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Approximation of subharmonic functions of slow growth

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Let u be a subharmonic function in $\mathbb{C}$, μ_u its Riesz measure. Suppose that $C_1 \le \mu(\{z : R < |z| \le R\psi(R)\} \le C_2$ $(R \ge R_1)$ for some positive constants C_1, C_2, and R_1, and a slowly growing to $+\infty$ function $\psi(r)$ such that $r/\psi(r) \nearrow +\infty$ $(r \to +\infty)$. Then there exist an entire function f, constants $K_1 = K_1(C_1, C_2)$, $K_2 = K_2(C_2)$ and a set $E \subset \mathbb{C}$ such that

$$|u(z) - \log|f(z)|| \le K_1 \log\psi(|z|), \quad z \to \infty, z \notin E,$$

and E can be covered by the system of discs $D_{z_k}(\rho_k)$ satisfying

$$\sum_{R<|z_k|<R\psi(R)} \frac{\rho_k \psi(|z_k|)}{|z_k|} < K_2\,,$$

as $R_2 \to +\infty$. We prove also that the estimate of the exceptional set is sharp up to a constant factor.

1. Introduction

We assume that the reader is familiar with principal notions of the theory of subharmonic functions [1]. Introduce some notations. Let $D_z(t)$ denote the disc $\{\zeta \in \mathbb{C} : |\zeta - z| < t\}$, $z \in \mathbb{C}$, $t > 0$. For a subharmonic function u in $\mathbb{C}$ we put $B(r,u) = \max\{u(z) : |z| = r\}$, $r > 0$, and define the order $\rho[u]$ by the equality $\rho[u] = \overline{\lim}_{r\to+\infty} \log B(r,u)/\log r$. Let also μ_u denote the Riesz measure associated with u and m denote the plane Lebesgue measure.

In 1985 R. Yulmukhametov [2] obtained the following significant result. For any subharmonic function u of order $\rho \in (0, +\infty)$ and $\alpha > \rho$ there exist an entire function f and a set $E_\alpha \subset \mathbb{C}$ such that

$$|u(z) - \log|f(z)|| \le C_\alpha \log|z|, \quad z \to \infty, z \notin E_\alpha, \tag{1.1}$$

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and E_α can be covered by the family of discs $D_{z_j}(t_j)$, $j \in \mathbb{N}$, satisfying $\sum_{|z_j|>R} t_j = O(R^{\rho-\alpha})$, $(R \to +\infty)$.

Recently, Yu. Lyubarskii and Eu. Malinnikova [3] have got rid of the assumption on finite order of growth and have obtained the best possible in some sense estimate for the left-hand side of (1.1) outside an exceptional set.

Theorem A. *Let $u(z)$ be a subharmonic function in $\mathbb{C}$. If for some $R_0 > 0$ and $q > 1$*

$$\mu_u(\{z : R < |z| \le qR\}) > 1, \quad R > R_0, \tag{1.2}$$

then there exists an entire function f satisfying

$$\sup_{R>0} R^{-2} \int\limits_{|z|<R} |u(z) - \log|f(z)||\, dm(z) < \infty,$$

and for each $\varepsilon > 0$ there exists $E_\varepsilon \subset \mathbb{C}$ with $\overline{\lim}_{R\to\infty} m(\{z \in E : |z| < R\})R^{-2} < \varepsilon$ such that $u(z) - \log|f(z)| = O(1)$, $z \notin E_\varepsilon$, $z \to \infty$.

In this paper we are going to approximate a subharmonic function u by the logarithm of an entire function in the uniform metric in the case when (1.2) does not hold. The methods of the works [2] and [3] are used.

Condition (1.2) implies that $n(R, u) \overset{\text{def}}{=} \mu_u(\overline{D_0(R)}) \ge C_0(q) \log R$ $(R \to +\infty)$. We shall consider functions u for which $n(R, u) = o(\log R)$ as $R \to +\infty$.

First, consider the limit case $n(R, u) = O(1)$, $R \to +\infty$. It is known that outside sufficiently small set the best possible estimate of left-hand side of (1.1) is $O(\log|z|)$, in general. In fact, let $u_0(z) = \frac{1}{2}\log|z|$. Suppose that an entire function f satisfies $|\log|f(z)| - u_0(z)| \le \alpha \log|z|$ on a sequence of circles $|z| = r_n \to +\infty$ $(n \to +\infty)$ for some positive number α. Then $B(r_n, \log|f|) \le (\alpha + \frac{1}{2})\log|r_n|$ as $r_n \to +\infty$, consequently, f is necessarily a polynomial, i.e., $\log|f(z)| \sim p\log|z|$ as $z \to \infty$ for some $p \in \mathbb{N} \cup \{0\}$. Therefore, $|\log|f(z)| - u_0(z)| \ge \left(\frac{1}{2} + o(1)\right)\log|z|$ $(z \to \infty)$.

