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Boundary equations in the contact dynamic problem for thermoelastic media

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The contact dynamic initial boundary value problem for thermoelastic media is under consideration. Its solution is represented by the dynamic analogues of thermoelastic single and double-layer potentials. This representation leads to the system of nonstationary boundary equations. The unique solvability of this system is proved in the one-parameter scale of Sobolev type function spaces .

1. Introduction

Reducing original problems to boundary equations is one of the basic methods for the numerical solution of the boundary value static problems for elastic bodies. The foundations of the potential theory in classical dynamic elasticity problems are established in [1–4]. Boundary equations in two main dynamic problems for thermoelastic media are studied in [5]. The goal of the paper is to study boundary equations in the contact dymamic problem for thermoelastic media by the methods developed in [2–4]. We begin with the statement of the problem.

Let Γ of class C^2 be a closed surface that divides $\mathbf{R}^3$ on the domains Ω^1 (interior) and Ω^2 (exterior). We denote by $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ a displacement at a point $x = (x_1, x_2, x_3)$ at moment t. $\theta(x,t)$ is the difference between present and initial $T_0 > 0$ medium temperature. Let each of the domains

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be occupied by homogeneous elastic medium with different elastic and heat parameters. Upper indices will mark all quantities that concern to thermoelastic media in Ω^1 or Ω^2 respectively.

The elastic coefficients a^s_{ijkl}, $i,j,k,l=1,2,3$, $s=1,2$, for each value of the indices satisfy the symmetry condition $a^s_{ijkl}=a^s_{jikl}=a^s_{ijlk}$ and the elliptic condition $a^s_{ijkl}\eta_{ij}\eta_{kl}\geq a^s_0\eta_{ij}\eta_{ij}$, $\forall\eta_{ij}=\eta_{ji}\in\mathbf{R}$ with a positive a^s_0. Here and later on the summation over repeated Latin indices is used. We denote by β^s_{ij}, $i,j=1,2,3$, $s=1,2$ the coefficients of the symmetric heat stress tensors. If volume forces and outer heat sources are absent, then the displacement and the temperature fields $U^s=(u^s,\theta^s)$ defined in the domains $G^s=\Omega^s\times\mathbf{R}_+$, $\mathbf{R}_+=(0,\infty)$ are the solutions of the problem S [1, 6]:

$$\left\{\begin{array}{ll}
\rho^1\partial_t^2u_i^1-\partial_j\left(a^1_{ijkl}\partial_l u^1_k\right)+\partial_j\left(\beta^1_{ij}\theta^1\right)=0,\quad i=1,2,3, & \multirow{2}{*}{$(x,t)\in G^1;$}\\
c^1_\varepsilon\partial_t\theta^1-\partial_k\left(\lambda^1_{kj}\partial_j\theta^1\right)+\beta^1_{kj}T_0\partial_j\partial_t u^1_k=0, & \\
\rho^2\partial_t^2u_i^2-\partial_j\left(a^2_{ijkl}\partial_l u^2_k\right)+\partial_j\left(\beta^2_{ij}\theta^2\right)=0,\quad i=1,2,3, & (x,t)\in G^2;\\
c^2_\varepsilon\partial_t\theta^2-\partial_k\left(\lambda^2_{kj}\partial_j\theta^2\right)+\beta^2_{kj}T_0\partial_j\partial_t u^2_k=0, & \\
u^s(x,0)=\partial_t u^s(x,0)=\theta^s(x,0)=0,\quad s=1,2, & x\in\Omega;\\
U^1(x,t)=U^2(x,t)+F(x,t), & (x,t)\in\Sigma^+=\Gamma\times\mathbf{R}_+,\\
\left(T^1U^1\right)(x,t)=\left(T^2U^2\right)(x,t)+G(x,t) &
\end{array}\right. \tag{1}$$

where $c^s_\varepsilon>0$ are the constant specific heat of the media, ρ^s are the constant densities of the media, $\{\lambda^s_{ij}\}^3_{i,j=1}$ are the positive definite symmetric tensors, $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$, $j=1,2,3$, T^s are the boundary differential operations on Σ^+

$$\left(T^sU\right)_i=\left\{\begin{array}{r}\left(a^s_{ijkl}\partial_k u_l-\beta^s_{ij}\theta\right)n_j(x),\quad i=1,2,3,\\ \left(\lambda^s_{lj}\partial_j\theta\right)n_l(x),\quad i=4.\end{array}\right.$$

