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On partial fraction expansion for meromorphic functions

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The paper is a short survey of results devoted to partial fraction expansion for meromorphic functions of one complex variable. In particular, this contains new results by the author on representation of a meromorphic function Φ on $\mathbb{C}$ in the form

$$\Phi(z) = \lim_{R\to\infty} \sum_{|b_k|<R} \Phi_k(z) + \alpha(z),$$

where $\{b_k\}_1^\infty$ is the sequence of all its poles arranged in the order of increase of the absolute values and tending to ∞,

$$\left\{ \Phi_k(z) = \sum_{n=1}^{N_k} \frac{A_{k,n}}{(z-b_k)^n},\ k = 1, 2, \dots \right\}$$

is the sequence of principal parts of the Laurent expansion of Φ near the poles, and α is an entire function.

For a meromorphic function Φ on $\mathbb{C}$, let $\Lambda = \{b_k\}_1^\infty$ be the sequence of all its poles arranged in the order of increase of the absolute values and tending to ∞. Let

$$\left\{ \Phi_k(z) = \sum_{n=1}^{N_k} \frac{A_{k,n}}{(z-b_k)^n}\,,\ k = 1, 2, \dots \right\} \tag{1}$$

be the sequence of principal parts of the Laurent expansion of Φ near the poles. The following assertion was proved by Mittag–Leffler (see, e.g. [1, p. 206]).

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Theorem. *In the notation of* (1) *any function* Φ *meromorphic in* $\mathbb{C}$ *admits an expansion*

$$\Phi(z)=\sum_{k=1}^{\infty}[\Phi_k(z)+P_k(z)]+\alpha(z),\quad z\in\mathbb{C}\setminus\Lambda, \tag{2}$$

where α *is an entire function and* $\{P_k(z)\}_1^{\infty}$ *is a sequence of polynomials.*

Let A be the class of entire functions such that the set $\Lambda_f=\{b_k\}$ of all zeros of any function $f\in A$ satisfies the condition $\sum\limits_k \Im(1/b_k)<\infty$. M.G. Krein (see, e.g. [2, p.116]) introduced the subclass $B=\{f\}\subset A$ of entire functions f for which $1/f(z)$ admits the following expansion:

$$\frac{1}{f(z)}=R(z)+\frac{c_{-1}}{z}+z^p\sum_k\frac{c_k}{b_k^p(z-b_k)},\quad z\in\mathbb{C}\setminus\Lambda_f,$$

where $p\in\mathbb{N}$, $R(z)$ is a polynomial, and

$$c_k=1/f'(b_k),\quad \sum_k\frac{|c_k|}{|b_k|^{p+1}}<\infty.$$

This is an expansion of the form (2) because

$$\frac{z^p}{a^p(z-a)}=\frac{1}{z-a}+\frac{1}{a}+\frac{z}{a^2}+\dots+\frac{z^{p-1}}{a^p},\quad \forall\, a\in\mathbb{C}\setminus\{0\}.$$

This formula was applied by M.G. Krein to description of the spectrum of boundary value problems for the Sturm–Liouville operator and to the study of selfadjoint operators in Hilbert space. M.G. Krein proved that every function $f\in B$ is an entire function of exponential type and satisfies the condition

$$\int_{-\infty}^{\infty}\frac{\ln^+|f(t)|}{1+t^2}<\infty.$$

M.V. Keldysh and I.V. Ostrovskii (see [3, Ch. 5, § 6]) investigated some asymptotic properties of Nevanlinna characteristics for meromorphic functions represented by a partial fraction series of the form

$$\sum_k\frac{c_k}{z-b_k},\quad b_k\to\infty,\quad \sum_k\frac{|c_k|}{|b_k|}<\infty.$$

In [4], in terms of conformal mappings, I.V. Ostrovskii described the class $K=\{f\}$ of entire functions that have only real zeros and admit the expansion

$$\frac{1}{f(z)}=c+\frac{c_{-1}}{z}+z\sum_k\frac{c_k}{b_k(z-b_k)},\quad \forall\, z\in\mathbb{C}\setminus\Lambda_f,$$

where c, c_1, c_k, b_k are real numbers, The class K concides with the class of entire functions that is constructed in the following way.

1. Take any domain G of the form

$$G = \{w \in \mathbb{C} : p\pi < \Re w < q\pi, \Im w > 0\} \setminus \left(\bigcup_{p<k<q} T_k \right),$$

where p, k, q are integers (possibly, $p = -\infty$, $q = \infty$), and

$$T_k = \{w \in \mathbb{C} :\ \Re w = k\pi,\ 0 \le \Im w \le h_k\},\ 0 \le h_k < \infty.$$

2. Consider a conformal mapping $z = F(w)$ of this domain G onto the halfplane $\{w \in \mathbb{C} :\ \Im w > 0\}$.

3. Consider all cuts T_k on the boundary of G. Their images under F are certain segments $[a_k, d_k] \subset \mathbb{R}$. For every k, take a point $b_k \in [a_k, d_k]$.

4. Consider any entire function f representable in the form

$$f(z) = \lim_{R\to\infty} dz^n \prod_{|b_k|\le R} \left(1 - \frac{z}{b_k}\right),$$

where $d = \text{const}$, $n = 0$ or 1, and the limit exists.

The paper is devoted to the investigation of conditions implying the existence of an expansion of the form (2) where

$$P_k(z) \equiv 0,\ z \in \mathbb{C}$$

for every $k \in \mathbb{N}$.