Let Φ be the class of all slowly growing to $+\infty$ functions ψ: $[1, +\infty) \to [2, +\infty)$, (in particular, $\psi(2r) \sim \psi(r)$ as $r \to +\infty$) such that $r/\psi(r) \nearrow +\infty$ as $r \to +\infty$. Remark that functions from Φ are necessary continuous. This follows from nondecrease of $\psi(r)$ and $r/\psi(r)$, $\psi(r) \in \Phi$.

Theorem 1. *Let u be a subharmonic function in $\mathbb{C}$, $\mu = \mu_u$. If for some $\psi \in \Phi$ there exist positive constants C_1, C_2, and R_1 satisfying*

$$(\forall R > R_1): \ C_1 \le \mu(\{z : R < |z| \le R\psi(R)\} \le C_2, \tag{1.3}$$

then there exist an entire function f, constants $K_1 = K_1(C_1, C_2)$, $K_2 = K_2(C_2)$ and a set $E \subset \mathbb{C}$ such that

$$|u(z) - \log|f(z)|| \le K_1 \log \psi(|z|), \quad z \to \infty, z \notin E, \tag{1.4}$$

and E can be covered by the system of discs $D_{z_k}(\rho_k)$ satisfying

$$(\forall R > R_2) : \sum_{R<|z_k|<R\psi(R)} \frac{\rho_k \psi(|z_k|)}{|z_k|} < K_2, \tag{1.5}$$

for some $R_2 > 0$.

2. Proof of Theorem 1

Under the assumptions on ψ we have [4, Ch. 1, Th. 1.2]

$$\log \psi(R) = \log \tilde{\psi}(R) + \int_1^R \beta(t)\, d\log t \equiv \lambda(R) \nearrow +\infty, \tag{2.1}$$

where $\tilde{\psi}$ and β are both continuous functions, and $\tilde{\psi}(t) \to \psi_0 > 0$ and $\beta(t) \ge 0$, $\beta(t) \to 0$ as $t \to +\infty$.

Define $\Psi_1(R) = R\psi(R)$, $\Psi_n(R) = \Psi_1(\Psi_{n-1}(R))$ for $n \in \mathbb{N}$, $\Psi_0(R) \equiv R$, $R > 1$. Without loss of generality, we may assume that $R_0 = \sup\{t \ge 0 : \operatorname{supp}\mu \cap D_0(t) = \varnothing\} > 0$. Now for $p = [2C_2] + 1$ where $[x]$ denotes the integer part of a real number x we define the sequence (T_n) by the conditions $T_0 = R_0$, $n(T_n - 0, u) \le pn \le n(T_n, u)$. Since $\mu(\mathbb{C}) = +\infty$, T_n is defined for all $n \in \mathbb{N}$.

We need estimates for T_n. Clearly, by the choice of p and (1.3) for some natural number $n_0(R)$

$$\frac{T_{n+1}}{T_n} \ge \psi(T_n) \text{ for } n \ge n_0(R_1). \tag{2.2}$$

According to (1.3) for $l = [p/C_1] + 1$, we have $T_{n+1} \le \Psi_l(T_n)$, $n \ge n_0(R_1)$. Thus,

$$\begin{aligned} T_{n+1} &\le \Psi_l(T_n) \\ &= \Psi_{l-1}(T_n)\psi(\Psi_{l-1}(T_n)) = \Psi_{l-2}(T_n)\psi(\Psi_{l-2}(T_n)) \exp\{\lambda(\Psi_{l-1}(T_n))\} \\ &= \Psi_{l-2}(T_n) \exp\{\lambda(\Psi_{l-2}(T_n)) + \lambda(\Psi_{l-1}(T_n))\} = \dots = T_n \exp\Big\{\sum_{j=0}^{l-1} \lambda(\Psi_j(T_n))\Big\}. \end{aligned} \tag{2.3}$$

Further, by (2.1) for any $j \in \mathbb{N}$ we have

$$\frac{\lambda(\Psi_j(R))}{\lambda(\Psi_{j-1}(R))} = 1 + \frac{\int_{\Psi_{j-1}(R)}^{\Psi_j(R)} \beta(t)\, d\log t + \log \frac{\tilde\psi(\Psi_j(R))}{\tilde\psi(\Psi_{j-1}(R))}}{\lambda(\Psi_{j-1}(R))}$$
$$= 1 + \frac{o(1)\log\psi(\Psi_{j-1}(R)) + o(1)}{\lambda(\Psi_{j-1}(R))} = 1 + o(1), \quad R \to +\infty.$$

Thus, by (2.3) we obtain

$$T_{n+1} \le T_n \exp\{(l + o(1))\lambda(T_n)\}, \quad n \to +\infty. \tag{2.4}$$

Denote $A_k = \{\zeta : T_k \le |\zeta| \le T_{k+1}\}$. According to the definition of (T_n), there exists a partition of μ such that $\mu = \sum_k \mu^{(k)}$, $\operatorname{supp}\mu^{(k)} \subset A_k$, $\mu^{(k)}(A_k) = p$. We put $r_k = \exp\Big\{\frac{1}{p}\int_{A_k} \log|\zeta|\, d\mu^{(k)}(\zeta)\Big\}$ and define the entire function f_1 by the equality $f_1(z) = \prod_{k=1}^{+\infty}\Big(1 - \frac{z}{r_k}\Big)^p$. Let $u_1(z) = \int_{|\zeta|\ge R_0} \log\Big|1 - \frac{z}{\zeta}\Big|\, d\mu(\zeta)$.