$F(x,t)$, $G(x,t)$ are given four-component vector fields on Σ^+. The assumtion of homogeneity of the equations and the initial conditions does not restrict the generality of the problem essentially since one can transfer existing nonhomogeneities to the boundary conditions. We remark that the problem S is posed formally, its correct formulation will be given after introducing the necessary function spaces.

2. Function spaces

To simplify notations we use the same symbols for space of vector and scalar functions and for their norms. Moreover, we use the same symbol U for functions and vector functions $U(x,t)$ in spaces of originals and $U(x,p)=\mathcal{L}U(x,t)$ for their Laplace transformations with respect to t, where $\mathcal{L}$ is the Laplace transformation operator. Let $H_m(\mathbf{R}^3)$, $m\in\mathbf{R}$, be the standard Sobolev spaces [7, 8]. For each

$p \in \mathbf{C}$, $m \in \mathbf{R}$ we introduce the spaces $H_{m,p}(\mathbf{R}^3)$ coinciding with $H_m(\mathbf{R}^3)$ as sets with their norms

$$\|u\|^2_{m,p} = \int\limits_{\mathbf{R}^3} (1 + |\xi|^2 + |p|^2)^m |\tilde{u}(\xi)|^2 d\xi,$$

where $\tilde{u}(\xi)$ is the distributional Fourier transformation of $u(x)$. By $H_m(\Omega^s)$ and $H_{m,p}(\Omega^s)$, $s = 1, 2$, we denote the spaces of restrictions to Ω^s of elements of $H_m(\mathbf{R}^3)$ and $H_{m,p}(\mathbf{R}^3)$, respectively, with their norms:

$$\|u\|_{m,\Omega^s} = \inf_{v \in H_m(\mathbf{R}^3), v|_{\Omega^s} = u} \|v\|_m \,, \qquad \|u\|_{m,p;\Omega^s} = \inf_{v \in H_{m,p}(\mathbf{R}^3), v|_{\Omega^s} = u} \|v\|_{m,p} \,,$$

where $\|v\|_m$ is the norm of $H_m(\mathbf{R}^3)$. The spaces $H_m(\Gamma)$ and $H_{m,p}(\Gamma)$ are introduced by the standard scheme using the resolution of identity and the corresponding local coordinates [7, 8]. Finally, we introduce the spaces $\mathcal{H}_{m,p}(\Omega^s) = H_{m,p}(\Omega^s) \times H_m(\Omega^s)$ and $\mathcal{H}_{m,p}(\Gamma) = H_{m,p}(\Gamma) \times H_m(\Gamma)$ of four-component vector functions $U = (u, \theta)$ whose first three components u belong to $H_{m,p}(\Omega^s)$ or $H_{m,p}(\Gamma)$ and the last component θ belongs to $H_m(\Omega^s)$ or $H_m(\Gamma)$, respectively. Norms in these spaces are defined by

$$\|U\|^2_{m,p;\Omega^s} = \|u\|^2_{m,p;\Omega^s} + \|\theta\|^2_{m,\Omega^s} \,, \qquad \|U\|^2_{m,p;\Gamma} = \|u\|^2_{m,p;\Gamma} + \|\theta\|^2_{m,\Gamma} \,.$$

We denote by γ^s the trace operators that map $\mathcal{H}_{m,p}(\Omega^s)$ onto $\mathcal{H}_{m-1/2,p}(\Gamma)$ continuously for $m > 1/2$.