Definition 1. *If, in the notation of (2), the series*

$$\sum_{k=1}^{\infty} \Phi_k(z) := \lim_{R\to\infty} \sum_{|b_k|<R} \Phi_k(z)$$

(see (1)) converges uniformly on every compact subset of $\mathbb{C} \setminus \Lambda$, we say that the meromorphic function Φ expands in partial fractions.

From the beginning, we study the case of a meromorphic function $\Phi(z) = 1/g(z)$, where g is an entire function of finite order $\rho > 0$.

The simplest example of such a function is the Gamma-function. It is representable in the form

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(z+k)} + \alpha(z),\ \alpha(z) = \int_1^{\infty} e^{-t} t^{z-1} dt.$$

Here $\alpha(z)$ is an entire function of order 1 with the generalized indicator

$$h_\alpha(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln|\alpha(re^{i\theta})|}{r\ln r} = \max\{0, \cos\theta\}, \quad \theta \in \mathbb{R}.$$

Definition 2. *Suppose that g is an entire of order $\rho > 0$ and of proximate order $\rho(r)$. Let $0 \le R_1 < R_2 < \dots < R_k < \dots$ be the sequence of radii of all circumferences that pass through zeros of g and are centered at 0. The function g is called a function of* <u>*locally regular growth*</u> *if there exists an increasing sequence of positive numbers $\{r_n\}_1^\infty$ such that for some $k \in \mathbb{N}$ we have*

$$R_{k+n} < r_n < R_{k+n+1}, \quad \forall n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} r_n^{-\rho(r_n)} \log|g(r_n e^{i\theta})| = h_g(\theta), \quad \theta \in \mathbb{R}, \tag{3}$$

where h_g is a generalized indicator of g and convergence in (3) is uniform in $\theta \in \mathbb{R}$.

Assume that $\Lambda_g = \{\lambda_k\}$ where Λ_g is the set of all zeros of g. Let A_g be the limit set for the set

$$\{\exp\{i \arg\lambda_k\} : \lambda_k \in \Lambda_g\}.$$

Put $B_g = \{e^{i\theta} : h_g(\theta) > 0\}$.

Theorem 1. *In the above notation suppose that g is an entire function of locally regular growth and its generalized indicator h_g satisfies the unequality*

$$\underline{\lim}_{\lambda_k\to\infty} h_g(\arg\lambda_k) > 0, \tag{4}$$

equivalent to the condition $A_g \subset B_g$. Then the function $1/g$ expands in partial fractions. Moreover (see(2)), α is an entire function in any closed cone K with center at 0, and the following two conditions are fulfilled: $\alpha(z) = o(1)$ as $z \to \infty$, $z \in K$, where K is any closed cone centered at 0, and

$$\mathfrak{A}_g \subset K \subset \mathfrak{B}_g,$$

where $\mathfrak{A}_g$, $\mathfrak{B}_g$ are cones centered at 0 and generated by the sets A_g, B_g, respectively.

R e m a r k. Strict inequality in (4) may fail to occur, as the following example shows. Consider the entire function $g_0(z) = z^{-1} \sin z$, $z \in \mathbb{C}$. It is of order 1 and of normal type, and its indicator is equal to $|\sin\theta|$, $\theta \in \mathbb{R}$. It is known (see, e.g. [1, p. 441]) that

$$\frac{z}{\sin\ z} = 1 + \lim_{m\to\infty} \left\{ 1 + \sum_{k=-m}^{k=m} (-1)^k \left[1 + \frac{\pi k}{z - \pi k} \right] \right\}$$

$$= 1 + 2 \lim_{m\to\infty} \sum_{k=1}^{k=m} (-1)^k \frac{z^2}{z^2 - \pi^2 k^2}.$$

This means that $P_k(z) \equiv 1$ in the expansion of the form (2).

Collorary. *Let g be an entire function of finite order $\rho > 0$ and of locally regular growth. Assume that its indicator is positive. Then its reciprocal $1/g$ expands in partial fractions and in the expansion (2) $\alpha(z) \equiv 0$.*

Now we present some conditions ensuring inequality (4) for entire functions with negative zeros.

Theorem 2. *Suppose that g is an entire function of locally and completely regular growth such that $\Lambda_g \subset (-\infty, 0)$. Let g be the Weierstrass canonical product. The function $1/g$ expands in partial fractions if its order ρ is not an integer, and $\rho \in \bigcup_{k=0}^{\infty} (k, k + 1/2)$ or if g is a function of order $\rho \in \mathbb{N}$ and of maximal type.*

R e m a r k. In the notation of Theorem 2 g is a *generator* of an analytic proximate order (see [5, § 5]), and it belongs to the class of functions treated in Theorems 1.11, 1.12 of the article [5]. For entire functions of order 1 and of maximal type Theorem 2 was proved in [5, § 6].

Now we study the case of a meromorphic function $\Phi(z) = f(z)/g(z)$, where f, g are entire functions of finite order without common zeros.

Definition 3. *Let $\rho_f(r)$ and $\rho_g(r)$ be proximate orders of the functions f and g. The function f is said to grow slower than g if either $r^{\rho_f(r)} = o(r^{\rho_g(r)})$ as $r \to \infty$ or $r^{\rho_f(r)}$ is equivalent to $r^{\rho_g(r)}$ as $r \to \infty$, but*

$$h_f(\theta) < h_g(\theta), \quad \forall \theta \in \mathbb{R},$$

where h_f, h_g are the generalized indicators of f and g.

Theorem 3. *Let g be an entire function satisfying the assumptions of Theorem 1. If a meromorphic function Φ is representable in the form*

$$\Phi(z) = f(z)/g(z) + \beta(z),$$

where an entire function f grows slower than g, and β is an entire function, then Φ expands in partial fractions.

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