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Topologies on full groups and normalizers of Cantor minimal systems

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We show how some notions and results of measurable dynamics can be applied to the theory of Cantor minimal systems. Motivated by measurable dynamics, we study the weak and uniform topologies on full groups and their normalizers. The article develops the approach and ideas of [BK].

Introduction

In this note, we are going to consider an interplay between topological and measurable dynamics. In the settings of topological dynamics, one deals with a topological space Ω (it is usually a compact metric space) and a group G acting on Ω by homeomorphisms. In measurable dynamics, the main object is $(X, \mathcal{B}, \mu, \Gamma)$ where $(X, \mathcal{B}, \mu)$ is a standard measure space and Γ is a countable group of measure-preserving or non-singular automorphisms. The core of both theories is the case of $\mathbb{Z}$-actions when dynamics is generated by a single transformation.

It is well known, and supported by many results, that topological and measurable dynamics are, strictly speaking, the completely different theories. One of the reason for this difference is that, from standpoint of topological dynamics, actions of G on Ω would be studied without throwing anything away since it

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may change the topology and dynamics on Ω crucially. In measurable dynamics, sets which are null with respect to a Γ-invariant (or quasi-invariant) measure are regarded as negligible. Moreover, it does not matter what underlying space is considered because all standard measure spaces are isomorphic. For Ω the situation is different: we cannot ignore the topological structure of Ω. But there is a class of topological spaces where we do can it. If Ω is a Cantor set, then its topological properties do not depend on a particular representative since all Cantor sets are homeomorphic. This fact and the fact that there exists a countable family of sets (namely, clopen sets) generating all Borel σ-algebra make the theory of Cantor dynamical systems close to measurable dynamics. This point of view can be supported by remarkable results on complete classification of Cantor minimal systems up to orbit equivalence [GPS1, GPS2, GW] as it was done earlier for ergodic automorphisms. Moreover, it turns out that such notions as full groups, normalizers, module play the roles that are similar to those in measurable dynamics.

In the first section, we consider some concepts of measurable dynamics, related to orbit equivalence, and point out their counterparts in the context of Cantor minimal systems. Because this section is written as a brief survey, we are not going to give all details, a part of obvious definitions and facts will be left to the reader. Section 2 is devoted to the study of normalizers of full groups for Cantor systems. We define a topology on the normalizer of a full group that converts the normalizer into a complete (Polish) space. It is shown that the *mod* map is a continuous group homomorphism.

1. Full groups in measurable and topological dynamics

1. Full groups. We begin with settling our notations. Throughout the paper, we will denote by

- $(X, \mathcal{B}, \mu)$ a standard measure space, i.e., $(X, \mathcal{B})$ is a standard Borel space and μ is a non-atomic measure on $(X, \mathcal{B})$;
- $Aut(X, \mathcal{B}, \mu)$ the group of all non-singular automorphisms;
- Γ a countable subgroup of $Aut(X, \mathcal{B}, \mu)$ (it will be called a group of automorphisms);
- Ω a Cantor set;
- $Homeo(\Omega)$ the group of all homeomorphisms of Ω;
- G a countable subgroup of $Homeo(\Omega)$.

We will consider only countable ergodic groups of automorphisms Γ of a standard measure space and countable minimal groups of homeomorphisms G of a Cantor set. The latter, as a rule, will be generated by a single homeomorphism. If a statement (or definition) can be formulated for automorphisms and homeomorphisms simultaneously, then we will refer to them as transformations.

Given a standard measure space $(X, \mathcal{B}, \mu, \Gamma)$, define the Γ-orbit of $x \in X$ as $\Gamma(x) = \{\gamma x : \gamma \in \Gamma\}$. Similarly, for a Cantor minimal (C.m.) system (Ω, G), $G(\omega) = \{g\omega : g \in G\}$ denotes the G-orbit of $\omega \in \Omega$. Let

$$[\Gamma] = \{R \in Aut(X, \mathcal{B}, \mu) : Rx \in \Gamma x, \quad \text{a.e. } x \in X\}.$$

Then $[\Gamma]$ is a subgroup in $Aut(X, \mathcal{B}, \mu)$ that is called the *full group* generated by Γ. If $\Gamma = (T^n : n \in \mathbb{Z})$, then the full group of Γ is denoted by $[T]$.