Consider

$$u_1(z) - \log|f_1(z)| = \sum_{k=1}^{+\infty} \int_{A_k} \Big(\log\Big|1 - \frac{z}{\zeta}\Big| - \log\Big|1 - \frac{z}{r_k}\Big|\Big)\, d\mu^{(k)}(\zeta) \equiv \sum_{k=1}^{+\infty} d_k(z).$$

Remark, that according to the definition of r_k, $\int_{A_k} \log\frac{r_k}{|\zeta|}\, d\mu^{(k)}(\zeta) = 0$, hence

$$d_k(z) = \int_{A_k} \Big(\log\Big|1 - \frac{\zeta}{z}\Big| - \log\Big|1 - \frac{r_k}{z}\Big|\Big)\, d\mu^{(k)}(\zeta). \tag{2.5}$$

Let $z \in A_n$. An estimate of d_k depends on k. First, let $k \ge n+2$. Then, for $\zeta \in A_k$ we have $|\zeta| \ge T_k \ge |z| T_k/T_{n+1}$, $r_k \ge |z| T_k/T_{n+1}$. Using that $T_{n+1}/T_n \to +\infty$ as $n \to +\infty$, we obtain

$$\Big|\log\Big|1 - \frac{z}{\zeta}\Big|\Big| + \Big|\log\Big|1 - \frac{z}{r_k}\Big|\Big| \le 2\Big(\frac{|z|}{|\zeta|} + \frac{|z|}{r_k}\Big) \le 4\frac{T_{n+1}}{T_k}, \quad n \to +\infty, z \in A_n, \zeta \in A_k.$$

Hence

$$\sum_{k \ge n+2} |d_k(z)| = O\Big(\frac{T_{n+1}}{T_{n+2}}\Big) = o(1), \quad n \to +\infty. \tag{2.6}$$

Similarly, using the representation (2.5) we deduce as $n \to +\infty$

$$\sum_{k \le n-2} |d_k(z)| \le 2 \sum_{k \le n-2} \int_{A_k} \Big(\frac{|\zeta|}{|z|} + \frac{r_k}{|z|}\Big) d\mu^{(k)}(\zeta)$$
$$\le 4p \sum_{k \le n-2} \frac{T_{k+1}}{T_n} = O\Big(\frac{T_{n-1}}{T_n}\Big) = o(1). \tag{2.7}$$

It remains to estimate $d_{n-1}(z)$, $d_n(z)$ and $d_{n+1}(z)$. Set $\varphi(t) \equiv t/\psi(t)$, $t \geq 1$. Since $\psi \in \Phi$, we have $\varphi(t) \nearrow +\infty$ and $\varphi(t) = o(t)$, $t \to \infty$. Let us estimate $d_n(z)$. Denote by $N_\varphi(\mu)$ the set of $\left(\frac{1}{\varphi(|z|)}, \varphi(|z|)\right)$-normal points z relatively to μ, i.e., $\mu_z(s) \stackrel{\text{def}}{=} \mu(\overline{D_z(s)}) \leq \frac{s}{\varphi(|z|)}$ for $s \in [0, \varphi(|z|)]$. For such z

$$\left|\int\limits_{A_n} \log\left|\frac{z-\zeta}{z}\right| d\mu^{(n)}(\zeta)\right| \leq \left(\int\limits_{A_n \cap D_z(\varphi(|z|))} + \int\limits_{A_n \setminus D_z(\varphi(|z|))}\right)\left|\log\left|\frac{z-\zeta}{z}\right|\right| d\mu^{(n)}(\zeta)$$
$$\equiv I_{1,n}(z) + I_{2,n}(z).$$

And

$$I_{1,n}(z) = \int\limits_0^{\varphi(|z|)} \left|\log\frac{s}{|z|}\right| d\mu_z^{(n)}(s) \leq \int_0^{\varphi(|z|)} \log\frac{|z|}{s}\, d\mu_z(s) \leq \int\limits_0^{\varphi(|z|)} \log\frac{|z|}{s}\frac{ds}{\varphi(|z|)}$$
$$= \frac{s\log\frac{|z|}{s}}{\varphi(|z|)}\Bigg|_0^{\varphi(|z|)} + \int\limits_0^{\varphi(|z|)} \frac{ds}{\varphi(|z|)} = \log\psi(|z|) + 1. \tag{2.8}$$

