Matematicheskaya fizika, analiz, geometriya
2002, v. 9, No. 3, p. 412–419

On exact inequalities of Kolmogorov type

V.A. Kofanov

Dnepropetrovsk National University
13 Naukovy line, Dnepropetrovsk, 490010, Ukraine
E-mail:abby@email.dp.ua

Received November 11, 2001
Communicated by I.V. Ostrovskii

New exact inequalities of Kolmogorov type for periodic functions are obtained. New exact inequalities of Bernstein type for trigonometric polinomials and splines are proved.

Dedicated to N.I. Akhiezer Centenary

1. Introduction

Let G be the real line $\mathbf{R}$ or the unite circle $\mathbf{T}$, which is realized as the interval $[-\pi, \pi]$ with coincident endpoints, or a finite interval $[a, b]$. We shall consider the spaces $L_p(G), 0 < p \le \infty$, of all measurable functions $x : G \to \mathbf{R}$ such that $\|x\|_{L_p(G)} < \infty$, where

$$\|x\|_{L_p(G)} := \left\{ \int_G |x(t)|^p \, dt \right\}^{1/p}$$

if $0 < p < \infty$ and

$$\|x\|_{L_\infty(G)} := \sup_{t \in G} \operatorname{vrai} |x(t)| .$$

For $x \in L_p(G)$ we set $E_0(x)_{L_p(G)} := \inf_{c \in \mathbf{R}} \|x - c\|_{L_p(G)}$. We will write $\| \cdot \|_p$ instead of $\|\cdot\|_{L_p(\mathbf{T})}$, $E_0(\cdot)_p$ instead of $E_0(\cdot)_{L_p(\mathbf{T})}$ and L_p instead of $L_p(\mathbf{T})$.

For a differentiable function $x \in L_p(\mathbf{R})$ or $x \in L_p$ we set

$$|||x|||_p := \sup \left\{ E_0(x)_{L_p[a,b]} \ : \ x'(t) \ne 0 \quad \forall t \in (a, b), \quad a, b \in \mathbf{R} \right\} . \qquad (1.1)$$

Mathematics Subject Classification 2000: 41A17, 26D10.

For $r \in \mathbf{N}$, $p > 0$ denote by $L_p^r(G)$ the space of all functions $x \in L_p(G)$ such that $x^{(r-1)}(x^{(0)} := x)$ is locally absolutely continuous and $x^{(r)} \in L_p(G)$. We set $W_\infty^r(G) := \{x \in L_\infty^r(G) : \|x^{(r)}\|_{L_\infty(G)} \leq 1\}$. Let $\varphi_0(t) = \operatorname{sgn} \sin t$, $t \in \mathbf{R}$, and let $\varphi_r(t)$ be its r^{th} 2π-periodic integral the mean value of which is equal to zero.

Exact inequalities of Kolmogorov type

$$\left\|x^{(k)}\right\|_q \leq C \, \|x\|_p^\alpha \left\|x^{(r)}\right\|_s^{1-\alpha} \tag{1.2}$$

for 2π-periodic functions $x \in L_s^r$, where $k, r \in \mathbf{N}$, $k < r; q, p, s \in [1, \infty]; \alpha \in (0, 1)$ are of great importance for many problems of analysis. It is well known [1] that inequality (1.2) holds for any $x \in L_s^r$ if and only if $\alpha \leq \alpha_{cr}$, where

$$\alpha_{cr} := \min\left\{1 - \frac{k}{r}, \ \ \frac{r - k - s^{-1} + q^{-1}}{r - s^{-1} + p^{-1}}\right\}.$$

Note that the inequalities of type (1.2) with the maximal exponent $\alpha = \alpha_{cr}$ are of the most interest.