In the context of C. m. systems, one similarly defines the full group $[G]$ as a subgroup in $Homeo(\Omega)$ that preserves *every* G-orbit in Ω. In other words, if $R \in [G]$, then Ω can be partitioned into the closed sets $E_g(R) = \{\omega \in \Omega : R(\omega) = g(\omega)\}$ (some of them may be empty) such that $\{g(E_g(R)) : \ g \in G\}$ is also a partition of Ω. It is noteworthy to remark that for connected topological spaces the notion of full group becomes meaningless because $[G]$ must coincide with G.

Consider also a subset $[[G]]$ of $[G]$ that is formed by all homeomorphisms $R \in [G]$ such that all sets $E_g(R)$, $g \in G$, are clopen. It is easily seen that $[[G]]$ is a countable subgroup in $[G]$. It is called the *topological full group*.

Definition 1.1. *Let $\Gamma_i \subset Aut(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$. Then Γ_1 and Γ_2 are orbit equivalent if there exists a measurable one-to-one map $\varphi : X_1 \to X_2$ such that $\varphi(\Gamma_1 x) = \Gamma_2(\varphi x)$ a.e. $x \in X_1$ and $\varphi^{-1} \circ \mu_2 \sim \mu_1$.*

It is clear that the definition of orbit equivalence of two C.m. systems (Ω_1, G_1) and (Ω_2, G_2) can be given by analogy with the following obviuos alternation: φ must be a homeomorphism from Ω_1 onto Ω_2 and the relation $\varphi(G_1 x) = G_2(\varphi x)$ must hold everywhere.

A group of transformations (either of a measure space or a Cantor set) is called *approximately finite* if it is orbit equivalent to a single transformation.

The following statement gives a nontrivial connection between orbit equivalence and full groups. We are not going to give the complete rigorous form for this theorem. The reader can easily supplement it (see, e.g. [HO, KW]).

Theorem 1.2. *Two groups of transformations are orbit equivalent if and only if their full groups are algebraically isomorphic.*

This theorem was mostly proven by Dye and Krieger for automorphisms of a measure space and by Giordano, Putnam, and Skau for C.m. systems [GPS2].

To every full group of transformations one can associate its *normalizer*. By definition, the normalizer $N[\Gamma]$ (or $N[G]$) is formed by all transformations T from $Aut(X, \mathcal{B}, \mu)$ (or from $Homeo(\Omega)$) that commute with the full group, that is $T[\Gamma]T^{-1} = [\Gamma]$ (resp. $T[G]T^{-1} = [G]$).

2. Weak and uniform topologies. The *uniform metric* d is defined on $Aut(X, \mathcal{B}, \mu)$ by $d(S,T) = \mu(\{x \in X : Sx \neq Tx\} \cup \{x \in X : S^{-1}x \neq T^{-1}x\})$. It turns $(Aut(X, \mathcal{B}, \mu), d)$ into a complete nonseparable metric space. Every full group is closed in the uniform topology d_u generated by d. Moreover, $([\Gamma], d_u)$ becomes a Polish space. The d_u-convergence of (T_n) to T means that the measure of the set where T_n differs from T goes to zero.

The *weak topology* δ_w on $Aut(X, \mathcal{B}, \mu)$ can be defined by the metric

$$\delta(S,T) = \sum_{i=1}^{\infty} 2^{-i}\mu(S(E_i) \,\Delta\, T(E_i)),$$

where (E_i) is a countable family of subsets generating $\mathcal{B}$. Equivalently, δ_w can be induced from the unitary group of $L^2(X, \mathcal{B}, \mu)$ equipped with the strong operator topology. $Aut(X, \mathcal{B}, \mu)$, endowed with δ_w, becomes a Polish space. In contrast to the uniform topology, full groups are not closed in the weak topology. A sequence (T_n) converges to T if for every measurable A, $\mu(T_n A \,\Delta\, TA) \to 0$. More facts about these topologies can be found in [Hal, R].