Observe, that by (2.4) for $z, \zeta \in A_n$, $\zeta \notin D_z(\varphi(|z|))$

$$\frac{1}{\psi(|z|)} \leq \frac{|z-\zeta|}{|z|} \leq \frac{|z|+|\zeta|}{|z|} \leq 1 + \frac{T_{n+1}}{T_n} \leq (\psi(|z|))^{l+o(1)}, \quad n \to +\infty.$$

Thus, $I_{2,n}(z) \leq (pl + o(1))\log\psi(|z|)$ $(n \to +\infty)$. Together with (2.8) this yields

$$\int\limits_{A_n}\left|\log\left|\frac{z-\zeta}{z}\right|\right| d\mu^{(n)}(\zeta) \leq p(l+1)\log\psi(|z|), n \to +\infty \tag{2.9}$$

for $z \in N_\varphi(\mu) \cap A_n$. Similarly,

$$\int_{A_n}\left|\log\left|\frac{z-r_n}{z}\right|\right| d\mu^{(n)}(\zeta) \leq p(l+1)\log\psi(|z|), \quad n \to +\infty, \tag{2.10}$$

for $z \in A_n \setminus \{\zeta : |r_n - \zeta| \leq \varphi(|\zeta|)\}$, i.e., for $\left(\frac{1}{\varphi(|z|)}, \varphi(|z|)\right)$-normal points z relatively to $\mu_{\log|f_1|}$.

Estimates of $d_{n-1}(z)$ is analogous. For $z \in N_\varphi(\mu) \cap A_n$ we have $I_{1,n-1}(z) \leq \log\psi(|z|) + 1$. If $\zeta \in A_{n-1} \setminus D_z(\varphi(|z|))$ then $-\log\frac{1}{\psi(|z|)} \leq \log\left|\frac{z-\zeta}{z}\right| \leq \log 2$, so $|I_{2,n-1}| \leq p\log\psi(|z|)$. Hence,

$$|d_{n-1}(z)| \leq \int\limits_{A_{n-1}} \left|\log\left|\frac{z-\zeta}{z}\right|\right| d\mu^{(n-1)}(\zeta) + \int\limits_{A_{n-1}} \left|\log\left|\frac{z-r_n}{z}\right|\right| d\mu^{(n-1)}(\zeta)$$
$$\leq 2(p+1)\log\psi(|z|), \quad n \to +\infty, \tag{2.11}$$

for $z \in N_\varphi(\mu + \mu_{\log|f_1|}) \cap A_n$.

Consider $d_{n+1}(z)$. For $z \in N_\varphi(\mu) \cap A_n$ we have

$$\int\limits_{A_{n+1}\cap D_z(\varphi(|z|))} \left|\log\left|\frac{\zeta - z}{\zeta}\right|\right| d\mu^{(n+1)}(\zeta)$$
$$\le \int\limits_{A_{n+1}\cap D_z(\varphi(|z|))} \left(\left|\log\left|\frac{\zeta - z}{z}\right| + \log\left|\frac{z}{\zeta}\right|\right|\right) d\mu^{(n+1)}(\zeta)$$
$$\le (1 + o(1)) \log \psi(|z|), \quad n \to +\infty.$$

Denote $B_{n+1} = \{\zeta \in A_{n+1} : |\zeta - z| \ge \varphi(|z|), |\zeta| < 2|z|\}$, $B'_{n+1} = \{\zeta \in A_{n+1} : |\zeta| \ge 2|z|\}$. Then $(z \in A_n)$

$$\int\limits_{A_{n+1}\setminus D_z(\varphi(|z|))} \left|\log\left|\frac{\zeta - z}{\zeta}\right|\right| d\mu^{(n+1)}(\zeta) = \Bigg(\int\limits_{B_{n+1}} + \int\limits_{B'_{n+1}}\Bigg)\left|\log\left|\frac{\zeta - z}{\zeta}\right|\right| d\mu^{(n+1)}(\zeta)$$
$$\le \log \psi(|z|)(p+1) + 2\int_{B_{n+1}} \left|\frac{z}{\zeta}\right| d\mu^{(n+1)}(\zeta) = (p + 1 + o(1)) \log \psi(|z|), \ n \to +\infty.$$

Similarly, for $z \in N_\varphi(\mu_{\log|f_1|}) \cap A_n$

$$\int\limits_{A_{n+1}} \left|\log\left|\frac{r_{n+1} - z}{r_{n+1}}\right|\right| d\mu^{(n+1)}(\zeta) \le (p + 1 + o(1)) \log \psi(|z|).$$