For each $\kappa > 0$ we denote by $\mathcal{H}_{\mathcal{L},1,k;\kappa}(\Omega^s)$, $k \in \mathbf{R}$, the spaces of four-component vector functions $U(x,p)$, $x \in \Omega^s$, $p \in \mathbf{C}_\kappa = \{p \in \mathbf{C} : \mathrm{Re}\, p = \sigma > \kappa\}$ that set a holomorphic map from $\mathbf{C}_\kappa$ into the space of four-component vector functions $H_1(\Omega^s)$ with the finite norms defined by

$$\|U\|^2_{1,k,\kappa;\Omega^s} = \sup_{\sigma > \kappa} \int\limits_{\mathbf{R}} (1 + |p|)^{2k} \|U\|^2_{1,p;\Omega^s} \, d\tau, \qquad p = \sigma + i\tau.$$

The spaces $\mathcal{H}_{\mathcal{L},m,k;\kappa}(\Gamma)$ of four-component vector functions defined on the boundary surface Γ with their norms

$$\|G\|^2_{m,k,\kappa;\Gamma} = \sup_{\sigma > \kappa} \int\limits_{\mathbf{R}} (1 + |p|)^{2k} \|G\|^2_{m,p;\Gamma} \, d\tau, \qquad p = \sigma + i\tau,$$

are introduced similarly. Finally, we introduce the spaces $\mathcal{H}_{r;1,k,\kappa}(G^s)$ and $\mathcal{H}_{r;m,k,\kappa}(\Sigma^+)$, formed by inverse Laplace transformations of elements of $\mathcal{H}_{\mathcal{L},1,k;\kappa}(\Omega^s)$ and $\mathcal{H}_{\mathcal{L},m,k;\kappa}(\Gamma)$ with their norms

$$\|U\|_{1,k,\kappa;G^s} = \|\mathcal{L}U\|_{1,k,\kappa;\Omega^s} \,, \; \|U\|_{m,k,\kappa;\Sigma^+} = \|\mathcal{L}U\|_{m,k,\kappa;\Gamma} \,,$$

respectively. We save the notation γ^s for the trace operators mapping continuously the spaces $\mathcal{H}_{r;1,k,\kappa}(G^s)$ onto $\mathcal{H}_{r;1/2,k,\kappa}(\Sigma^+)$.

Now we give the correct formulation of the problem S. The solution of this problem is the pair $\left\{U^1, U^2\right\}$, $U^s = (u^s, \theta^s) \in \mathcal{H}_{r;1,0,\kappa}(G^s)$, $s = 1, 2$, that $\gamma^1 U^1 = \gamma^2 U^2 + F$ and that satisfies the equality

$$\int\limits_{G^1} (-\rho^1 \partial_t u_i^1 \partial_t v_i + a_{ijkl}^1 \partial_l u_k^1 \partial_j v_i + \beta_{ij}^1 \partial_i \theta^1 v_i - c_\varepsilon^1 \theta^1 \partial_t \eta$$
$$+\lambda_{kj}^1 \partial_j \theta^1 \partial_k \eta - \beta_{kj}^1 T_0 \partial_t u_k^1 \partial_j \eta) dx dt + \int\limits_{G^2} (-\rho^2 \partial_t u_i^2 \partial_t v_i$$
$$+a_{ijkl}^2 \partial_l u_k^2 \partial_j v_i + \beta_{ij}^2 \partial_i \theta^2 v_i - c_\varepsilon^2 \theta^2 \partial_t \eta + \lambda_{kj}^2 \partial_j \theta^2 \partial_k \eta$$
$$-\beta_{kj}^2 T_0 \partial_t u_k^2 \partial_j \eta) dx dt = \int\limits_{\Sigma^+} (G_i v_i + G_4 \eta)\, ds dt$$

for each finite vector function $V = (v(x,t), \eta(x,t)) \in C^\infty\left(\mathbf{R}^3 \times \overline{\mathbf{R}_+}\right)$.