We will discuss in this paper the inequalities of the form

$$\left\|x^{(k)}\right\|_q \leq C|||x|||_p^\alpha \left\|x^{(r)}\right\|_\infty^{1-\alpha} \tag{1.3}$$

for 2π-periodic functions $x \in L_\infty^r$, where $k, r \in \mathbf{N}$, $k < r; q \in [1, \infty]; p \in (0, \infty]$; $\alpha \in (0, 1)$ and the value $|||x|||_p$ is defined by (1.1). It is easy to see that the maximal exponent α in the inequality of the form (1.3) is $\alpha = (r - k)/(r + 1/p)$. In this paper some new exact inequalities of form (1.3) for functions $x \in L_\infty^r$ and for any $q \in [1, \infty]$, $p \in (0, \infty]$ are obtained, where $\alpha = (r - k)/(r + 1/p)$ (see Theorem 3). By means of Theorem 3 a new exact inequality of Bernstein's type for trigonometric polynomials τ of order $\leq n$ and for any $p \in (0, \infty], q \in [1, \infty]$ is proved (Theorem 4). An analog of Theorem 4 for polynomial splines also is obtained (Theorem 5).

For $r \in \mathbf{R}$, $\lambda > 0$ we set $\varphi_{\lambda,r}(t) := \lambda^{-r}\varphi_r(\lambda t + a_r)$, where a_r is chosen such that the spline $\varphi_{\lambda,r}(t)$ increases on $[-\pi/2\lambda, \pi/2\lambda]$. We need a modification of the Kolmogorov's comparision theorem [2].

Theorem 1. *(see [3]). Let $r \in \mathbf{N}$, $x \in W_\infty^r(\mathbf{R})$ and λ is chosen satisfying condition*

$$|||x|||_\infty = \|\varphi_{\lambda,r}\|_\infty.$$

Let then $[a, b]$ be an interval such that $x'(t) \neq 0$ $\forall t \in (a, b)$; $x'(a) = x'(b) = 0$.

If points $t \in [a, b]$ and $y \in [-\pi/2\lambda, \pi/2\lambda]$ are chosen such that

$$|x(b) - x(t)| = |\varphi_{\lambda,r}(\pi/2\lambda) - \varphi_{\lambda,r}(y)|$$

or such that

$$|x(t)-x(a)| = |\varphi_{\lambda,r}(y)-\varphi_{\lambda,r}(-\pi/2\lambda)|,$$

then

$$|x'(t)| \le |\varphi'_{\lambda,r}(y)|.$$

Repeating the proof of Ligun's inequality [4], but applying Theorem 1 instead of Kolmogorov's comparision theorem, we obtain the following amplification of Ligun's inequality.

Theorem 2. *Let $k, r \in \mathbf{N}$, $k < r$. Then for any function $x \in L^r_\infty$ the inequality*

$$\left\|x^{(k)}\right\|_q \le \frac{\|\varphi_{r-k}\|_q}{\|\varphi_r\|_\infty^{1-k/r}} |||x|||_\infty^{1-k/r} \left\|x^{(r)}\right\|_\infty^{k/r}$$

holds. The inequality becomes equality for functions $x(t) = a\varphi_r\,(nt+b)$, a, $b \in \mathbf{R}$, $n \in \mathbf{N}$.

R e m a r k. It is obvious that $|||x|||_\infty \le \|x\|_{L_\infty(\mathbf{R})}$. Moreover, for any $M > 0$ there exists a function $x \in L^r_\infty(\mathbf{R})$ such that $\frac{\|x\|_{L_\infty(\mathbf{R})}}{|||x|||_\infty} > M$.

2. Some new exact inequalities of Kolmogorov type

Theorem 3. *Let $r, k \in \mathbf{N}$; $k < r$; $q \in [1, \infty]$, $p \in (0, \infty]$. Then for any function $x \in L^r_\infty$ the following inequalities hold:*

$$\left\|x^{(k)}\right\|_q \le \frac{\|\varphi_{r-k}\|_q}{|||\varphi_r|||_p^{\frac{r-k}{r+1/p}}} |||x|||_p^{\frac{r-k}{r+1/p}} \left\|x^{(r)}\right\|_\infty^{\frac{k+1/p}{r+1/p}} \tag{2.1}$$

and

$$|||x|||_\infty \le \frac{\|\varphi_r\|_\infty}{|||\varphi_r|||_p^{\frac{r}{r+1/p}}} |||x|||_p^{\frac{r}{r+1/p}} \left\|x^{(r)}\right\|_\infty^{\frac{1/p}{r+1/p}}. \tag{2.2}$$