Thus, for $z \in N_\varphi(\mu + \mu_{\log|f_1|}) \cap A_n$

$$|d_{n+1}(z)| \le (2p + 3) \log \psi(|z|), \quad n \to +\infty. \tag{2.12}$$

Applying (2.6), (2.7), (2.9)–(2.12), we deduce that

$$|\log|f_1(z)| - u_1(z)| \le 2p(l + 4) \log \psi(|z|) = K_1(C_1, C_2) \log \psi(|z|) \tag{2.13}$$

for $z \to \infty$, $z \in N_\varphi(\mu_u + \mu_{\log|f|})$. Since $u(z) - u_1(z)$ is harmonic in $\mathbb{C}$, there exists an entire function f_2 such that $\operatorname{Re} f_2(z) = u(z) - u_1(z)$. Define $f(z) = f_1(z)\exp\{f_2(z)\}$. Then by (2.13)

$$|\log|f(z)| - u(z)| = |\log|f_1(z)| - u_1(z)| \le K_1(C_1, C_2) \log \psi(|z|), \tag{2.14}$$

for $z \to \infty$, $z \in N_\varphi(\mu_u + \mu_{\log|f|})$. Finally, estimate the size of the set $\mathbb{C} \setminus N_\varphi(\mu_u + \mu_{\log|f|})$. If $z \in A_n \setminus N_\varphi(\mu_u)$, then there exists $\tau \in (0, \varphi(|z|))$ such that $\mu_z(\tau) > \tau_z/\varphi(|z|)$. Cover every such point by the disc $D_z(\tau_z)$. According to the Lemma of balls covering [5, Lemma 3.2, p. 246] there exists at most countable subcovering $\{D_{z_k}(\tau_{z_k})\}$ of finite multiplicity q, where q is an absolute constant. Then

$$\sum_{z_k\in A_n} \frac{\tau_{z_k}}{\varphi(|z_k|)} \le \sum_{z_k\in A_n} \mu_{z_k}(\tau_{z_k}) \le 3pq.$$

This together with (2.14) yields us (1.4) and (1.5) with $K_1(C_1, C_2) \le (2C_2 + 1)((2C_2+1)/C_1+6)$, $K_2(C_2) \le 6(2C_2+1)q$. Theorem 1 is proved.

3. Sharpness of the estimate of an exceptional set

We are going to prove that estimate (1.5) of the exceptional set for (1.4) is sharp up to the constant factor.

For $\psi \in \Phi$ define

$$u(z) = u_\psi(z) = \frac{1}{2}\sum_{k=1}^{+\infty} \log\left|1 - \frac{z}{r_k}\right|, \tag{3.1}$$

where $r_0 = 2$, $r_{k+1} = r_k\psi(r_k)$, $k \in \mathbb{N}\cup\{0\}$. Thus, μ_u satisfies condition (1.4) with $C_1 = C_2 = \frac{1}{2}$.

Theorem 2. *Let $\psi \in \Phi$ be such that $\log r = O(\psi(r))$ $(r \to +\infty)$. For any entire function f and any covering of the set $E_f = \{\zeta : |u_\psi(\zeta) - \log|f(\zeta)|| \ge \frac{1}{6}\log\psi(|\zeta|)\}$ by the discs $\{D_k\} = \{D_{\zeta_k}(\sigma_k)\}$ we have*

$$\varlimsup_{R\to+\infty} \sum_{R\le|\zeta_k|<R\psi(R)} \frac{\sigma_k\psi(|\zeta_k|)}{|\zeta_k|} \ge 1. \tag{3.2}$$

P r o o f. Let f be an entire function. Without loss of generality we may assume that $f(0) \ne 0$. We have

$$|u_\psi(z) - \log|f(z)|| \le \frac{1}{6}\log\psi(|z|), \quad z \notin E_f. \tag{3.3}$$

Assume that there exists a covering $\{D_{\zeta_k}(\sigma_k)\}$ of E_f such that (3.2) does not hold. Recall that $\varphi(t) = t\psi^{-1}(t)$, and $\varphi(t) = o(t)$, $\varphi(t) \uparrow +\infty$ as $t \uparrow +\infty$.

Let $E_f^* = \{r > 0 : \partial D_0(r) \cap E_f \ne \varnothing\}$ be the circular projection of E_f. Show that $[r, r+4\varphi(r)]\setminus E_f^* \ne \varnothing$ for all sufficiently large r. In fact, suppose the converse. Then for arbitrary $R_1 > 0$ there exists $r > R_1$ such that $[r, r+4\varphi(r)] \subset E_f^*$. Since (3.2) does not hold, $\sigma_k \le \varphi(|\zeta_k|)$. Thus, for $R = r - \varphi(r)$

$$\begin{aligned}\sum_{R\le|\zeta_k|<R\psi(R)} \frac{\sigma_k\psi(|\zeta_k|)}{|\zeta_k|} &\ge \frac{\psi(\frac{3}{2}r)}{3r/2} \sum_{r-\varphi(r)\le|\zeta_k|\le 3r/2} \sigma_k \\ &\ge \frac{1}{3}\frac{\psi(r)}{r}\,\mathrm{mes}(E_f^*\cap[r, 3r/2]) \ge \frac{4}{3},\end{aligned}$$

a contradiction with our assumption on E_f. So, $[r, r+4\varphi(r)] \setminus E_f^* \neq \varnothing$ for all $r > R_2$ and some $R_2 > 0$. In view of (3.3) this implies $B(r, \log|f|) = O(B(r, u_\psi) + \log\psi(r))$ $(r \to +\infty)$, and by the Hadamard decomposition theorem we obtain $f(z) = c\prod_{n=1}^{+\infty}(1 - z/a_n)$, where $c \in \mathbb{C} \setminus \{0\}$ and the product is absolutely convergent on compacts in $\mathbb{C}$.