3. Thermoelastic potentials

Denote by $\mathcal{T}^s$, $s = 1, 2$, the 4×4 matrix differential operators of thermoelasticity that correspond to the medium parameters in the domains Ω^s. Let $\Phi^s(x,t)$ be the fundamental solutions for the operators $\mathcal{T}^s$ which are equal to zero as $t < 0$. $\Phi^s(x,t)$ are the 4×4 matrices satisfying

$$\begin{cases} \mathcal{T}^s \Phi^s(x,t) = \delta(x,t) I, & (x,t) \in \mathbf{R}^4, \\ \Phi^s(x,t) = 0, & x \in \mathbf{R}^3, t < 0, \end{cases}$$

where $\delta(x,t)$ is the Dirac function and I is the unit matrix. We introduce the thermoelastic single-layer potential with defined on $\Sigma = \Gamma \times \mathbf{R}$ four-component density $\alpha(x,t)$ by

$$(V^s \alpha)(x,t) = \int\limits_{\Sigma} \Phi^s(x - y, t - \tau)\, \alpha(y, \tau)\, ds_y d\tau.$$

The properties of the single-layer potential are studied in [6]. It was shown that at least for smooth and finite on Σ densities this potential satisfy the homogeneous thermoelastic equation outside Γ. Moreover, if the density is equal to zero when $t < 0$ then the potential satisfies zero initial data.

We represent the solution of the problem S by the single-layer potentials $U^s(x,t) = (V^s \alpha^s)(x,t)$, $(x,t) \in G^s$. This representation leads to the boundary equation system

$$\begin{aligned} \left(V^1 \alpha^1\right)(x,t) &= \left(V^2 \alpha^2\right)(x,t) + F(x,t), \\ \left(T^1 V^1 \alpha^1\right)(x,t) &= \left(T^2 V^2 \alpha^2\right)(x,t) + G(x,t), \end{aligned} \qquad (x,t) \in \Sigma^+. \tag{2}$$

The goal of the paper is to prove the unique solvability of the system (2).

After the transition to the Laplace transformation with respect to the time variable the single-layer potential takes the form

$$\left(V_p^s\alpha\right)(x,p) = \int\limits_{\Gamma_0} \Phi^s\,(x-y,p)\,\alpha\,(y,p)\,ds_y,$$

where $\Phi^s\,(x,p)$ is the fundamental solution for the operator $\mathcal{T}_p^s$ that arises from the operator $\mathcal{T}^s$ by the Laplace transformation.

4. Properties of the basic boundary operators

Accomplishing the Laplace transformation in (1) we obtain the problem S_p that consists in seeking $U^s\,(x,p) = (u^s\,(x,p)\,,\theta^s\,(x,p))) \in \mathcal{H}_{1,p}\,(\Omega^s)$ that satisfy

$$\begin{cases} \rho^1 p^2 u_i^1 - \partial_j\left(a_{ijkl}^1\partial_l u_k^1\right) + \partial_j\left(\beta_{ij}^1\theta^1\right) = 0, \quad i=1,2,3, & \\ c_\varepsilon^1 p\theta^1 - \partial_k\left(\lambda_{kj}^1\partial_j\theta^1\right) + \beta_{kj}^1 T_0 p\partial_j u_k^1 = 0, & x\in\Omega^1; \\ \rho^2 p^2 u_i^2 - \partial_j\left(a_{ijkl}^2\partial_l u_k^2\right) + \partial_j\left(\beta_{ij}^2\theta^2\right) = 0, \quad i=1,2,3, & \\ c_\varepsilon^2 p\theta^2 - \partial_k\left(\lambda_{kj}^2\partial_j\theta^2\right) + \beta_{kj}^2 T_0 p\partial_j u_k^2 = 0, & x\in\Omega^2; \\ U^1\,(x,p) = U^2\,(x,p) + F\,(x,p)\,, & \\ \left(T^1U^1\right)(x,p) = \left(T^2U^2\right)(x,p) + G\,(x,p)\,, & x\in\Gamma. \end{cases} \tag{3}$$

