}:|\xi-x|<\frac{1}{2},|\eta|<\frac{1}{2}\right\}\right)\ge\delta.$$

Then for any subharmonic function $u(z), z \in \mathbb{C}$, that is nonpositive on E and of finite degree not exceeding one, i.e.,

$$\limsup_{|z|\to\infty} \frac{u(z)}{|z|} \le 1,$$

the inequality

$$\sup\{u(x) : x \in \mathbb{R}\} \le \frac{C}{\ln(1/\delta)}$$

holds. Here $C < \infty$ is an absolute constant.

To apply this theorem to the estimate of $|A_{\mathbf{k}}(\omega)|$, the preimage $E = \tau^{-1}(\mathbf{e}) \subset \mathbb{R}^{n-p}$ that is 2π-periodic with respect to each of coordinates should satisfy the following condition: Ronkin's Γ-capacity of

$$E \cap \left\{\theta = \left(\theta^{(1)}, \dots, \theta^{(p)}\right) : \theta_k^{(j)} \in [0, 2\pi], k = 1, \dots, n_j - 1,\ j = 1, \dots, p\right\}$$

has to be positive. The concepts of Γ-projection and Γ-capacity were introduced by L. Ronkin [10].

Let $E \subset \mathbb{C}^m$ be a compact. Denote by $\Delta\left(E; z_1', \dots z_{m-1}'\right)$ the intersection of E with the complex affine subspace $\left\{z = (z_1, \dots, z_m) : z_1 = z_1', \dots, z_{m-1} = z_{m-1}'\right\}$. This intersection also is a compact. Define Γ-projection $\Gamma_m^{m-1}(E)$ of E onto the space $\mathbb{C}^{m-1}$ of variables $z_1, \dots, z_{m-1}$ as the set of all such $(z_1, \dots, z_{m-1})$ that the logarithmic capacity

$$cap\left(\Delta\left(E; z_1, \dots z_{m-1}\right)\right) > 0.$$

Γ-projection of E onto $\mathbb{C}^{m-2}$ we define as

$$\Gamma_m^{m-2}(E) = \Gamma_{m-1}^{m-2}\left(\Gamma_m^{m-1}(E)\right)$$

and so on. At last, $\Gamma_m^1(E) = \Gamma_2^1\left(\Gamma_m^2(E)\right)$. Besides, let us agree that $\Gamma^1(E) = E$. $\Gamma_m^1(E)$, which is now defined for each whole number m, is Ronkin's Γ-projection of E. It occurs that the logarithmic capacity of $\Gamma_m^1(E)$ cannot be chosen as a measure of the massiveness of $E \subset \mathbb{C}^m$. The reason is that it strongly depends on the numeration of coordinates. V. Ivanov and L. Ronkin independently constructed examples [10] of compacts with such capacity equal to 0 for one order of numeration and positive for another So, it is a good idea to define this measure as the maximum of logarithmic capacities of $\Gamma_m^1(E)$ over all possible permutations of variables. This new measure is called Ronkin's Γ-capacity and is denoted by Γ-$cap(E)$.

Any compact set $E \subset \mathbb{R}^m$ of positive Lebesgue measure has, of course, positive Γ-*cap* (E) but there exist a lot of compacts E of Γ-*cap* $(E) > 0$ and of Lebesgue measure equal to 0.

Let us call a set $\mathbf{e} \in \mathbb{S}$ "good" if it is closed and the intersection of its preimage E upon τ (defined above) and the cube

$$\left\{\theta = \left(\theta^{(1)}, \dots, \theta^{(p)}\right) : \theta_k^{(j)} \in [0, 2\pi], k = 1, \dots, n_{j-1}, j = 1, \dots, p\right\}$$

has positive Γ-capacity, say, δ. Because of periodicity of τ , we have that Γ-capacity of the intersection of E with the cube

$$\left\{\theta = \left(\theta^{(1)}, \dots, \theta^{(p)}\right) : \theta_k^{(j)} \in \left[a_k^{(j)}, a_k^{(j)} + 4\pi\right], k = 1, \dots, n_{j-1}, j = 1, \dots, p\right\}$$

is at least δ for any $\mathbf{a} = \left(a^{(1)}, \dots, a^{(p)}\right) \in \mathbb{R}^{n-p}$. Besides, this preimage is 2π-periodic with respect to each coordinate.