Let $N(r, u_\psi) = \int_0^r \mu_u(\overline{D_0(t)})t^{-1}\, dt$, $N(r, f) = \int_0^r \frac{n(t,f)-n(0,f)}{t}\, dt + n(0, f)\log r$, where $n(t, f)$ is the number of zeros f according to the multiplicity in $\overline{D_0(t)}$, be Nevanlinna counting functions of u_ψ and f.

Lemma 1. *Suppose $f(0) \neq 0$. Then, under the hypotheses of Theorem 2*

$$|N(r, u_\psi) - N(r, f)| \leq \frac{1}{6}\log\psi(r) + O(1), \tag{3.4}$$

$$|n(r, u_\psi) - n(r, f)| \leq 1, \quad r \to +\infty. \tag{3.5}$$

P r o o f o f L e m m a 1. For any $r \notin E_f^*$ we have from (3.3) $-\frac{1}{6}\log\psi(r) \leq u_\psi(re^{i\theta}) - \log|f(re^{i\theta})| \leq \frac{1}{6}\log\psi(r)$. Integrating this inequalities with respect to θ on $[0, 2\pi]$ and using Jensen's formula [1, Ch. 3.7], we obtain

$$-\frac{1}{6}\log\psi(r) \leq N(r, u_\psi) - N(r, f) - \log|f(0)| \leq \frac{1}{6}\log\psi(r), \quad r \notin E_f^*. \tag{3.6}$$

Let now r be an arbitrary number greater than R_2, one can find $r^* \in [r, r + 4\varphi(r)] \setminus E_f^*$. Then, by (3.6) as $r \to +\infty$

$$\begin{aligned} N(r, f) &\leq N(r^*, f) \leq N(r^*, u_\psi) + \frac{1}{6}\log\psi(r^*) + O(1) \\ &\leq N(r, u_\psi) + n(r^*, u_\psi)\log\frac{r^*}{r} + \frac{1}{6}\log\psi(r) + O(1) \\ &\leq N(r, u_\psi) + n(r, u_\psi)\psi^{-1}(r) + \frac{1}{6}\log\psi(r) + O(1) \\ &= N(r, u_\psi) + \frac{1}{6}\log\psi(r) + O(1), \end{aligned} \tag{3.7}$$

because $n(r, u_\psi)\psi^{-1}(r) \leq \psi^{-1}(r)\log r = O(1)$ as $r \to +\infty$. Further, there exists $r^{**} \notin E_f^*$ such that $r \geq r^{**} \geq r - 4\varphi(r^{**}) = r - (4 + o(1))\varphi(r)$ $(r \to +\infty)$, and

$$\begin{aligned} N(r, u_\psi) - N(r, f) &\leq N(r, u_\psi) - N(r^{**}, f) \\ &\leq N(r^{**}, u_\psi) - N(r^{**}, f) + n(r, u_\psi)\log\frac{r}{r^{**}} \\ &\leq \frac{1}{6}\log\psi(r^{**}) + O(\log r\psi^{-1}(r)) + O(1) = \frac{1}{6}\log\psi(r) + O(1), \quad r \to +\infty. \end{aligned}$$

The last inequality together with (3.7) yields (3.4).

Let us prove (3.5). Suppose the contrary, i.e. there exists a sequence (t_k) such that $t_k \to +\infty$ $(k \to +\infty)$ and $|n(t_k, u_\psi) - n(t_k, f)| > 1$. Then either i) $n(\tau_k, u_\psi) - n(\tau_k, f) > 1$ or ii) $n(\tau_k, f) - n(\tau_k, u_\psi) > 1$ hold on a subsequence (τ_k) of (t_k), where $\tau_k \to +\infty$ as $k \to +\infty$.