The solvability of this boundary value problem is proved easily by the standard methods. Denote by $(\cdot,\cdot)_{0,\Omega^s}$, $\|\cdot\|_{0,\Omega^s}$ the inner product and the norm of the space $L^2(\Omega^s)$, by the $(\cdot,\cdot)_{0,\Gamma}$, $\|\cdot\|_{0,\Gamma}$ the inner product and the norm of the space $L^2(\Gamma)$. We represent the boundary differential expression T^sU as the pair $((T^sU)_e,(T^sU)_t)$ where $(T^sU)_e$ are the first three components of the T^sU and the $(T^sU)_t$ is its fourth component. From (3) it follows that

$$\begin{aligned} \rho^s p^2\,\|u^s\|_{0,\Omega^s}^2 + E^s\,(u^s,u^s) - \left(\theta^s,\beta_{ij}^s\partial_j u_i^s\right)_{0,\Omega^s} &= (-1)^{s-1}\left((T^sU^s)_e\,,u^s\right)_{0,\Gamma}\,, \\ c_\varepsilon^s\bar{p}\,\|\theta^s\|_{0,\Omega^s}^2 + \Lambda^s\,(\theta^s,\theta^s) + T_0\bar{p}\left(\theta^s,\beta_{ij}^s\partial_j u_i^s\right)_{0,\Omega^s} &= (-1)^{s-1}\left(\theta^s,(T^sU)_t\right)_{0,\Gamma}\,, \\ s=1,2, & \end{aligned} \tag{4}$$

where

$$E^s(u,v) = (a_{ijkl}^s\partial_k u_l,\partial_j v_i)_{0,\Omega^s}, \qquad \Lambda^s(\theta,\eta) = (\lambda_{kj}^s\partial_k\theta,\partial_j\eta)_{0,\Omega^s}.$$

Multiplying the first equation in (4) by $T_0^s|p|^{-2}\bar{p}^2$, the second one by $|p|^{-2}\bar{p}$ and adding the results one obtains

$$
\begin{aligned}
&\rho^1 T_0 |p|^2 \left\|u^1\right\|_{0,\Omega^1}^2 + T_0 E^1\left(u^1, u^1\right) + c_\varepsilon^1 \left\|\theta^1\right\|_{0,\Omega^1}^2 + \sigma^{-1}\Lambda^1\left(\theta^1, \theta^1\right) \\
&+\rho^2 T_0 |p|^2 \left\|u^2\right\|_{0,\Omega^2}^2 + T_0 E^2\left(u^2, u^2\right) + c_\varepsilon^2 \left\|\theta^2\right\|_{0,\Omega^2}^2 + \sigma^{-1}\Lambda^2\left(\theta^2, \theta^2\right) \\
&= \sigma^{-1}\mathrm{Re}\left\{T_0\bar{p}\left\{\left((T^1U^1)\right)_e, u^1\right)_{0,\Gamma} - \left((T^2U^2)\right)_e, u^2\right)_{0,\Gamma}\right\} \\
&+ \left(\theta^1, \left(T^1U^1\right)_t\right)_{0,\Gamma} - \left(\theta^2, \left(T^2U^2\right)_t\right)_{0,\Gamma}\right\}.
\end{aligned}
\tag{5}
$$

We denote by the same letter c any positive constants arising in estimates that do not depend on parameter $p \in \mathbf{C}_\kappa$. Note that the constants c may depend on κ. Using the Corn inequality [7] and ellipticity of the tensors $\{a_{ijkl}^s\}_{i,j,k,l=1}^3$ and $\{\lambda_{ij}^s\}_{i,j=1}^3$ for each $p \in \mathbf{C}_\kappa$ we obtain the estimate

$$
\begin{aligned}
\|U^1\|_{1,p;\Omega^1}^2 + \|U^2\|_{1,p;\Omega^2}^2 \quad \le \quad & c\left\{|p|\left|\left((T^1U^1)_e, u^1\right)_{0,\Gamma} - \left((T^2U^2)_e, u^2\right)_{0,\Gamma}\right|\right. \\
&\left.+\left|\left(\theta^1, (T^1U^1)_t\right)_{0,\Gamma} - \left(\theta^2, (T^2U^2)_t\right)_{0,\Gamma}\right|\right\}.
\end{aligned}
$$