For the sake of simplicity, let us assume that $p = 2$. So, $n_1 = n_2 = 2$ and $n = 4$. In this case, E is a subset of $\mathbb{R}^2$. Because of periodicity of E with respect to each variable, the logarithmic capacity of the intersection of its Γ-projection $\Gamma_2^1(E)$ on the space $\mathbb{C}$ of variable z_2 and any segment $[b, 4\pi + b] \subset \mathbb{R}_{(x_2)}$ is bounded from below by some positive constant δ_2 for a certain periodic subset of $\mathbb{R}_{(x_1)}$ whose part in each segment $[a, 4\pi + a] \subset \mathbb{R}_{(x_1)}$ has logarithmic capacity bounded from below by another positive constant δ_1. It means that for each $x_2^0 \in \Gamma_2^1(E)$ the entire function $G_{k_1}(z_1) = G_{\mathbf{k}}\left(z_1, x_2^0\right)$ of exponential type k_1 is bounded on relatively dense by capacity subset of real line by

$$\frac{CR^{k_1+k_2}}{\mathbf{k}1}.$$

By Levin–Logvinenko–Sodin's theorem it means that this function is bounded on the whole real line by

$$\frac{C\exp\{\gamma_1 k_1\} R^{k_1+k_2}}{\mathbf{k}1}, \tag{10}$$

where $\gamma_1 < \infty$ does not depend on $\mathbf{k}$. Let us fix now any real x_1^0. The entire function $H_{k_2}(z_2) = G_{\mathbf{k}}\left(x_1^0, z_2\right)$ of exponential type k_2 is bounded by (10) on the relatively dense by logarithmic capacity subset of real line. Therefore, by the same theorem, it is bounded on the whole real line by

$$\frac{C\exp\{\gamma_1 k_1 + \gamma_2 k_2\} R^{k_1+k_2}}{\mathbf{k}1}.$$

It means that $G_{\mathbf{k}}(x_1, x_2)$ is bounded by this constant on $\mathbb{R}^2$. Starting from this moment, we can proceed as we did it proving Theorem 8 to finish the proof of the following statement:

Theorem 16. *The statement of Theorem 8 remains true if one changes the restriction on Lebesgue measure of* $\mathbf{e} \in \mathbb{S}$ *to the condition of* $\mathbf{e}$ *being "good".*

4. Proof of Theorem 9

As before, Fourier transform $\hat{f}(\mathbf{x})$ of f has the following formal presentation:

$$\hat{f}(\mathbf{x}) \sim \sum_{\mathbf{k} \in (\mathbb{Z}_+)^p} \frac{1}{\mathbf{k}!} \prod_{j=1}^{p} \left(x_1^{(j)} \frac{\partial}{\partial x_1^{(j)}} + \cdots + x_{n_p}^{(j)} \frac{\partial}{\partial x_{n_p}^{(j)}} \right)^{k_j} f(0) = \sum_{\mathbf{k} \in (\mathbb{Z}_+)^p} A_{\mathbf{k}}(\mathbf{x}).$$

For each $\omega \in \mathbf{e}$ the function $A_{\mathbf{k}}(\mathbf{x})$ satisfies the estimate

$$|A_{\mathbf{k}}(\omega)| = \left| \frac{1}{\mathbf{k}!} \frac{\partial^{k_1 + \cdots + k_p}}{\partial \lambda_1^{k_1} \ldots \partial \lambda_p^{k_p}} \hat{f}(\lambda \cdot \omega)_{|\lambda = 0} \cdot \right|$$

$$\leq \frac{1}{\mathbf{k}!} \left| \int_{\mathbb{R}^p} (it_1)^{k_1} \cdots (it_p)^{k_p} \tilde{f}(\omega, t) \exp\{i \langle \lambda, t \rangle\} \, dt \right|$$

$$\leq \frac{C_\omega}{\mathbf{k}!} \int_{\mathbb{R}^p} |t_1|^{k_1} \cdots |t_p|^{k_p} \exp\{-C_\omega (|t_1| + \cdots + |t_p|)^\rho\} \, dt$$

$$\leq 2C_\omega \prod_{j=1}^{p} \frac{1}{k_j!} \int_0^\infty u^{k_j} \exp\left\{-C_\omega p^{\rho-1} u^\rho\right\} du$$

$$\leq C^{k_1 + \cdots + k_p} \prod_{j=1}^{p} k_j^{k_j/\rho - k_j}.$$

Here $C < \infty$ does not depend on $\mathbf{k}$ (but does depend on ω) and $k_j = 0$ contributes a factor equal to 1 to the product. Estimating $A_{\mathbf{k}}(\omega)$ by means of Levin–Schaeffer's theorem, we see that this estimate, is valid for all $\omega \in \mathbb{S}$ but possibly for different C. Taking, if necessary, larger C, we get that the inequality

$$\left| \sum_{k_1 + \cdots + k_p = k} A_{\mathbf{k}}(\omega) \right| \leq C^k k^{k(1-\rho)/\rho}$$

holds for each $\omega \in \mathbb{S}$. Once again, using product of Poisson's kernels for halfplane, we see that this inequality remains true for $\mathbf{z} \in \mathbb{C}^n$, $\|z\| \leq 1$, if we take larger but still finite C. According to Goldberg's Theorem mentioned above, the series

$$\sum_{\mathbf{k} \in (\mathbb{Z}_+)^p} A_{\mathbf{k}} (\mathbf{z})$$

converges for all $\mathbf{z} \in \mathbb{C}^n$ to entire function $g(\mathbf{z})$ of normal type with respect to order $\frac{\rho}{\rho-1}$. The same quasianalytic reasoning as before shows that $g(\mathbf{x}) = \hat{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and, therefore is an element of $L^2(\mathbb{R}^n)$. Theorem 9 is proved.

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