It is natural to try to find out what topologies are analogous to the uniform and weak topologies on $Homeo(\Omega)$ for C.m. systems. Saying about the topological properties of full groups, we should restrict ourself by the case of approximately finite groups; full groups for more general transformation groups have not been investigated yet. In [BK] we considered the most known and studied the sup-topology τ_w on $Homeo(\Omega)$ defined by the metric

$$p(T,S) = \sup_{\omega\in\Omega} d(T(\omega), S(\omega)) + \sup_{\omega\in\Omega} d(T^{-1}(\omega), S^{-1}(\omega))$$

as a topological version of the weak topology (later on we will see more reasons to call the sup-topology as weak one). Then $(Homeo(\Omega), \tau_w)$ is a complete separable metric space. If (T_n) converges to T in τ_w, then for every clopen $F \subset \Omega$, $T_n(F) = T(F)$ for all sufficiently large n. It was remarked in [GPS2] that full groups are not closed in the weak topology. Moreover, if we take the closure of the topological full group $[[T]]$ in τ_w, then it, in general, does not contain $[T]$ and is not contained in $[T]$ as well (see [BK] for details).

We follow [BK] where a topological analogue of the uniform topology was introduced. The definition is motivated by the measurable case. It can be done even in the context of one-to-one Borel maps. Let now Ω be a compact metric space and denote by $Bor(\Omega)$ the set (group) of all one-to-one Borel maps of Ω onto itself. Let $M_1(\Omega)$ be the set of all Borel probability measures on Ω.

Definition 1.3. *The uniform topology τ_u on $Bor(\Omega)$ is defined by the family $\mathcal{U} = \{U(T;\mu_1,\dots,\mu_p;\varepsilon)\}$ of open neighborhoods (the base of topology): given $\varepsilon > 0$, $\mu_1,\dots,\mu_n \in M_1(\Omega)$, and $T \in Bor(\Omega)$, set*

$$U(T;\mu_1,\dots,\mu_n;\varepsilon) = \{S \in Bor(\Omega) : \mu_i(E(S,T)) < \varepsilon,\ i = 1,\dots,n\},$$

where $E(S,T) = \{\omega \in \Omega : S(\omega) \neq T(\omega)\} \cup \{\omega \in \Omega : S^{-1}(\omega) \neq T^{-1}(\omega)\}$.

It is shown in [BK] that $(Bor(\Omega), \tau_u)$ becomes a nonseparable complete topological group and $Homeo(\Omega) \subset Bor(\Omega)$ is not closed with respect to τ_u. We consider the relative topology on $Homeo(\Omega)$ denoted again by τ_u. A sequence (T_n) is τ_u-converging to $S \in Bor(\Omega)$ if and only if for every $\omega \in \Omega$ there exists $n(\omega) \in \mathbb{N}$ such that $T_n(\omega) = S(\omega)$ and $T_n^{-1}(\omega) = S^{-1}(\omega)$ for all $n \geq n(\omega)$. Then the τ_u-closure of $[T]$ in $Homeo(\Omega)$ coincides with $[T]$. If we take the τ_u-closure of the topological full group $[[T]]$, then we get the full group $[T]$. These facts show that the uniform topology τ_u has natural properties related to full groups (in contrast to the weak topology τ_w).

Recall that, in measurable dynamics, weak convergence means convergence on every measurable set. For homeomorphisms of a Cantor set, a weakly converging sequence must eventually stabilize on every clopen set. This fact is a motivation for the following definition. We introduce one more topology (call it τ for a moment) on $Homeo(\Omega)$ by defining its base. Given clopen sets $F_1,\dots,F_n$, set

$$W(T;F_1,\dots,F_n) = \{S \in Homeo(\Omega) : S(F_i) = T(F_i),\ i = 1,\dots,n\}.$$

The next proposition will justify the name of weak topology for τ_w.

Proposition 1.4. *τ is equivalent to τ_w on $Homeo(\Omega)$.*

P r o o f. It suffices to consider the two topologies at the identity map $\mathbb{I}$ only. Fix some $\varepsilon > 0$ and let $\mathcal{P} = (F_i : i = 1,\dots,n)$ be a partition of Ω into clopen sets such that $\mathrm{diam}(F_i) < \varepsilon$, $i = 1,\dots,n$. If $S \in W(\mathbb{I};F_1,\dots,F_n)$, then $S(F_i) = F_i$ and therefore

$$\sup_{\omega\in\Omega} d(S\omega,\omega) + \sup_{\omega\in\Omega} d(S^{-1}\omega,\omega) \leq 2\varepsilon.$$

This proves that $W(\mathbb{I};F_1,\dots,F_n) \subseteq B_{2\varepsilon}(\mathbb{I})$ where $B_{2\varepsilon}(\mathbb{I})$ is the ball centered at $\mathbb{I}$ of the radius 2ε in metric p.