Let $U = (u, \theta) \in \mathcal{H}_{1,p}\left(\Omega^s\right)$ be the solution of the problem

$$
\left\{
\begin{array}{ll}
\rho^s p^2 u_i - \partial_j\left(a_{ijkl}^s \partial_l u_k\right) + \partial_j\left(\beta_{ij}^s \theta\right) = 0, \quad i = 1,2,3, & \\
 & x \in \Omega^s; \\
c_\varepsilon^s p\theta - \partial_k\left(\lambda_{kj}^s \partial_j \theta\right) + \beta_{kj}^s T_0^s p \partial_j u_k = 0, & \\
U(x,p) = F(x,p), & x \in \Gamma,
\end{array}
\right.
\tag{6}
$$

where $F = (f, \xi) \in \mathcal{H}_{1/2,p}(\Gamma)$, $\Psi = (\psi, \zeta) \in \mathcal{H}_{1/2,p}(\Gamma)$, $V = (v, \eta) \in \mathcal{H}_{1,p}(\Omega^s)$ is an extention of Ψ into Ω^s: $\gamma^s V = \Psi$. Poincaré–Steklov operators acting on $F \in \mathcal{H}_{1/2,p}(\Gamma)$ are introduced by

$$
\begin{aligned}
&\left(\mathcal{N}_p^s F, \Psi\right)_{0,\Gamma} = \left(\left(\mathcal{N}_p^s F\right)_e, \psi\right)_{0,\Gamma} + \left(\left(\mathcal{N}_p^s F\right)_t, \zeta\right)_{0,\Gamma}, \qquad s = 1,2, \\
&(-1)^{s-1}\left(\left(\mathcal{N}_p^s F\right)_e, \psi\right)_{0,\Gamma} = \rho^s p^2 (u,v)_{0,\Omega^s} + E^s(u,v) - \left(\theta, \beta_{ij}^s \partial_j v_i\right)_{0,\Omega^s}, \\
&(-1)^{s-1}\left(\left(\mathcal{N}_p^s F\right)_t, \zeta\right)_{0,\Gamma} = c_\varepsilon^s p (\theta, \eta)_{0,\Omega^s} + \Lambda^s(\theta, \eta) + T_0^s p\left(\beta_{ij}^s \partial_j u_i, \eta\right)_{0,\Omega^s}.
\end{aligned}
$$

The continuity of the operatiors $\mathcal{N}_p^s$ and the estimate

$$
\|\mathcal{N}^s F\|_{-1/2,p;\Gamma} \le c\,|p|^2 \|F\|_{1/2,p;\Gamma}
\tag{7}
$$

are proved in [5].

Lemma 1. *For each $p \in \mathbf{C}_\kappa$, $\kappa > 0$, the operator $\mathcal{N}_p^1 - \mathcal{N}_p^2$ is an isomorphism from $\mathcal{H}_{1/2,p}(\Gamma)$ to $\mathcal{H}_{-1/2,p}(\Gamma)$. The estimate*

$$
\|F\|_{1/2,p;\Gamma} \le c\,|p| \left\|\left(\mathcal{N}_p^1 - \mathcal{N}_p^2\right) F\right\|_{-1/2,p;\Gamma}
\tag{8}
$$

holds.

P r o o f. The continuity of the operator $\mathcal{N}_p^1 - \mathcal{N}_p^2$ is the consequence of the continuity of the operators $\mathcal{N}_p^s$. The continuity of the inverce operator $\left(\mathcal{N}_p^1 - \mathcal{N}_p^2\right)^{-1}$ follows from the trace theorem and (5):

$$\begin{aligned} &\|F\|^2_{1/2,p;,\Gamma} \le c\{|p| \left\| \left(\left(\mathcal{N}_p^1 - \mathcal{N}_p^2\right) F\right)_e \right\|_{-1/2,p;\Gamma} \|f\|_{1/2,p;\Gamma} \\ &+ \|\xi\|_{1/2;\Gamma} \left\| \left(\left(\mathcal{N}_p^1 - \mathcal{N}_p^2\right) F\right)_t \right\|_{-1/2;\Gamma} \} \\ &\le c\,|p|\,\|F\|_{1/2,p;\Gamma} \left\| \left(\mathcal{N}_p^1 - \mathcal{N}_p^2\right) F\right\|_{-1/2,p;\Gamma}, \end{aligned}$$