The inequalities (2.1) and (2.2) are the best possible and become equalities for functions $x(t) = a\varphi_r(nt+b)$; $a, b \in \mathbf{R}$, $n \in \mathbf{N}$.

P r o o f. Fix any $x \in L^r_\infty$. Taking into account the homogeneity of the inequalities (2.1) and (2.2), we can assume that

$$\|x^{(r)}\|_\infty = 1. \tag{2.3}$$

Let us choose λ satisfying condition

$$|||x|||_\infty = \|\varphi_{\lambda,r}\|_\infty. \tag{2.4}$$

Let us prove that

$$|||x|||_p \ge \frac{1}{2^{1/p}} E_0(\varphi_{\lambda,r})_{L_p[0,2\pi/\lambda]}. \tag{2.5}$$

Since x is the periodic function, there exists the interval $[a,b]$ such that $x'(t) \neq 0$ $\forall t \in (a,b)$ and

$$|||x|||_\infty = E_0(x)_{L_\infty[a,b]}. \tag{2.6}$$

Without loss of generality we assume that the function x increases on $[a,b]$. Denote by $c_p = c_p(x)$ the constant of the best L_p- approximation of the contraction of the function x on the interval $[a,b]$, i.e., such constant that $E_0(x)_{L_p[a,b]} = \|x(t) - c_p(x)\|_{L_p[a,b]}$. It is clear that $x(t) - c_p(x)$ has an zero on $[a,b]$. Denote this zero by z. So

$$x(z) = c_p. \tag{2.7}$$

Let us choose $u \in \left[-\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda}\right]$ such that

$$\varphi_{\lambda,r}\left(\frac{\pi}{2\lambda}\right) - \varphi_{\lambda,r}(u) = x(b) - x(z). \tag{2.8}$$

By (2.4) and (2.6)

$$\varphi_{\lambda,r}(u) - \varphi_{\lambda,r}\left(-\frac{\pi}{2\lambda}\right) = x(z) - x(a). \tag{2.9}$$

It follows from (2.8) and (2.9) that for any $t \in [z,b]$ (or $t \in [a,z]$) there exists $y \in \left[u, \frac{\pi}{2\lambda}\right]$ (or $y \in \left[-\frac{\pi}{2\lambda}, u\right]$) such that

$$\varphi_{\lambda,r}\left(\frac{\pi}{2\lambda}\right) - \varphi_{\lambda,r}(y) = x(b) - x(t) \tag{2.10}$$

or

$$\varphi_{\lambda,r}(y) - \varphi_{\lambda,r}\left(-\frac{\pi}{2\lambda}\right) = x(t) - x(a). \tag{2.11}$$

By Theorem 1

$$|x'(t)| \le |\varphi'_{\lambda,r}(y)|, \tag{2.12}$$

moreover,

$$b - z \ge \frac{\pi}{2\lambda} - u, \quad z - a \ge u + \frac{\pi}{2\lambda}. \tag{2.13}$$

It follows from (2.8)–(2.12) that

$$x(b-s) - x(z) \ge \varphi_{\lambda,r}\left(\frac{\pi}{2\lambda} - s\right) - \varphi_{\lambda,r}(u) \ge 0, \quad s \in \left[0, \frac{\pi}{2\lambda} - u\right] \tag{2.14}$$

and

$$x(a+s)-x(z)\le\varphi_{\lambda,r}\left(-\frac{\pi}{2\lambda}+s\right)-\varphi_{\lambda,r}(u)\le 0,\quad s\in\left[0,\frac{\pi}{2\lambda}+u\right]. \tag{2.15}$$