Conversely, let a neighborhood $W(\mathbb{I}; F_1, \dots, F_n)$ be given. Then the sets F_i and $\Omega \setminus F_i$ $(i = 1, \dots, n)$ generate the clopen partition $\mathcal{P} = (E_j : j \in \Lambda)$. Take some positive $\varepsilon < \varepsilon_0$ where

$$\varepsilon_0 = \min\{\min(\mathrm{dist}(E_k, E_j) : k, j \in \Lambda,\ k \neq j),\ \min(\mathrm{diam}(E_j) : j \in \Lambda)\}.$$

Then for any $S \in B_\varepsilon(\mathbb{I})$ we have that $S(E_j) = E_j,\ j \in \Lambda$, i.e., each atom of $\mathcal{P}$ is fixed. Therefore $S(F_i) = F_i,\ i = 1, \dots, n$, because each F_i is a union of some E_j's. This proves that $S \in W(\mathbb{I}; F_1, \dots, F_n)$. ■

The problem of investigation of topological and algebraic structures of full groups has been rather popular in measurable dynamics (see, e.g. [D, Hal, R]). As far as we know, almost nothing is clarified about those properties of full groups for C.m. systems with respect to the two topologies. For example, it is unknown if full groups are contractible, if they have closed normal subgroups etc.

3. Saturated Cantor minimal systems. Let (Ω, T) be a C.m. system. We say that two clopen sets A and B are $[[T]]$-*equivalent* (resp. $[T]$-*equivalent*) if there exists $\gamma \in [[T]]$ (resp. $\gamma \in [T]$) such that $\gamma(A) = B$. It was proved in [GW] that if $\mu(A) = \mu(B)$ for every T-invariant probability measure μ, then A and B are $[T]$-equivalent. It was proved in [GPS2] that two clopen sets A and B are $[[T]]$-equivalent if and only if $\chi_A(x) - \chi_B(x)$ is a coboundary.

The following definition was given in [BK]. Let $M_1(T)$ denote the set of all T-invariant measures.

Definition 1.5. *We say that a C.m. system (Ω, T) is saturated if any two clopen sets A and B from X such that $\mu(A) = \mu(B)$, $\mu \in M_1(T)$, are $[[T]]$-equivalent.*

It turns out that this property of C.m. systems has several equivalent formulations [BK]. We mention here only one: (Ω, T) is saturated if and only if the closure of $[[T]]$ in the weak topology τ_w contains $[T]$.

In measurable dynamics every ergodic automorphism is automatically mod 0 saturated. The following assertion, which is not hard to deduce from the results proved in [GW] and [BK], can be treated as a topological variant of the above statement.

Theorem 1.6. *For any C.m. system (Ω, T) there exists a minimal homeomorphism $S \in [T]$ orbit equivalent to T and such that the C.m. system (Ω, S) is saturated.*

2. Normalizer of Cantor minimal systems

Let (Ω, G) be a C.m. system and let $N[G]$ denote the normalizer of $[G]$. Recall that $R \in N[G]$ if and only if $R(G\omega) = G(R\omega)$ for all $\omega \in \Omega$. We denote by $M_1(G)$ the set of G-invariant probability measures on Ω. Let $Homeo(M_1(G)) = \{\varphi \in Homeo(\Omega) : \varphi(M_1(G)) = M_1(G)\}$. It is obvious that $N[G] \subset Homeo(M_1(G))$.

We introduce a new topology on $N[G]$. Next definition is again motivated by measurable dynamics.