First, consider case i). In this case for any k and $t \in [\tilde{\tau}_k, \tau_k]$, where $\tilde{\tau}_k \psi(\tilde{\tau}_k) = \tau_k$,

$$n(t, u_\psi) - n(t, f) \geq n(\tau_k, u_\psi) - n(\tau_k, f) + n(t, u_\psi) - n(\tau_k, u_\psi) \geq \frac{1}{2}\,.$$

Hence,

$$|N(\tilde{\tau}_k, u_\psi) - N(\tilde{\tau}_k, f)| \geq \int_{\tilde{\tau}_k}^{\tau_k} \frac{n(t, u_\psi) - n(t, f)}{t}\, dt - |N(\tau_k, u_\psi) - N(\tau_k, f)|$$
$$\geq \frac{1}{2} \log \frac{\tau_k}{\tilde{\tau}_k} - \frac{1}{6} \log \psi(\tau_k) + O(1) = \Big(\frac{1}{3} + o(1)\Big) \log \psi(\tilde{\tau}_k), \quad k \to +\infty,$$

a contradiction with (3.4). Thus, case i) is impossible. In the case ii), similarly, $n(t, f) - n(t, u_\psi) \geq \frac{1}{2}$ for $t \in [\tau_k, \tau_k \psi(\tau_k)]$. Then,

$$|N(\tau_k \psi(\tau_k), f) - N(\tau_k \psi(\tau_k), u_\psi)|$$
$$\geq \int_{\tau_k}^{\tau_k \psi(\tau_k)} \frac{n(t, f) - n(t, u_\psi)}{t}\, dt - |N(\tau_k, f) - N(\tau_k, u_\psi)|$$
$$\geq \frac{1}{2} \log \psi(\tau_k) - \frac{1}{6} \log \psi(\tau_k) + O(1) > \Big(\frac{1}{3} + o(1)\Big) \log \psi(\tau_k \psi(\tau_k)), \quad k \to +\infty,$$

which contradicts to (3.4). Hence, the case ii) is impossible too, so (3.5) holds. ■

Let $a_1^k, \ldots, a_{m_k}^k$ be zeros of f lying in $\{\zeta : r_{k-1} < |\zeta| \leq r_{k+2}\}$, then

$$m_k \leq [n(r_{k+2}, u_\psi) - n(r_{k-1}, u_\psi)] + 2 = 3. \tag{3.8}$$

For $|z| = r \in [r_k/2, 2r_{k+1}]$ we get

$$\log |f(z)| - \log |C| - N(r, f) = \sum_{|a_m| \leq r} \log \Big|1 - \frac{a_m}{z}\Big| + \sum_{|a_m| > r} \log \Big|1 - \frac{z}{a_m}\Big|$$
$$= \sum_{|a_j^k| \leq r} \log \Big|1 - \frac{a_j^k}{z}\Big| + \sum_{|a_j^k| > r} \log \Big|1 - \frac{z}{a_j^k}\Big| + O\Big(\sum_{|a_m| \leq r_{k-1}} \frac{|a_m|}{r} + \sum_{|a_m| > r_{k+2}} \frac{r}{|a_m|}\Big). \tag{3.9}$$

Since, by (3.5) $|n(r_l,f)-n(r_{l-1},f)|\le 2$, we have

$$\sum_{|a_m|\le r_{k-1}}\frac{|a_m|}{r}+\sum_{|a_m|>r_{k+2}}\frac{r}{|a_m|}$$
$$=\sum_{|a_m|\le r_0}\frac{|a_m|}{r}+\sum_{l=1}^{k-1}\sum_{r_{l-1}<|a_m|\le r_l}\frac{|a_m|}{r}+\sum_{l=k+2}^{+\infty}\sum_{r_l<|a_m|\le r_{l+1}}\frac{r}{|a_m|}$$
$$\le o(1)+\sum_{l=1}^{k-1}\frac{2r_l}{r}+\sum_{l=k+2}^{+\infty}\frac{2r}{r_l}=O\Big(\frac{r_{k-1}}{r}+\frac{r}{r_{k+2}}\Big)+o(1)=o(1),\quad k\to+\infty.$$

Substituting this estimate in (3.9), we obtain ($|z|=r\in[r_k/2,2r_{k+1}]$)

$$\log|f(z)|-N(r,f)=\sum_{|a_j^k|\le r}\log\Big|1-\frac{a_j^k}{z}\Big|+\sum_{|a_j^k|>r}\log\Big|1-\frac{z}{a_j^k}\Big|+O(1),\quad k\to+\infty. \tag{3.10}$$

Analogously,

$$u_\psi(z)-N(r,u_\psi)=\frac{1}{2}\log\Big|1-\frac{r_k}{z}\Big|+\frac{1}{2}\log\Big|1-\frac{z}{r_{k+1}}\Big|+O(1),$$
$$k\to+\infty,|z|\in[r_k/2,2r_{k+1}]. \tag{3.11}$$

Now, consider two cases ($|z|\in[r_k/2,2r_{k+1}]$): i) There is no a_j^k such that r_k or r_{k+1} lies in $D_a(|a|/2)$ for $a=a_j^k$; ii) There exists a_j^k such that $r_k\in D_a(|a|/2)$ or $r_{k+1}\in D_a(|a|/2)$ for $a=a_j^k$.