Using (2.7) and (2.13)–(2.15), we have

$$|||x|||_p^p\ge\|x-c_p\|^p_{L_p[a,b]}=\|x-x(z)\|^p_{L_p[a,b]}=\int_z^b|x(s)-x(z)|^p ds+\int_a^z|x(z)-x(s)|^p ds$$

$$=\int_0^{b-z}|x(b-s)-x(z)|^p ds+\int_0^{z-a}|x(z)-x(a+s)|^p ds$$

$$\ge\int_0^{\frac{\pi}{2\lambda}-u}\left|\varphi_{\lambda,r}\left(\frac{\pi}{2\lambda}-s\right)-\varphi_{\lambda,r}(u)\right|^p ds+\int_0^{\frac{\pi}{2\lambda}+u}\left|\varphi_{\lambda,r}\left(-\frac{\pi}{2\lambda}+s\right)-\varphi_{\lambda,r}(u)\right|^p ds$$

$$=\int_u^{\frac{\pi}{2\lambda}}|\varphi_{\lambda,r}(s)-\varphi_{\lambda,r}(u)|^p ds+\int_{-\frac{\pi}{2\lambda}}^{u}|\varphi_{\lambda,r}(s)-\varphi_{\lambda,r}(u)|^p ds$$

$$=\int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}}|\varphi_{\lambda,r}(s)-\varphi_{\lambda,r}(u)|^p ds=\frac{1}{2}\int_{-\frac{\pi}{\lambda}}^{\frac{\pi}{\lambda}}|\varphi_{\lambda,r}(s)-\varphi_{\lambda,r}(u)|^p ds\ge\frac{1}{2}E_0\left(\varphi_{\lambda,r}\right)^p_{L_p[0,2\pi/\lambda]}.$$

The inequality (2.5) is proved.

On the other hand, it follows from (2.4) and (2.3) by Theorem 2 that

$$\|x^{(k)}\|_q\le\lambda^{-(r-k)}\|\varphi_{r-k}\|_q. \tag{2.16}$$

Let us prove (2.1). Set $\alpha=\frac{r-k}{r+1/p}$. Taking into account the evident equality

$$E_0(\varphi_{\lambda,r})_{L_p[0,2\pi/\lambda]}=\lambda^{-r-1/p}E_0(\varphi_r)_p,\quad |||\varphi_r|||_p=2^{-1/p}E_0(\varphi_r)_p,$$

and applying (2.16) and (2.5), we obtain

$$\frac{\left\|x^{(k)}\right\|_q}{|||x|||_p^\alpha}\le\frac{\lambda^{-(r-k)}\left\|\varphi_{r-k}\right\|_q}{\left[2^{-1/p}E_0(\varphi_{\lambda,r})_{L_p[0,2\pi/\lambda]}\right]^\alpha}=\frac{\lambda^{-(r-k)}\left\|\varphi_{r-k}\right\|_q}{\left[2^{-1/p}\lambda^{-r-1/p}E_0(\varphi_r)_p\right]^\alpha}=\frac{\|\varphi_{r-k}\|_q}{|||\varphi_r|||_p^\alpha}.$$

The inequality (2.1) follows from the last inequality in view of (2.3). In a similar manner one can obtain (2.2) using (2.3)–(2.5). The exactness of the inequalities (2.1) and (2.2) is evident. Theorem is proved.

R e m a r k s. 1. The inequality (2.2) is modification of the inequality

$$E_0(x)_\infty \le \frac{\|\varphi_r\|_\infty}{E_0(\varphi_r)_p^{\frac{r}{r+1/p}}} E_0(x)_p^{\frac{r}{r+1/p}} \left\|x^{(r)}\right\|_\infty^{\frac{1/p}{r+1/p}},$$

that has been obtained in [5].