Definition 2.1. *Given $R \in N[G]$, $\mu_1, \dots, \mu_k \in M_1(\Omega)$, and $\varepsilon > 0$, define*

$$U(R; \mu_1, \dots, \mu_k; \varepsilon) = \{P \in N[G] : p(R, P) < \varepsilon,\ \mu_i(E(R\gamma R^{-1}, P\gamma P^{-1})) < \varepsilon,\ i = 1, \dots, k,\ \gamma \in G\}.$$

Then the sets $\{U(R; \mu_1, \dots, \mu_k; \varepsilon)\}$ determine a base of a topology (call it λ) on $N[G]$.

By definition, a sequence (R_n) is λ-converging to R if: (1) $p(R_n, R) \to 0$ and (2) $\forall g \in G,\ R_n g R_n^{-1} \to R g R^{-1}$ in τ_u. It is equivalent to the following two conditions:

- ($\forall$ clopen $F \subseteq \Omega$) ($\exists n(F) \in \mathbb{N}$) such that $R_n(F) = R(F)$ if $n > n(F)$ (see Proposition 1.4);
- ($\forall \omega \in \Omega$) ($\forall g \in G$) ($\exists n(\omega, g) \in \mathbb{N}$) such that for all $n > n(\omega, g)$ one has $R_n g R_n^{-1}(\omega) = R g R^{-1}(\omega)$ or, equivalently, $(R^{-1}R_n)g(\omega) = g(R^{-1}R_n)(\omega)$, i.e., $R^{-1}R_n$ eventually commutes with every $g \in G$.

Theorem 2.2. *$(N[G], \lambda)$ is a complete topological group. If G is approximately finite, then $(N[G], \lambda)$ is separable.*

We first prove a simple lemma that will be used below. It was shown in [BK] that the weak and uniform topologies are not comparable. Nevertheless, one can prove

Lemma 2.3. *Suppose a sequence (T_n) of homeomorphisms of Ω converges to a homeomorphism T in the weak topology and simultaneously (T_n) converges to a homeomorphism T' in the uniform topology. Then $T = T'$.*

P r o o f. We have that for any given $\varepsilon > 0$ and $\omega_0 \in \Omega$, there exists sufficiently large n such that

$$\sup_{\omega \in \Omega} d(T_n(\omega), T(\omega)) < \varepsilon \quad \text{and} \quad T_n(\omega_0) = T'(\omega_0). \qquad (*)$$

If we assumed that $T \neq T'$, then we would find an open subset V and some $r > 0$ such that $d(T(\omega), T'(\omega)) > r$ for all $\omega \in V$. It will lead to a contradiction with $(*)$ if we take ω_0 from V. ■

P r o o f o f T h e o r e m 2.2. We will apply the idea elaborated in [HO]. Consider the space $Y = Homeo(\Omega) \times \prod_{g \in G}[G]$ endowed with the topology $\tau_w \times \prod_{g\in G} \tau_u$. Then Y becomes a complete topological space. Define $\imath : N[G] \to Y$ by $\imath(R) = (R, (RgR^{-1})_{g \in G})$ (recall that G is countable). It is clear that $\imath$ is an injective map and, moreover, $\imath$ is a homeomorphism from $(N[G], \lambda)$ onto $\imath(N[G])$ endowed with the relative topology. To prove the first statement, we should verify only that $\imath(N[G])$ is closed in Y. Let $\imath(R_n)$ be a convergent sequence. Since $(Homeo(\Omega), \tau_w)$ is a complete space, then R_n must converge to a homeomorphism T in the weak topology. Obviously, for every $g \in G$, $p(R_n g R_n^{-1}, TgT^{-1}) \to 0$ as $n \to \infty$. It is clear that every $R_n g R_n^{-1}$ belongs to $[G]$ and $(R_n g R_n^{-1})$ converges in τ_u to a homeomorphism $S(g)$ from $[G]$. It follows from the definition of the topology on Y along with the assumption that $\imath(R_n)$ is convergent in Y. By Lemma 2.3, $S(g) = TgT^{-1}$. It proves that $\imath(R_n) \to (T, (TgT^{-1})_{g\in G}) = \imath(T)$, as $n \to \infty$, and $T \in N[G]$.