that is $\|F\|_{1/2,p;\Gamma} \le c\,|p| \left\| \left(\mathcal{N}_p^1 - \mathcal{N}_p^2\right) F\right\|_{-1/2,p;\Gamma}$. To complete the proof it is sufficient to verify that the range of the operator $\mathcal{N}_p^1 - \mathcal{N}_p^2$ is dense in the space $\mathcal{H}_{-1/2,p}(\Gamma)$. If it is not true we find a nonzero element $\Psi = \{\psi, \zeta\} \in \mathcal{H}_{1/2,p}(\Gamma)$ such that $\left(\left(\mathcal{N}_p^1 - \mathcal{N}_p^2\right) F, \Psi\right)_{0,\Gamma} = 0$ for each $F \in \mathcal{H}_{1/2,p}(\Gamma)$. We take $F = (T_0 p\psi, \zeta)$ and construct the solutions of (6) $U^s \in \mathcal{H}_{1,p}(\Omega^s)$. From (5) it follows that $U^s = 0$, hence $F = 0$ and $\Psi = 0$. This contradiction completes the proof.

We introduce operators $\mathcal{V}_p^s$ by

$$\mathcal{V}_p^s \alpha = \left(V_p^s \alpha\right)(x,p), \qquad x \in \Gamma,$$

on the smooth densities. In [5] the properties of these operators are studied and the following lemma was proved.

Lemma 2. *For each $p \in \mathbf{C}_\kappa$, $\kappa > 0$, the operators $\mathcal{V}_p^s$ can be extended by continuity to the isomorphisms between the spaces $\mathcal{H}_{-1/2,p}(\Gamma)$ and $\mathcal{H}_{1/2,p}(\Gamma)$. For each $\alpha \in \mathcal{H}_{-1/2,p}(\Gamma)$ the estimate*

$$\|\alpha\|_{-1/2,p;\Gamma} \le c\,|p|^2 \left\|\mathcal{V}_p^s \alpha\right\|_{1/2,p;\Gamma} \tag{9}$$

holds.

5. The solvability of the system of boundary equation

Theorem 1. *The boundary equation system (2) is uniquely solvable for any vector functions $F \in \mathcal{H}_{r;1/2,k+2,\kappa}(\Sigma^+)$, $G \in \mathcal{H}_{r;-1/2,k,\kappa}(\Sigma^+)$, $k \in \mathbf{R}$, $\kappa > 0$. Its solving operator for any $k \in \mathbf{R}$, $\kappa > 0$, is the continuous mapping:*

$$\begin{aligned} &\{F, G\} \in \mathcal{H}_{r;1/2,k+2,\kappa}(\Sigma^+) \times \mathcal{H}_{r;-1/2,k,\kappa}(\Sigma^+) \\ &\to \left\{\alpha^1, \alpha^2\right\} \in \mathcal{H}_{r;-1/2,k-3,\kappa}(\Sigma^+) \times \mathcal{H}_{r;-1/2,k-3,\kappa}(\Sigma^+). \end{aligned}$$

P r o o f. After transition to the Laplace transformation the system (2) takes the form

$$\left\{ \begin{aligned} \mathcal{V}_p^1 \alpha^1 &= \mathcal{V}_p^2 \alpha^2 + F, \\ \mathcal{N}_p^1 \mathcal{V}_p^1 \alpha^1 &= \mathcal{N}_p^2 \mathcal{V}_p^2 \alpha^2 + G. \end{aligned} \right.$$