2. In addition to the inequality (2.1) the following inequality holds:

$$\left\|x^{(k)}\right\|_q \le \frac{\|\varphi_{r-k}\|_q}{E_0(\varphi_r)_p^{\frac{r-k}{r+1/p}}} E_0(x)_p^{\frac{r-k}{r+1/p}} \left\|x^{(r)}\right\|_\infty^{\frac{k+1/p}{r+1/p}}. \tag{2.17}$$

Its proof is analogous to the proof of the inequality (2.1). However the exponent $\alpha = (r-k)/(r+1/p)$ in (2.17) is not the greatest in the case $q < \infty$ or $p < \infty$. On the other hand, the same exponent α in the inequality (2.1) is the best possible .

The inequality (2.1) in the case $q = \infty$ is the modification of the inequality

$$\left\|x^{(k)}\right\|_\infty \le \frac{\|\varphi_{r-k}\|_\infty}{E_0(\varphi_r)_p^{\frac{r-k}{r+1/p}}} E_0(x)_p^{\frac{r-k}{r+1/p}} \left\|x^{(r)}\right\|_\infty^{\frac{k+1/p}{r+1/p}},$$

that has been obtained in [6].

3. Some new exact inequalities of Bernstein type

Denote by $\mathcal{T}_n$ the space of all trigonometric polynomials of order $\le n$.

Theorem 4. *Let $k, n \in \mathbf{N}$; $q \in [1,\infty]$, $p \in (0,\infty]$. Then for any polynomial $\tau \in \mathcal{T}_n$ the inequality holds:*

$$\|\tau^{(k)}\|_q \le n^{k+1/p} \cdot \frac{\|\cos(\cdot)\|_q}{|||\cos(\cdot)|||_p} |||\tau|||_p. \tag{3.1}$$

The inequality (3.1) is the best possible on $\mathcal{T}_n$ and becomes equality for polynomials $\tau(t) = a\cos(nt+b)$, $a, b \in \mathbf{R}$, $n \in \mathbf{N}$.

P r o o f. Let us choose $r \in \mathbf{N}$, $r > k$. Applying Theorem 3, we have

$$\left\|\tau^{(k)}\right\|_q \le \frac{\|\varphi_{r-k}\|_q}{|||\varphi_r|||_p^\alpha} |||\tau|||_p^\alpha \left\|\tau^{(r)}\right\|_\infty^{1-\alpha}, \tag{3.2}$$

where $\alpha = \frac{r-k}{r+1/p}$. Estimating $\left\|\tau^{(r)}\right\|_\infty$ in (3.2) with the help of Bernstein's inequality (see for example [7, p. 20]) $\|\tau^{(k)}\|_\infty \le n^r \|\tau\|_\infty$, we obtain

$$\left\|\tau^{(k)}\right\|_q \le \frac{\|\varphi_{r-k}\|_q}{|||\varphi_r|||_p^\alpha} |||\tau|||_p^\alpha \left(n^r \|\tau\|_\infty\right)^{1-\alpha}. \tag{3.3}$$

Note that

$$|||\cos(\cdot)|||_p = 2^{-\frac{1}{p}} E_0(\cos(\cdot))_p; \quad E_0(\cos(\cdot))_p = \|\cos(\cdot)\|_p$$

(last equality is evident in the case $p \geq 1$; as for the case $p < 1$ see [8]). Hence, letting $r \to \infty$ in (3.3) and taking into account that

$$r(1-\alpha) = r\left(1 - \frac{r-k}{r+1/p}\right) = \frac{r}{r+1/p}\left(k + \frac{1}{p}\right) \to k + \frac{1}{p}$$

and

$$\|\varphi_r\|_p \to \frac{4}{\pi}\|\cos(\cdot)\|_p; \quad |||\varphi_r|||_p \to \frac{4}{\pi}|||\cos(\cdot)|||_p,$$

we get (3.1). The exactness of the inequality (3.1) is evident. Theorem is proved.

The inequality (3.1) in the case $q = \infty$ is the modification of the inequality

$$\|\tau^{(k)}\|_\infty \leq \frac{n^{k+1/p}}{\|\cos(\cdot)\|_p}\|\tau\|_p,$$

that has been obtained in [6].