To see that the second statement holds, it suffices to note that $[G]$ is separable in the uniform topology when G is approximately finite. Indeed, $[[G]]$ is dense in $[G]$ [BK]. ■

For approximately finite groups of automorphisms the *mod* map from $N[\Gamma]$ onto the centralizer of associated flow plays a very important role in many problems of measurable dynamical systems [BG1, BG2, H]. In [GPS2], a topological counterpart of the *mod* map has been offered. Let (Ω, G) be a C.m. system and let $C(\Omega, \mathbb{Z})$ denote the countable group of $\mathbb{Z}$-valued continuous functions. Define $Z(G) = \{f \in C(\Omega, \mathbb{Z}) : \int_\Omega f d\mu = 0, \ \forall \mu \in M_1(G)\}$. Then, we will denote by $\hat{f}$ the image of $f \in C(\Omega, \mathbb{Z})$ in the qoutient group $C(\Omega, \mathbb{Z})/Z(G)$. Given $R \in N[G]$ and a clopen set F, define

$$mod(R)(\hat{\chi}_F) = \hat{\chi}_{R(F)}. \qquad (**)$$

The *mod* map is well defined because $R(M_1(G)) = M_1(G)$ and therefore $R(Z(G)) = Z(G)$. Since every function from $C(\Omega, \mathbb{Z})$ is a finite linear combination of some characteristic functions, $mod(R)$ can be extended to an automorphism of $C(\Omega, \mathbb{Z})/Z(G)$. Note that this group automorphism preserves the cone of positive functions, therefore it can be considered as an ordered group automorphism. Thus, we get a group homomorphism

$$mod(R) : N[G] \to Aut(C(\Omega, \mathbb{Z})/Z(G))$$

(in fact, *mod* is defined on the set of all homeomorphisms that preserve $M_1(G)$).

Repeating the proof given in [GPS2] for a single minimal homeomorphism, one can show that

$$ker(mod) = \overline{[G]}^{\tau_w}$$

(here bar means the closure in the weak topology). But as we have seen, the weak topology does not reflect properties of full groups and normalizers. In particular, the closure $\overline{[G]}^{\tau_w}$ does not belong to $N[G]$. It would be interesting to find out if the above formula holds for the λ-topology.

Let H be a (countable, abelian) group and let $Aut(H)$ denote all group automorphisms of H. Given $\alpha \in Aut(H)$ and $h_1, \dots, h_n$ from H, define $V(\alpha; h_1, \dots, h_n) = \{\beta \in Aut(H) : \beta(h_i) = \alpha(h_i)\}$. Then the collection $\{V(\alpha; h_1, \dots, h_n)\}$ is a base of a Hausdorff topology σ that turns $Aut(H)$ into a topological group.

Proposition 2.4. *Let (Ω, G) be an approximately finite C. m. system. Then the map $mod : (N[G], \lambda) \to Aut(C(\Omega, \mathbb{Z})/Z(G))$ is continuous.*

P r o o f. We first note that the *mod* map is onto due to [GPS2]. Thus, for every group automorphism α there exists $R \in N[G]$ such that $mod(R) = \alpha$. It suffices to check that if $R_n \to R$, then $mod(R_n) \to mod(R)$. Take a neighborhood $V(mod(R); \hat{f}_1, \dots, \hat{f}_k)$ and show that for sufficiently large n, $mod(R_n) \in V(mod(R); \hat{f}_1, \dots, \hat{f}_k)$. Let $f_i \in C(\Omega, \mathbb{Z})$ be a representative of $\hat{f}_i$, $i = 1, \dots, k$. Then

$$f_i = \sum_{j \in J} a_j^i \chi(F_j^i),$$

where $a_j^i \in \mathbb{Z}$ and F_j^i is clopen, $i = 1, \dots, k$, $j \in J$. It follows from the λ-convergence of (R_n) that $p(R_n, R) \to 0$ and, therefore, $R_n(F)$ eventually equals $R(F)$ for any clopen F. Take n_0 so large that $R_n(F_j^i) = R(F_j^i)$ for all $n > n_0$ and $i = 1, \dots, k$, $j \in J$. From $(**)$ and the above observation, we get that $mod(R_n)(\hat{f}_i) = mod(R)(\hat{f}_i)$, $i = 1, \dots, k$, i.e., $mod(R_n) \in V(mod(R); \hat{f}_1, \dots, \hat{f}_k)$. ∎

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