In the case i) there is no zeros of f in $D_{r_k}(r_k/3)$. Therefore, for $z\in D_{r_k}(r_k/6)$ using (3.10), (3.11) and (3.4), we obtain ($|z|=r$)

$$\log|f(z)|-u_\psi(z)=N(r,f)+\sum_{|a_j^k|\le r}\log\Big|1-\frac{a_j^k}{z}\Big|+\sum_{|a_j^k|>r}\log\Big|1-\frac{z}{a_j^k}\Big|-N(r,u_\psi)$$
$$-\frac{1}{2}\log\Big|1-\frac{z}{r_k}\Big|+O(1)\ge\Big(-\frac{1}{6}+o(1)\Big)\log\psi(r)-m_k\log 7-\frac{1}{2}\log\Big|1-\frac{z}{r_k}\Big|$$
$$=-\Big(\frac{1}{6}+o(1)\Big)\log\psi(r)-\frac{1}{2}\log\Big|1-\frac{z}{r_k}\Big|,\quad r\to+\infty, \tag{3.12}$$

and for $z\in D_{r_k}(2\varphi(r_k))$

$$\log|f(z)|-u_\psi(z)\ge\Big(\frac{1}{3}+o(1)\Big)\log\psi(r),\quad r\to+\infty. \tag{3.13}$$

Hence, $D_{r_k}(2\varphi(r_k))\subset E_f$.

Case ii). We may assume, that $r_k \in D_a(|a|)$, where $a = a^k_{j_0}$, $1 \le j_0 \le m_k$, is a nearest to r_k zero of f. If $|a - r_k| > 3\varphi(|a|)$, then for $z \in D_a(2\varphi(|a|))$ we have

$$u_\psi(z) - \log|f(z)| \ge N(r, u_\psi) - N(r, f) + \frac{1}{2}\log\Big|1 - \frac{z}{r_k}\Big| - \log\Big|1 - \frac{z}{a^k_{j_0}}\Big|$$

$$- \sum_{|a^k_j| \le r, j \ne j_0} \log\Big|1 - \frac{a^k_j}{z}\Big| - \sum_{|a^k_j| > r, j \ne j_0} \log\Big|1 - \frac{z}{a^k_j}\Big| + O(1) \ge -\Big(\frac{1}{6} + o(1)\Big)\log\psi(r)$$

$$+\frac{1}{2}\log\frac{|r_k - z|}{|z - a|} - \frac{1}{2}\log\Big|1 - \frac{z}{a}\Big| - \sum_{j \ne j_0, |a^k_j| \le r} \log\frac{|a^k_j| + r}{r} - \sum_{|a^k_j| > r, j \ne j_0} \log\frac{|a^k_j| + r}{|a^k_j|}$$

$$\ge \Big(\frac{1}{2} - \frac{1}{6} + o(1)\Big)\log\psi(r) + O(1) = \Big(\frac{1}{3} + o(1)\Big)\log\psi(r), \quad r \to +\infty. \tag{3.14}$$

Finally, if $|a - r_k| \le 3\varphi(|a|)$, then consider the annulus $\Omega_k = D_a(6\varphi(|a|)) \setminus D_a(4\varphi(|a|))$. For $\zeta \in \Omega_k$ we have $|\zeta - r|/|\zeta - a| \in [\frac{1}{6}, 6]$ and similarly to (3.14) we obtain

$$u_\psi(z) - \log|f(z)| \ge \Big(\frac{1}{3} + o(1)\Big)\log\psi(r), \quad z \in \Omega_k,\ k \to +\infty. \tag{3.15}$$

Suppose again that $\{D_{\zeta_m}(\sigma_m)\}$ is a covering of E_f. It follows from (3.13)–(3.15) that $E_f \cap \{\zeta : r_k/2 \le |\zeta| \le 3r_k\}$ contains a disc $D_{\xi_k}(2\varphi(|\xi_k|))$. If we assume that (3.2) does not hold, then for all $m \ge m_0$ we have $\sigma_m \le \varphi(|\zeta_m|)$ for some $m_0 \in \mathbb{N}$, so there exists ζ_{l_k} with

$$|\zeta_{l_k} - \xi_k| \le \sigma_{l_k} = o(|\xi_k|), \quad k \to +\infty. \tag{3.16}$$

Since $D_{\xi_k}(2\varphi(|\xi_k|) \subset \bigcup_{l \in L_k} D_{\zeta_l}(\sigma_l)$, where L_k is the set of all indices l_k satisfying (3.16), we have $\sum_{l \in L_k} \sigma_l \ge 2\varphi(|\xi_k|)$. Thus, using (3.16), we get

$$\varlimsup_{r \to +\infty} \sum_{R \le |\zeta_m| < R\psi(R)} \frac{\sigma_m \psi(|\zeta_m|)}{|\zeta_m|} \ge \varlimsup_{k \to +\infty} \sum_{\frac{r_k}{2} < |\zeta_m| < 3r_k} \frac{\sigma_m \psi(|\zeta_m|)}{|\zeta_m|}$$

$$\ge \varlimsup_{k \to +\infty} \inf_{l \in L_k} \frac{\psi(|\zeta_l|)}{|\zeta_l|} 2\varphi(|\xi_k|) = 2.$$

Inequality (3.2) is proved. ∎

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