The inequality (3.1) in the case $q = p = \infty$ is the amplification of Bernstein's inequality (see for example [7, p. 20]). In the case $q < \infty, p = \infty$ it is the amplification of Taikov's inequality [9].

Let $S_{n,r},\ n, r \in \mathbf{N}$ be the set of all 2π-periodic polynomial splines of the order r defect 1 with knots at the points $k\pi/n,\ n \in \mathbf{N},\ k \in \mathbf{Z}$.

In the same manner one can prove the following analog of Theorem 4.

Theorem 5. *Let $n, k, r \in \mathbf{N},\ k < r$; $q \in [1, \infty]$, $p \in (0, \infty]$. Then for any spline $s \in S_{n,r}$ the inequality holds:*

$$\|s^{(k)}\|_q \leq n^{k+\frac{1}{p}} \cdot \frac{\|\varphi_{r-k}\|_q}{|||\varphi_r|||_p}|||s|||_p. \tag{3.4}$$

The inequality (3.4) is the best possible on $S_{n,r}$ and becomes equality for splines $s(t) = a\varphi_r(nt), a \in \mathbf{R},\ n \in \mathbf{N}$.

The inequality (3.4) in the case $q = \infty$ is the modification of the inequality

$$\|s^{(k)}\|_\infty \leq n^{k+\frac{1}{p}} \cdot \frac{\|\varphi_{r-k}\|_\infty}{E_0(\varphi_r)_p} E_0(s)_p,$$

that has been obtained in [6].

The inequality (3.4) in the case $q = p = \infty$ is the amplification of Tikhomirov's inequality [10]. In the case $q < \infty, p = \infty$ it is the amplification of Ligun's inequality [11].

References

[1] *B.E. Klots*, Approximation of differentiable functions by more smooth functions. — *Mat. Zametki* (1977), v. 21, No. 1, p. 21–32. (Russian)

[2] *A.N. Kolmogorov*, Selected papers. Mathematics and Mechanics. Nauka, Moscow (1985). (Russian)

[3] *V.A. Kofanov*, The amplification of comparision theorem and of Kolmogorov's inequality and their applications. — *Ukr. Mat. J.* (2000), v. 54, No. 10, p. 1348–1355. (Russian)

[4] *A.A. Ligun*, Inequalities for upper bounds of functionals. — *Anal. Math.* (1976), v. 2, No. 1, p. 11–40. (Russian)

[5] *V.F. Babenko, V.A. Kofanov, and S.A. Pichugov*, Inequalities of Kolmogorov type and some their applications in approximation theory. — *Rendiconti del Circolo Matematico di Palermo. Serie II, Suppl.* (1998), v. 52, p. 223–237.

[6] *V.F. Babenko, V.A. Kofanov, and S.A. Pichugov*, Inequalities for norms of intermediate derivatives of periodic functions and their applications. — *East J. Appr.* (1997), v. 3, No. 3, p. 351–376.

[7] *A. Zigmund*, Trigonometrical Series. Cambridge University Press, New York, (1959), V. II.

[8] *V.F. Babenko, V.A. Kofanov, and S.A. Pichugov*, On the best approximation of the perfect Euler's splines in L_p, $0 < p < 1$. — Int. Conf. "Computer simulation", (Abstracts), Dneprodzerjinsk (2001), p. 9. (Russian)

[9] *L.V. Taikov*, On generalization of Bernstein's inequality. — *Proc. Steclov Inst. Math.* (1965), No. 78, p. 43–47. (Russian)

[10] *V.M. Tikhomirov*, The width of the sets in functional spaces and the theory of the best approximations. — *Uspekhi Mat. Nauk* (1960), v. 15, No. 3, p. 81–120. (Russian)

[11] *A.A. Ligun*, The exact inequalities for spline functions and the best quadrature formulae. — *Mat. Zametki* (1976), v. 19, No. 6, p. 913–926. (Russian)