The consequence of this system is the equation $\left(\mathcal{N}_p^1-\mathcal{N}_p^2\right)\mathcal{V}_p^2\alpha^2=G-\mathcal{N}_p^1F$, whose solvability for any $p\in\mathbf{C}_\kappa$ follows from Lemmas 1 and 2. Using the scheme developed in [2–4] one can make sure that if $\{F(x,p),G(x,p)\}$ are holomorphic maps from $\mathbf{C}_\kappa$ to $H_{1/2}(\Gamma)\times H_{-1/2}(\Gamma)$, then $\alpha^2=\left(\mathcal{V}_p^2\right)^{-1}\left(\mathcal{N}_p^1-\mathcal{N}_p^2\right)^{-1}\left(G-\mathcal{N}_p^1F\right)$ is a holomorphic map from $\mathbf{C}_\kappa$ to $H_{-1/2}(\Gamma)$. Finally, from the estimates (7–9) it follows that for $\alpha^2(x,t)=\mathcal{L}^{-1}\alpha^2(x,p)$ the inequality

$$\left\|\alpha^2(x,t)\right\|^2_{-1/2,k-3,\kappa;\Sigma^+}=\sup_{\sigma>\kappa}\int\limits_{\mathbf{R}}(1+|p|)^{2(k-3)}\left\|\alpha^2(x,p)\right\|^2_{-1/2,p;\Gamma}d\tau$$
$$\le c\sup_{\sigma>\kappa}\int\limits_{\mathbf{R}}\left((1+|p|)^{2(k+2)}\left\|F(x,p)\right\|^2_{1/2,p;\Gamma}+(1+|p|)^{2k}\left\|G(x,p)\right\|^2_{-1/2,p;\Gamma}\right)d\tau$$
$$=\left\|F(x,t)\right\|^2_{1/2,k+2,\kappa;\Sigma^+}+\left\|G(x,t)\right\|^2_{-1/2,k,\kappa;\Sigma^+}$$

holds. Analogous estimates for $\alpha^1=\left(\mathcal{V}_p^1\right)^{-1}\left(\left(\mathcal{N}_p^1-\mathcal{N}_p^2\right)^{-1}\left(G-\mathcal{N}_p^1F\right)+F\right)$ complete the proof.

Boundary equations (2) being solved, we construct functions $U^s(x,t)=(V^s\alpha^s)(x,t)$ with obtained densities.

Theorem 2. *Let $F\in\mathcal{H}_{r;1/2,k,\kappa}(\Sigma^+)$, $G\in\mathcal{H}_{r;-1/2,k,\kappa}(\Sigma^+)$, $k\in\mathbf{R}$, $\kappa>0$. $\left\{\alpha^1(x,t),\alpha^2(x,t)\right\}$ is the solution of the boundary equation system (2). Then the vector functions $\left\{U^1(x,t),U^2(x,t)\right\}=\left\{(V^1\alpha^1)(x,t),(V^2\alpha^2)(x,t)\right\}$ are elements of $\mathcal{H}_{r;1,k-1,\kappa}(G^1)\times\mathcal{H}_{r;1,k-1,\kappa}(G^2)$ and they are the solution of S when $k\ge1$.*

P r o o f. From the inequality

$$||U^s(x,p)||_{1,p;\Omega^s}\le c|p|\left\{||F(x,p)||_{1/2,p;\Gamma}+||G(x,p)||_{-1/2,p;\Gamma}\right\},\quad s=1,2,$$

that is the consequence of the definition of the Poincaré–Steklov operators and (5), and from the trace theorem it follows that

$$||U^s(x,t)||^2_{1,k-1,\kappa;G^s}=\sup_{\sigma>\kappa}\int_{\mathbf{R}}(1+|p|)^{2(k-1)}\left\|U^s(x,p)\right\|^2_{1,p;\Omega^s}d\tau$$
$$\le\sup_{\sigma>\kappa}\int_{\mathbf{R}}(1+|p|)^{2k}\left\{\left\|F(x,p)\right\|^2_{1/2,p;\Gamma}+\left\|G(x,p)\right\|^2_{-1/2,p;\Gamma}\right\}d\tau$$
$$=\left\|F(x,t)\right\|^2_{1/2,k,\kappa;\Sigma^+}+\left\|G(x,t)\right\|^2_{-1/2,k,\kappa;\Sigma^+}.$$

These inequalities and the easily verified statement that $U(x,p)$ is a holomorphic map from $\mathbf{C}_\kappa$ to $H_1(\Omega^\pm)$ complete the proof.

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